SATURATION ASSUMPTIONS FOR A 1D CONVECTION-DIFFUSION MODEL

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Abstract. We refer to the saturation assumptions on the finite element approximation for a one dimensional convection-diffusion model. By examining piecewise linear finite elements with refined mesh by half and hierarchical bases, we verify the saturation results, respectively.

1. Introduction

The saturation assumption has been an essential part in establishing the efficient a posteriori error estimators. One of the most popular a posteriori error estimators is the hierarchical estimators introduced by Bank and Weiser [4] for an elliptic problem and symmetric problem. The saturation assumption asserts that the best approximation error in the energy norm with piecewise quadratic finite elements is strictly smaller than that of piecewise linear elements. In this note, we will verify that the saturation assumptions hold true for the piecewise linear hierarchical elements on a singularly perturbed one-dimensional convection-diffusion equation. A good overview for hierarchical basis functions can be found in [7]. The advantages of hierarchical bases are principally connected to

the possibility of obtaining a posteriori error estimators from the analysis of the solution components of high level([2]). Bank and Smith [3] presented an analysis of an a posteriori error estimator based on the use of hierarchical basis functions under the presumption of the saturation assumption between the uniform and hierarchical mesh refinements. However, the saturation assumption is difficult to ascertain, in practice. Dörfler and Nochetto [5] have used the comparison technique with the residual estimator to circumvent the obscure saturation assumption instead.

Let us consider the simple one dimensional convection-diffusion problem

(1.1)
$$
\begin{aligned}
-\epsilon u''(x) + u'(x) &= 1, \qquad 0 < x < 1, \\
u(0) &= 1, \qquad u(1) = 0,
\end{aligned}
$$

where $0 < \epsilon < 1$ is a small positive parameter called the singular perturbation parameter. The exact solution of the problem (1.1) is given by

$$
(1.2) \qquad u(x) = x - K \left(2\exp(-\frac{1-x}{\epsilon}) - \left(1 + \exp(-\frac{1}{\epsilon})\right) \right),
$$

where $K = \left(1 - \exp(-\frac{1}{\epsilon})\right)^{-1}$. And we also have

$$
(1.3) \qquad u'(x) = 1 - \frac{2K}{\epsilon}\exp(-\frac{1-x}{\epsilon}).
$$

We know that it presents a boundary layer of size $O(\epsilon)$ near the right end boundary $x = 1$ if ϵ is small as shown in Figure 1.

Let us take the solution space $V = H_0^1((0, 1))$ for the variational problem of (1.1): Seek $u \in V$ such that

$$
(1.4) \qquad \int_0^1 (\epsilon u' \, v' - u \, v') \, dx = \int_0^1 v \, dx, \, \forall v \in V.
$$

We begin approximation of (1.4) by the Galerkin method with linear finite elements on a uniform mesh. More precisely, we choose a positive integer N, and we set $h = 1/N$, $x_j = jh$, for $j = 0, 1, \cdots, N$. Then we approximate the variational problem with the space

$$
V_h = \left\{ v \in \mathcal{C}([0, 1]) \,|\, v\Big|_{[x_j, \, x_{j+1}]} \in P_1, \ j = 0, 1, \cdots, N-1 \right\}.
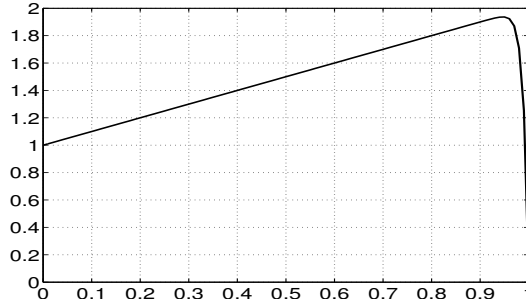$$

FIGURE 1. Exact solution with a boundary layer at $x = 1$ when $\epsilon = 0.01$

Thus the solution will be of the form

$$(1.5) \qquad u_h(x) = \sum_{j=0}^{N-1} u_j \, \phi_j(x) \,, \;\; u_j = u(x_j) \,,$$

where $\phi_j \in V_h$ are linear Lagrange basis functions. It is well known that the standard Galerkin discretizaton gives rise to unstable oscillations unless the exact solution is regular and the discretization parameter is sufficiently small.

Rossi [6] achieved the saturation assumption for a one-dimensional convection-diffusion model in a different setting to ours by considering an artificial diffusion in conjunction with stabilization techniques in the uniform mesh. The purpose of this paper is to present some direct proofs for the saturation assumptions to the model. This paper is organized as follows. In section 2, we will verify that the saturation assumption is fully satisfied under the piecewise linear elements in mesh refinement by half, and in section 3 the saturation for the piecewise linear hierarchical elements shall be presented.

