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Computational Thinking based Mathematical Program for Free Semester System[1,2]

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In recent years, coding education has been globally emphasized and the Free Semester System will be executed to the public schools in Korea from 2016. With the introduction of the Free Semester System and the rising demand of Computational Thinking (CT) capacity, this research aims to design 'learning environment' in which learners can design and construct mathematical objects through computers and print them out through 3D printers. Furthermore, it will design learning mathematics by constructing the figurate number patterns from 'soma cubes' in the playing context and connecting those to algebraic and combinatorial patterns, which will allow students to experience mathematical connectivity. It is expected that the activities of designing figurate number patterns suggested in this research will not only strengthen CT capacity in relation to mathematical thinking but also serve as a meaningful program for the Free Semester System in terms of career experience as 3D printers can be widely used.

Keywords: computational thinking, coding education, free semester, 3D printer, 3D turtle representation, figurate number pattern
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1. INTRODUCTION

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Computational Thinking (CT) is a key capacity required for creative problem-solving in the 21st century, and there is an increasing emphasis on the coding education to strengthen CT. In Korea, the ‘Free Semester System’ will be introduced in middle schools with the aim of letting students find their “dreams” and “talents” while they are liberated from the burden of taking exams for one semester and go through real-life experience of problem-solving, communication and career exploration. Promoting CT is in line with the purpose of the ‘Free Semester System’ as it is a core capacity needed for nurturing complicated and creative thinking in connection with mathematics, science and art and is related to various careers needed for the future society. Chapter 1 will discuss how to enhance CT demanded by the contemporary society under the introduction of the ‘Free Semester System’ and how it will be connected to mathematics curriculum.

1.1. Computational Thinking and Coding Education

There have been many attempts by American schools in 1980s to introduce programming in their curriculums, but the demand for programming education fell in mid-1990s with the introduction of CD-ROM. However, the world is suddenly witnessing an increasing emphasis on coding education for the last couple of years. In the United States, non-profit organizations such as code.org are providing free on-line coding education programs and Bill Gates, the founder of Microsoft, and Mark Zuckerberg, the founder of Facebook, have also joined in the campaigns such as ‘Hour of Code’ which aim for enhancing computer programming and coding education. Moreover, in the UK, software coding will be included in the mandatory course for every school curriculum from this year’s fall semester and teachers in elementary, middle and high schools will be trained to become coding experts. With the rise in the emphasis of coding education these days in Korea, the Ministry of Science, ICT and Future Planning introduced the ‘Software Professional Training Course’ while developing its education model through Science, Technology and Society program.

These days the term ‘coding’ is replacing ‘programming.’ The existing programming education stressed specialized ‘computational language’ such as Java/Java Script, Python, and C/C++, HTML, mostly used by computer scientists and programming experts. These days, however, coding education emphasizes the importance of enhancing ‘CT.’ Cuny, Snyder & Wing (2010) defined CT as “the thought processes involved in formulating problems and their solutions so that the solutions are represented in a form that can be effectively carried out by an information-processing agent” and the key is the capacity for ‘abstractionism’ and ‘automation.’ ‘Abstractionism’ is the capacity to collect and analyze data needed for analysis and problem-solving while building problem-solving models after selecting necessary factors. ‘Automation’ is the capacity to realize these models into computing system and automating the problem-solving (Lee, 2014). CT provides special perspective to solve problems as it expands the principle of computer science into other areas, divides problems into smaller sectors, reorganizes their relationships and constructs again an algorithm covering the entire structure (Kafai & Burke, 2013). As an important tool to let students go beyond simplistic learning limited to one sector and enable comprehensive and complicated thoughts, CT is hailed as a key capacity for creative problem solving required in the 21st century.

1.2. Computational Thinking and Free Semester

In line with the emphasis on coding education, structural changes are emerging in the education system of Korean public schools. In 2013, the Ministry of Education announced the 'Plan for Implementing Pilot Free Semester System' for middle schools. The Free Semester System 'introduces flexibility into the curriculum for middle school students to help them find their genuine "dreams" and "talents" by freeing them from any types of tests during one semester and letting them go through various experience like career exploration.' At present, it is being tested at various research schools and will be expanded to the entire country from 2016.