2. Saturation for piecewise linear elements in mesh refinement

We will examine the piecewise linear elements with mesh refinements by half to verify the saturation assumption for the model problem.

THEOREM 2.1. *Let $u_h \in V_h$ and $u_{\frac{h}{2}} \in V_{\frac{h}{2}}$ be finite element solutions of the problem (1.4). Then the saturation assumption holds true in the*

sense that there exists $1/2 \leq \beta < 1$ independent of the mesh size h such that

(2.1)
$$\|u - u_{\frac{h}{2}}\| \leq \beta \|u - u_h\|,$$

where $\|\cdot\|$ denotes the energy norm.

Proof. To prove (2.1), at first we evaluate the error in the subinterval $[x_j, x_{j+1}]$, that is the integral

$$\mathcal{I}_j(h) = \int_{x_j}^{x_{j+1}} |u'(x) - u'_h(x)|^2 \, dx\,.$$

By (1.5), it follows that

(2.2)
$$\begin{aligned}
u_h{'}(x)\Big|_{\left[x_j,\, x_{j+1}\right]} \\
= \frac{1}{h}(u_{j+1} - u_j) \\
= 1 - \frac{2K}{h}\exp(-\frac{1}{\epsilon})\left(\exp(\frac{h}{\epsilon}) - 1\right)\exp(\frac{jh}{\epsilon})\,.
\end{aligned}$$

Hence from (1.2), (1.3) and (2.2), we have

$$\begin{aligned}
&\mathcal{I}_j(h) \\
&= \int_{x_j}^{x_{j+1}} |u'(x) - u'_h(x)|^2 \, dx \\
&= 4K^2\exp(-\frac{2}{\epsilon})\int_{x_j}^{x_{j+1}} \left|-\frac{1}{\epsilon}\exp(\frac{x}{\epsilon}) + \frac{1}{h}\left(\exp(\frac{h}{\epsilon}) - 1\right)\exp(\frac{jh}{\epsilon})\right|^2 dx \\
&= 4K^2\exp(-\frac{2}{\epsilon})\left[\frac{1}{2\epsilon}\left(\exp(\frac{2(j+1)h}{\epsilon}) - \exp(\frac{2jh}{\epsilon})\right) \right. \\
&\qquad\qquad\qquad\qquad \left. -\frac{1}{h}\left(\exp(\frac{h}{\epsilon}) - 1\right)^2\exp(\frac{2jh}{\epsilon})\right] \\
&= 4K^2\left[\frac{1}{2\epsilon}\left(1 - \exp(-\frac{2h}{\epsilon})\right) - \frac{1}{h}\left(1 - \exp(-\frac{h}{\epsilon})\right)^2\right] \\
&\qquad\qquad\qquad\qquad \cdot \exp(-\frac{2}{\epsilon})\exp(\frac{2}{\epsilon}(j+1)h)\,.
\end{aligned}$$

Using $N = 1/h$, it is followed that

$$\sum_{j=0}^{N-1} \exp(\frac{2}{\epsilon}(j+1)h) = \exp(\frac{2h}{\epsilon}) \sum_{j=0}^{N-1} \left(\exp(\frac{2h}{\epsilon})\right)^j$$

$$= \exp(\frac{2h}{\epsilon}) \frac{1 - \exp(\frac{2}{\epsilon})}{1 - \exp(\frac{2h}{\epsilon})}.$$

Thus the approximation error in the energy norm shall be

$$\|u - u_h\|^2 = \sum_{j=0}^{N-1} \mathcal{I}_j(h)$$

$$(2.3) \qquad = 4K^2 \left[\frac{1}{2\epsilon} \left(1 - \exp(-\frac{2h}{\epsilon})\right) - \frac{1}{h} \left(1 - \exp(-\frac{h}{\epsilon})\right)^2 \right]$$

$$\cdot \exp(-\frac{2}{\epsilon}) \exp(\frac{2h}{\epsilon}) \frac{1 - \exp(\frac{2}{\epsilon})}{1 - \exp(\frac{2h}{\epsilon})}.$$