When introducing the Free Semester System, the Korean Educational Development Institute (KEDI) suggested enhancing problem-solving, communication, and discussion as a way to diversify 'teaching and learning methods.' Furthermore, the Seoul Metropolitan Office of Education is recommending running classes through convergence and integration during the free semester by connecting the curriculum to career and converging different subjects. However, it seems that the system has already faced significant difficulties in schools that have introduced them. Teachers rarely have any manuals related to the Free Semester. In performance evaluation, majority of the subjects focus on 'studying scholars, writing journals on book reading related to careers,' far from being future career education connected to the curriculum (Cho, So, Jung & Lee, 2014).

The world of job is changing fast. Historically, many jobs have been disappearing and emerging. The Industrial Revolution brought a significant number of industrial workforces whereas the advancement of the information technology created a lot of IT jobs. It is a huge challenge to devise curriculum related to career education in the midst of the fast changing society. There is a growing need for mathematics education to adjust itself to the changing society. During the Free Semester System, math education should be more than just solving problems in the textbooks. It should ultimately contribute to the career exploration as students use math in the process of convergent problem solving and find answers to the questions of 'why do we have to learn math?' and 'how is math applied to different sectors?'

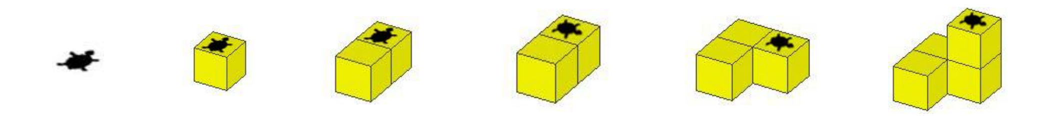
As a key capacity to nurture complicated and creative thinking in connection with math, science and art and highly relevant to various jobs needed in the future society, CT is in line with the fundamental spirit of the 'Free Semester System.' Moreover, 3D printers, which are becoming widely commercialized these days, provide specific methods of realizing virtual artifact that has been composed through CT. With the introduction of the Free Semester System and the rising demand of CT capacity, this research aims to design 'learning environment' in which learners can design and construct mathematical objects through computers and print them out through 3D printers. Furthermore, it will design learning mathematics by constructing the figurate number patterns from 'soma cubes' in the playing context and connecting those to algebraic and combinatorial patterns, which will allow students to experience mathematical connectivity. It is expected that the activities of designing figurate number patterns suggested in this research will not only strengthen CT capacity in relation to mathematical thinking but also serve as a meaningful program for the Free Semester System in terms of career experience as 3D printers can be widely used.

2. 3D CUBE PATTERN DESIGN AND 3D PRINTER

Through constructionism learning theory, Papert (1980) stresses the importance of 'learning by design' whereby children are naturally led to the process of composing knowledge mentally by engaging in activities that make physically meaningful objects. After all, learning is an active process in which learners gain new understanding on the world around them through active exploration and educators' role should be providing an appropriate environment in which learners engage in exploring, creating and doing the meaningful world (Resnick, 2002; Jenkins, 2012). In this context, Papert devised LOGO MicroWorlds where learners can construct geometric objects by simply giving basic commands such as "forward" and "rotate." On the basis of LOGO's turtle metaphor, this research devises a 'learning environment,' where learners can construct and explore 3D objects, and suggests 'mathematical context' where it can be used and studied.

2.1. 3D Turtle Representation System and 3D Printer

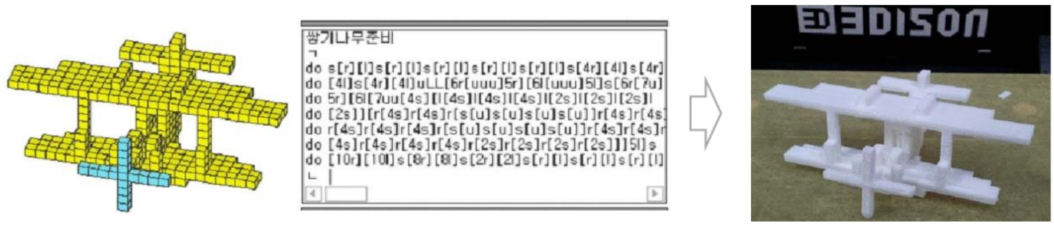
LOGO has been serving as a specific method and tool for realizing constructionism as it allows learners to design and construct virtual artifact through turtle agents. Furthermore, LOGO-based activities are based on CT as they enable learners to take geometric properties into abstract level through embodied simulation, which is in turn automatically changed into a language that turtle agents can understand. Cho, Kim, Song & Lee (2010) and Cho, Song & Lee (2011) focused on the fact that the existing LOGO was too difficult for students as they are programming languages. They therefore changed the ‘action command’ into simple ‘action symbols’ and designed ‘3D turtle representation system’ so that three-dimensional objects can be constructed. In other words, the existing LOGO used a metaphor of drawing figures through lines but the three-dimension LOGO uses a metaphor of making spatial objects as turtle agents pile up cubes. As shown in Figure 1, 3D turtle representation system is composed of action letters s and t that move from front to rear, action letters R and L that rotate to right or left, and action letters u and d that move from up and down. The representation system has been developed into three versions depending on whether the turtle agents move on absolute coordinate or relative coordinate (Lee et al., 2010).



initial state	s	s	R	s	u

Figure 1. 3D turtle representation system

Thanks to the development of 3D printers, it has become possible to realize the artifacts made at virtual space through the 3D turtle representation system into concrete forms. Figure 2 shows the airplane made through the turtle representation system and actually printed out through 3D printer. With the spread of 3D printers, the traditional embodied simulation process that served as a communication channel between turtle agents and learners has now evolved into communication between humans and machines; and artifacts composed in the virtual world can now be realized in a real world.



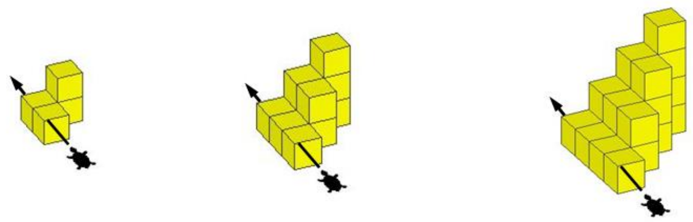
(a) airplane made in virtual world using turtle representation	(b) airplane printed by 3D printer

Figure 2. 3D artifact

2.2. Figurate Number Pattern Design and Computational Thinking

Recently developed software such as Minecraft and Google Sketchup enables modeling 3D objects easily by simply using a computer mouse and those objects can be printed out through 3D printers. Yet, the turtle representation introduced above lets users to design 3D objects not through a mouse but through symbol coding. The 3D objects therefore show not only the final ‘product’ but also the ‘process’ of how the product was made and learners naturally get to use CT in the process. Therefore the ‘hard fun’ learners experience using the turtle representation is clearly different from the ‘soft fun’ learners experience through a computer mouse.

For instance, let us suppose that we are making a triangular pyramid number step by step by using the turtle representation as it is shown in Figure 3. Some learners may make 3D objects by randomly moving the course of the turtle. However, if learners are asked to make 3D objects with the minimum use of letters, how should they approach making a 3D object? Figure 3 (a) can be easily made as it is a simple form composed of just four cubes. But as we move on to Figure 3 (b) and (c), the number of letters needed increases. In this case, to use the letter as few as possible, students should approach the cube not as a ‘shape’ but as a ‘pattern.’ In other words, students need to figure out and focus on the regularity of the object, such as which part is being repeated and which part is being gradually increased. As we can see from Figure 3, letter X substitutes for repetitive patterns to express the triangular pyramid number compressively and expressed general terms using recursions of ‘for next’ and symbol n. This kind of structural expression is the result of generalization of regularity by seeing objects as ‘pattern’ rather than ‘shape.’