Using this result, we can evaluate the error in the refined mash as follows

$$\|u - u_{\frac{h}{2}}\|^2 = \sum_{j=0}^{2N-1} \mathcal{I}_j\left(\frac{h}{2}\right)$$

$$= 4K^2 \left[\frac{1}{2\epsilon} \left(1 - \exp(-\frac{h}{\epsilon})\right) - \frac{2}{h} \left(1 - \exp(-\frac{h}{2\epsilon})\right)^2 \right]$$

$$\cdot \sum_{j=0}^{2N-1} \exp(-\frac{2}{\epsilon}(1 - (j+1)\frac{h}{2}))$$

$$= 4K^2 \left[\frac{1}{2\epsilon} \left(1 - \exp(-\frac{h}{\epsilon})\right) - \frac{2}{h} \left(1 - \exp(-\frac{h}{2\epsilon})\right)^2 \right]$$

$$\cdot \exp(-\frac{2}{\epsilon}) \exp(\frac{h}{\epsilon}) \frac{1 - \exp(\frac{2}{\epsilon})}{1 - \exp(\frac{h}{\epsilon})}.$$

Now, we consider the ratio

$$r(h, \epsilon) = \frac{\|u - u_{\frac{h}{2}}\|^2}{\|u - u_h\|^2} \, .$$

If we prove that there exists β which is independent on h such that $r(h, \epsilon) \leq \beta^2 < 1$ then (2.1) is verified. Taking $x = h/\epsilon$, the ratio is simplified into

$$
\begin{aligned}
r(h, \epsilon) &= \frac{x(1 - \exp(-x)) - 4\left(1 - \exp(-\frac{x}{2})\right)^2}{x\left(1 - \exp(-2x)\right) - 2\left(1 - \exp(-x)\right)^2} \frac{1 - \exp(-2x)}{1 - \exp(-x)} \\
&: = R(x) \, .
\end{aligned}
$$

Simple considerations give rise that $R(x)$ is a strictly increasing function. Also it is easy to check that

$$\lim_{x \to 0^+} R(x) = \frac{1}{4}, \quad \lim_{x \to \infty} R(x) = 1 \, .$$

Whenever h is small enough, we have a sharper result

$$\beta^2 = 1/4 + o(1)$$

by considering the expansion

$$R(x) = \frac{1}{4} + \frac{3x^2}{160} - \frac{89x^4}{179200} + \mathrm{O}(x^6) \, .$$

Therefore, taking advantage from the bound $h \leq 1$, we deduce

$$r(h, \epsilon) \leq \beta^2 = R\left(\frac{1}{\epsilon}\right) < 1 \, ,$$

so that we have proved (2.1).   □

It is obvious that $V_{\frac{h}{2}} \subset V_h \subset V$ and $u_{\frac{h}{2}} \in V_{\frac{h}{2}}$. The bases of $V_{\frac{h}{2}}$ are composed of piecewise linear functions as in Figure 2 (a).

3. Saturation for piecewise linear hierarchical architecture

In this section, we will examine the case of linear hierarchical bases by supplementing the linear Lagrange basis functions $\bar{\phi}_{j+\frac{1}{2}}$ at the middle points $x_{j+\frac{1}{2}}$ of subinterval as shown in Figure 2 (b).

Let us denote the approximation space by piecewise linear hierarchical bases to $\bar{V}_h$. It is obvious that $\bar{V}_h = V_h \bigoplus$ linear span of $\{\bar{\phi}_{j+\frac{1}{2}}\}$.
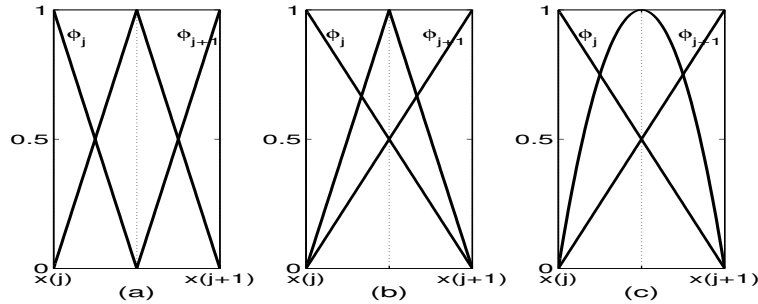
FIGURE 2. (a) linear elements in mesh refinement by half (b) linear hierarchical elements (c) linear quadratic hierarchical elements

Compared with the pure mesh refinements, the main advantages of taking hierarchical bases come from the facts that it is easy to update the results based on the previous coarser mesh as well as it reduces the computational effort. For this reason, some studies has been done to develop a posteriori error estimators by using hierarchical bases([1], [3]).

LEMMA 3.1. *Let $\bar{u}_h \in \bar{V}_h$ be the finite element solutions by piecewise linear hierarchical bases for the problem (1.4). Then the error in the energy norm is followed by*

$$\|u - \bar{u}_h\|^2$$

(3.1)
$$= 4K^2 \exp(-\frac{2}{\epsilon}) \left[\frac{1}{2\epsilon} \left(1 - \exp(-\frac{h}{\epsilon}) \right) - \frac{2}{h} \left(1 - \exp(-\frac{h}{2\epsilon}) \right)^2 \right]$$
$$\cdot \left(1 + \exp(-\frac{h}{\epsilon}) \right) \exp(\frac{2h}{\epsilon}) \frac{1 - \exp(\frac{2}{\epsilon})}{1 - \exp(\frac{2h}{\epsilon})} \, .$$

Proof. Hierarchical approximation in the second level just supplements $\bar{\phi}_{j+\frac{1}{2}}$, so that the approximation will be

$$\bar{u}_h(x) = \sum_{j=0}^{N-1} \left[u_j \phi_j(x) + \delta_{j+\frac{1}{2}} \bar{\phi}_{j+\frac{1}{2}}(x) \right], \ u_0 = u(x_0) = 1 \, .$$

Hence using

$$
\begin{aligned}
\bar{u}_h(x_{j+\frac{1}{2}}) &= u_j \phi_j(x_{j+\frac{1}{2}}) + u_{j+1}\phi_j(x_{j+\frac{1}{2}}) + \delta_{j+\frac{1}{2}}\bar{\phi}_{j+\frac{1}{2}}(x_{j+\frac{1}{2}}) \\
&= \frac{1}{2}u_j + \frac{1}{2}u_{j+1} + \delta_{j+\frac{1}{2}}\,,
\end{aligned}
$$

we have

$$
\delta_{j+\frac{1}{2}} = u_{j+\frac{1}{2}} - \frac{1}{2}(u_j + u_{j+1})\,.
$$

For the estimation of the error, we need to evaluate $\bar{u}_h{}'(x)\big|_{\left[x_j,\,x_{j+\frac{1}{2}}\right]}$ and

$\bar{u}_h{}'(x)\big|_{\left[x_{j+\frac{1}{2}},\,x_{j+1}\right]}$ separately. Direct computation leads to

$$
\begin{aligned}
&\bar{u}_h{}'(x)\Big|_{\left[x_j,\,x_{j+\frac{1}{2}}\right]} \\
&= u_j \phi_j{}'(x) + u_{j+1}\phi_{j+1}{}'(x) + \delta_{j+\frac{1}{2}}\bar{\phi}_{j+\frac{1}{2}}{}'(x)\Big|_{\left[x_j,\,x_{j+\frac{1}{2}}\right]} \\
&= -\frac{1}{h}u_j + \frac{1}{h}u_{j+1} + \frac{2}{h}\delta_{j+\frac{1}{2}} \\
&= -\frac{1}{h}u_j + \frac{1}{h}u_{j+1} + \frac{2}{h}\left(u_{j+\frac{1}{2}} - \frac{1}{2}(u_j + u_{j+1})\right) \\
&= \frac{2}{h}\left(u_{j+\frac{1}{2}} - u_j\right) \\
&= \frac{2}{h}\left[\left(x_{j+\frac{1}{2}} - x_j\right) - 2K\exp(-\frac{1}{\epsilon})\left(\exp(\frac{h}{2\epsilon}) - 1\right)\exp(\frac{jh}{\epsilon})\right] \\
&= 1 - \frac{4K}{h}\exp(-\frac{1}{\epsilon})\left(\exp(\frac{h}{2\epsilon}) - 1\right)\exp(\frac{jh}{\epsilon})\,.
\end{aligned}
$$

Likewise, one can also derive

$$
\bar{u}_h{}'(x)\Big|_{\left[x_{j+\frac{1}{2}},\,x_{j+1}\right]} = 1 - \frac{4K}{h}\exp(-\frac{1}{\epsilon})\left(\exp(\frac{h}{2\epsilon}) - 1\right)\exp(\frac{(j+\frac{1}{2})h}{\epsilon})\,.
$$

Let us put

$$
\bar{\mathcal{I}}_h = \int_{x_j}^{x_{j+1}} |\,u'(x) - \bar{u}_h{}'(x)|^2 \; dx\,.
$$