Right type triangular pyramidal number			
	(a) 2nd	(b) 3rd	(c) 4th
turtle representation using certain patterns	do s do s[Rs[u]]	do s do s[Rs[u]] do s[Rs[u]s[uu]]	do s do s[Rs[u]] do s[Rs[u]s[uu]] do s[Rs[u]s[uu]s[uuu]]
turtle representation using substitution and recursion	n=1 X='s[(m)u](m=m+1)' for k=0 to n do (m=1)s[R(k)X] next	n=2 X='s[(m)u](m=m+1)' for k=0 to n do (m=1)s[R(k)X] next	n=3 X='s[(m)u](m=m+1)' for k=0 to n do (m=1)s[R(k)X] next

‘[ ]’is a symbol to save a turtle’s position and recall it. For example, when we command 'A[B]C', the turtle makes A, memorizes its position, makes B, then comes back to the remembered position, and makes C.

Figure 3. Right type triangular pyramidal number and their turtle representations

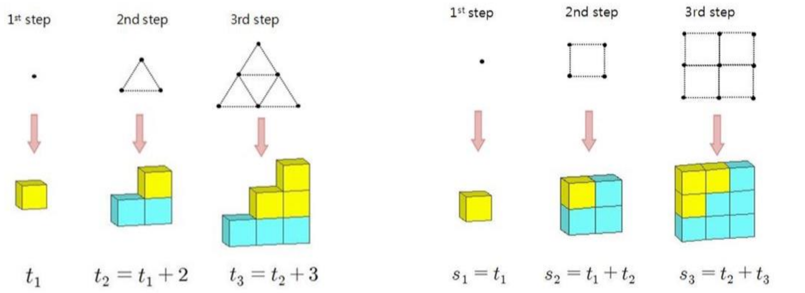
Now, just by changing the value of n, we can easily make a triangular pyramid at any steps, whether in 10th or 20th steps. As 'substitution,' 'recursion,' and 'generalization' are all key concepts in mathematics, activities of designing figurate number patterns can be evaluated as an application of CT to math curriculum. As students print out the 3D objects they make through 3D printers, they will also enjoy 'hard fun.'

3. FIGURATE NUMBER PATTERN AND ITS APPLICATION TO MATHEMATICAL INQUIRE

'Figurate number' means 'the number arranged to correspond to the shape of a figure' and was used as a tool to express numbers in ancient Greece, long before letters were invented. Figurate numbers can form plane figures such as triangular and quadrangle, three-dimensional solids such as triangular pyramid and quadrangular pyramid, and fractal figures such as snowflake and Menger Sponge. Figurate numbers serve as an entry tool for children to approach abstract 'numbers' through concrete 'figures,' giving them intuitive understanding on algebraic patterns. It also contains plenty of mathematical contexts such as sequence and series, flow chart, recurrence formula and combinations. That is why many mathematicians have been highly interested in figurate numbers and have studied how to apply and use them in education. Nonetheless, there have been limitations in expressing three-dimensional solid figures as figurate numbers have been generally expressed on two-dimensional paper with dots and circles. Education using figurate numbers, therefore, have been developed only at basic levels such as triangular number and quadrangular number. For more complicated figurate numbers, calculation with 'formula' has been given much more emphasis than 'figures.' Chapter 3 generalizes figurate number patterns from three types of soma cubes and suggests mathematical contexts where these generalizations can be studied from 'algebraic' and 'combinatorial' perspectives.

3.1. Polygonal Number: Triangular Number and Quadrangular Number

Triangle is one of the most basic figures that make up polygons and all types of polygons can be made from triangles. Similarly, triangular number forms the basis of plane figurate numbers. Triangular number is the number that corresponds with the figure of a triangle. Let us note the first-step triangular number as t_1, the second-step triangular number as t_2, the third-step triangular number as t_3, and the n-th step triangular number as t_n. As we can see from Figure 4(a), the second-step triangular number t_2 is made by adding two cubes on the first-step triangular number t_1 and the third-step triangular number t_3 is made by adding three cubes on the second-step triangular number t_2. Such pattern shows that the triangular number is an aggregate of a series of natural numbers. For instance, the first-step triangular number t_1 is 1, the second-step triangular number t_2 is 1+2, the third-step triangular number t_3 is 1+2+3, and the n-th step triangular number t_n is an aggregate of a consecutive sequence of natural numbers starting from 1 to n. From this, we can get the idea of getting an aggregate of a consecutive sequence of natural numbers. For example, Figure 5 shows that an aggregate of two fifth-step triangular numbers $2t_5$ is 5×6. If we generalize this to n-th step triangular number, an aggregate of two triangular numbers $2t_n$ is n(n+1) and if we divide the two sides by 2, it becomes $t_n=n(n+1)/2$. This formula is also famously known as the one suggested by Mathematician Carl Friedrich Gauss in his early childhood as he figured out how to add consecutive natural numbers from one to ten.