Then using

$$\int_{x_j}^{x_{j+\frac{1}{2}}} |\, u'(x) - \bar{u}_h{}'(x)|^2 \, dx$$

$$= \int_{x_j}^{x_{j+\frac{1}{2}}} \left| \left(1 - \frac{2K}{\epsilon} \exp(-\frac{1-x}{\epsilon}) \right) \right.$$

$$\left. - \left(1 - \frac{4K}{h} \exp(-\frac{1}{\epsilon}) \left(\exp(\frac{h}{2\epsilon}) - 1 \right) \exp(\frac{jh}{\epsilon}) \right) \right|^2 \, dx$$

$$= 4K^2 \exp(-\frac{2}{\epsilon}) \left[\frac{1}{2\epsilon} \left(\exp(\frac{h}{\epsilon}) - 1 \right) - \frac{2}{h} \left(\exp(\frac{h}{2\epsilon}) - 1 \right)^2 \right] \exp(\frac{2jh}{\epsilon})$$

and

$$\int_{x_{j+\frac{1}{2}}}^{x_{j+1}} |\, u'(x) - \bar{u}_h{}'(x)|^2 \, dx$$

$$= 4K^2 \exp(-\frac{2}{\epsilon}) \left[\frac{1}{2\epsilon} \left(\exp(\frac{h}{\epsilon}) - 1 \right) \right.$$

$$\left. - \frac{2}{h} \left(\exp(\frac{h}{2\epsilon}) - 1 \right)^2 \right] \exp(\frac{(2j+1)h}{\epsilon}),$$

we can obtain

$$\bar{\mathcal{I}}_h = \int_{x_j}^{x_{j+1}} |\, u'(x) - \bar{u}_h{}'(x)|^2 \, dx$$

$$= 4K^2 \exp(-\frac{2}{\epsilon}) \left[\frac{1}{2\epsilon} \left(\exp(\frac{h}{\epsilon}) - 1 \right) - \frac{2}{h} \left(\exp(\frac{h}{2\epsilon}) - 1 \right)^2 \right]$$

$$\cdot \exp(\frac{2jh}{\epsilon}) \left(\exp(\frac{h}{\epsilon}) + 1 \right).$$

By summing up these results over the whole interval, the error estimation (3.1) for the hierarchical approximation is established. □

We are now ready to show the saturation result for the piecewise linear hierarchical bases.

THEOREM 3.2 *Let $\bar{u}_h \in \bar{V}_h$ be the finite element solutions by piecewise linear hierarchical bases for the problem (1.4). Then the saturation assumption holds true in the sense that there exists $1/2 \le \bar{\beta} < 1$ independent of the mesh size h such that*

(3.2) $$\|u - \bar{u}_h\| \le \bar{\beta} \, \|u - u_h\|.$$

Proof. Let us consider the ratio

$$\bar{r}(h, \epsilon) = \frac{\|u - \bar{u}_h\|^2}{\|u - u_h\|^2}.$$

Using (2.3), (3.1), and by taking $x = h/\epsilon$, the ratio is simplified into

$$\bar{r}(h, \epsilon)$$

$$= \frac{\left[\frac{1}{2\epsilon} \left(1 - \exp(-\frac{h}{\epsilon}) \right) - \frac{2}{h} \left(1 - \exp(-\frac{h}{2\epsilon}) \right)^2 \right] \left(1 + \exp(-\frac{h}{\epsilon}) \right)}{\left[\frac{1}{2\epsilon} \left(1 - \exp(-\frac{2h}{\epsilon}) \right) - \frac{1}{h} \left(1 - \exp(-\frac{h}{\epsilon}) \right)^2 \right]}$$

$$= \frac{\left[x \left(1 - \exp(-x) \right) - 4 \left(1 - \exp(-\frac{x}{2}) \right)^2 \right] \left(1 + \exp(-x) \right)}{\left[x \left(1 - \exp(-2x) \right) - 2 \left(1 - \exp(-x) \right)^2 \right]}$$

$$= : \bar{R}(x).$$

Simple considerations also give rise that $\bar{R}(x)$ is a strictly increasing function. Moreover, it is not difficult to check that

$$\lim_{x \to 0^+} \bar{R}(x) = \frac{1}{4}, \quad \lim_{x \to \infty} \bar{R}(x) = 1.$$

Whenever h is small enough, we also have a sharper result

$$\bar{\beta}^2 = 1/4 + o(1)$$

Therefore, taking advantage from the bound $h \leq 1$, we deduce

$$\bar{r}(h, \epsilon) \leq \bar{\beta}^2 = \bar{R} \left(\frac{1}{\epsilon} \right) < 1,$$

which completes the theorem. $\square$

Using our result, one may also examine the piecewise linear-quadratic hierarchical bases as in Figure 2 (c). In this case, the following piecewise quadratic functions $\tilde{\phi}_{j+\frac{1}{2}}$ are supplemented in the second level

$$\tilde{\phi}_{j+\frac{1}{2}}(x) = \begin{cases} 1 - 4 \left(\dfrac{x - x_{j+\frac{1}{2}}}{h} \right)^2 & x_j \leq x \leq x_{j+1}, \\ 0, & \text{else}. \end{cases}$$

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