(a) Triangular number (b) Quadrangular number

Figure 4. Polygonal number

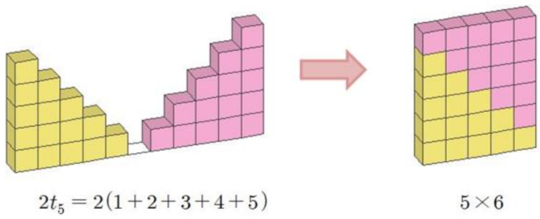


Figure 5. Sum of triangular numbers

Let is now expand our discussion from triangular number to quadrangular number. Quadrangular number is the number that corresponds to the figure of quadrangular and can be constructively approached from triangular numbers. As we can see from Figure 4(b), the first-step quadrangular s_1 is the same with the first-step triangular number t_1, the second-step quadrangular number s_2 is an aggregate of the first-step triangular number t_1 and the second-step triangular number t_2, and the third-step quadrangular number s_3 is an aggregate of the second-step triangular number t_2 and the third-step triangular number t_3. If we generalize this pattern, the n-th step quadrangular number s_n is an aggregate of the n–1-th step triangular number t_{n-1} and the n-th step triangular number t_n.

Since quadrangular number is a square number of the length of the square's side, the n-th step quadrangular number s_n is n^2. Furthermore, quadrangular number is an aggregate of consecutive gnomon[3] numbers. For example, as we can see from Figure 6, the fifth-step quadrangular number s_5 is the sum of five steps of consecutive gnomon numbers. In this case, gnomon number is a figurate number showing odd numbers for each step and is the sum of five consecutive odd numbers starting from 1. If we generalize it, n-th step quadrangular number is an aggregate of consecutive odd numbers from 1 to n.

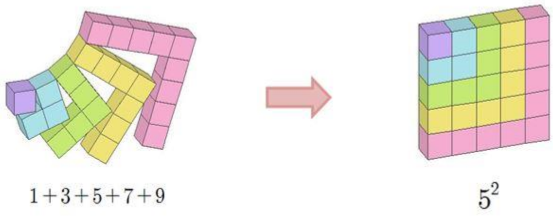
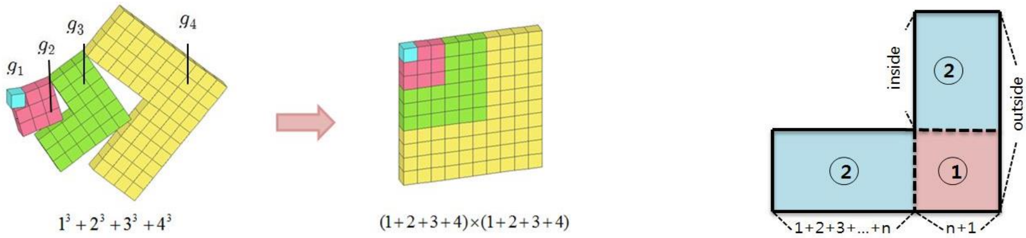


Figure 6. Sum of odd numbers and quadrangular numbers

Based on the idea of Figure 6 that quadrangular number is an aggregate of consecutive gnomon numbers, let us think of a quadrangular number in which the length of one side is composed of a sum of consecutive natural numbers. At this point, we need to think of gnomon numbers previously mentioned. As we can see from Figure 7(a) the 1st quadrangular number s_1 is the same with the 1st gnomon number g_1 and the (1+2)-th quadrangular number $s_{(1+2)}$ equals the sum of the 1st gnomon number g_1 and the 2nd gnomon number g_2 and the (1+2+3)-th quadrangular number $s_{(1+2+3)}$ is the sum of the 1st gnomon number $g_{1,}$ the 2nd gnomon number g_2 and the 3rd gnomon number g_3. If we generalize this, a quadrangular number whose length of a side is an aggregate of consecutive natural numbers from 1 to n is an aggregate of n consecutive gnomon numbers. What mathematical facts can we find from this?



(a) Gnomon numbers and quadrangular numbers (b) (n+1)th gnomon

Figure 7. Sum of gnomon numbers and quadrangular numbers

Since gnomon is L-shaped, we can divide it into part ① and part ② like Figure 7(b).

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[3] The term "gnomon" was originated from carpenter's square and the famous mathematician Euclid extended the term to the plane figure formed by removing a similar parallelogram from a corner of a larger parallelogram

Part ① of gnomon numbers for each step is the same with quadrangular numbers for each step. Since inside of part ② of gnomon numbers for each step is adjacent to the side of a quadrangular number of the previous step, we can think it in terms of the relationship with the quadrangular number of the previous step. First, we know that the sum of the natural numbers from 1 to n is n(n+1)/2. If we understand this from the perspective of figures, it can be understood that there are n/2 blocks whose length is n+1. In other words, n/2 quantities of quadrangular number blocks whose lengths are n+1 can be attached outside of the n-th gnomon. For instance, the outside length of the 2nd gnomon number is an aggregate of two consecutive natural numbers starting from 1. Accordingly, 2/2 blocks whose lengths are (2+1) can be attached. In other words, the 1 quantity of the 3rd quadrangular number can be attached to the outside of the 2nd gnomon number. Moreover, as the outside length of the 3rd gnomon number is equal to the sum of three consecutive natural numbers starting from 1, 3/2 blocks whose lengths are (3+1) can be attached. In other words, 1.5 quantities of the 4th quadrangle numbers can be attached to the outside of the 3rd gnomon number. Let us generalize this. Since the outside length of the n-th gnomon number is the sum of consecutive natural numbers starting from 1, n/2 quantities of quadrangular numbers whose lengths are (n+1) can be attached. In other words, n/2 quantities of quadrangles in the (n+1)-th can be attached to the outside of the n-th gnomon number. However, as we can see from Figure 7(b), Part ② has two areas. Therefore, n quantities of the (n+1)-th quadrangular numbers can be attached to Part ② area and since there is only one (n+1)-th quadrangular number in Part ①, a total of (n+1) quadrangles in the (n+1)-th can be attached to the (n+1)-th gnomon. The (n+1)-th quadrangular number is $(n+1)^2$ and since there are (n+1) quantities of them we have $(n+1)^3$ blocks at the (n+1)-th gnomon. Moreover, a quadrangular number whose length is the sum of consecutive natural numbers from 1 to (n+1) equals the sum of consecutive gnomon numbers from 1 to (n+1). From this we know that an aggregate of consecutive cubes from 1 to n equals the square value of an aggregate of consecutive natural numbers from 1 to n.

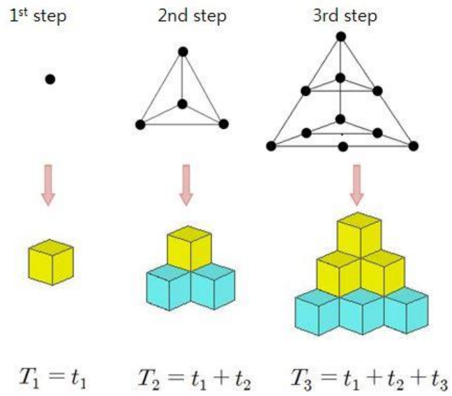
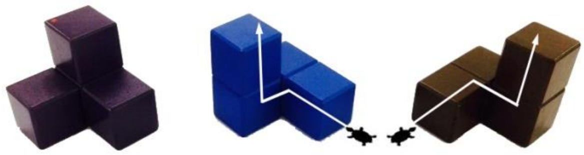


Figure 8. Sum of triangular numbers and triangular pyramid numbers

3.2. Three Types of Soma Pieces and 3D Figurate Numbers

Now let us expand the relationship between triangular numbers and quadrangular numbers in plane figurate numbers into three-dimensional figurate numbers such as triangular pyramid numbers and quadrangular pyramid numbers. Triangular pyramid number is the number that corresponds with the shape of the triangular pyramid and it can be approached constructively from triangular numbers of each step. If we suppose that the n-th step triangular pyramid number is T_n, we can see from Figure 8 that the first-step triangular pyramid number T_1 is equal to the first-step triangular number t_1, and the second-step triangular pyramid number T_2 is made by piling up the first-step triangular number t_1 and the second-step triangular number t_2. The third-step triangular pyramid number T_3 is also made by piling up the first-step triangular number t_1, the second-step triangular number t_2, and the third-step triangular number t_3. If we generalize this, the n-th step triangular pyramid number T_n is made by accumulating triangular numbers from the first-step triangular number t_1 to the n-th step triangular number t_n.



(a) symmetric (b) left type (b) right type

Figure 9. Three types of soma pieces

Soma cubes that have been used for children's playing activities such as Lego blocks or block buildings are composed of seven soma cubes. Let us think of three soma cubes composed of four cubes like Figure 9. Symmetric soma cubes of Figure (a) have the shape of the second-step triangular pyramid number. Figure (b) and (c) are modified forms of symmetric soma cubes. If we put in into turtle metaphors, (b) can be called left type and (c) can be called right type because (b) is a metaphor of a turtle going forward and climbing leftward and (c) is a metaphor of a turtle going forward and climbing rightward. We can expand this idea into triangular pyramid numbers of the third-step and fourth-step, etc. In other words, three types of soma cubes are basic structures for three types of triangular pyramid numbers and can be generalized into symmetric triangular pyramid, left type triangular pyramid, and right type triangular pyramid patterns. Now let us examine the relationship of three-dimensional figurate numbers using left type and right type triangular pyramid numbers.

From Figure 4(b) of plane figurate numbers, we can infer that the n-th step quadrangular number s_n is an aggregate of the (n-1)-th step triangular number t_{n-1} and the n-th step triangular number t_n. From Figure 8 we also know that three-dimensional figurate numbers are composed by piling up the corresponding plane figurate numbers. Therefore, we can expand and apply the relationship between triangular number and quadrangular number in a plane figurate numbers into the relationship between triangular pyramid number and quadrangular pyramid number into three-dimensional figurate numbers. For example, Figure 10 shows that the fourth-step quadrangular pyramid number S_4 is the sum of the third-step left type triangular pyramid number T_3 and the fourth-step right type triangular pyramid number T_4.

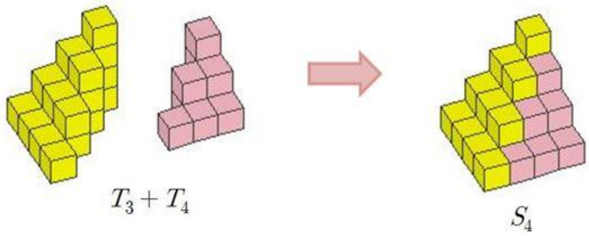


Figure 10. Sum of triangular pyramid numbers and quadrangular pyramid numbers

Furthermore, let us consider the figurate number composed of three triangular pyramid numbers. First, prepare two second-step left type triangular pyramid numbers and one second-step right type triangular pyramid number. As shown in Figure 11, rotate by 90 degrees the left type triangular pyramid number on the left side towards the right type triangular pyramid number on the middle, and take the left type triangular pyramid number on the right side to parallel movement and attach it to the right type triangular pyramid number on the middle. Then, we come up with a figure in the shape of triangular pillar with the base line being the second-step triangular number t_2 and the height being 2+2. And similar patterns are repeated to the third and fourth steps. If we generalize it in the n-th step, the aggregate of the three n-th step triangular pyramid numbers $3T_n$ becomes a triangular pillar whose base line is the n-th step triangular number t_n and whose height is (n+2). Moreover, since we can know the n-th step triangular number t_n from Figure 5, the n-th step triangular pyramid number T_n can be inductively inferred. Furthermore, if we know the triangular pyramid number T_n, we can also get the n-th step quadrangular pyramid number S_n from the relationship between the triangular pyramid number and the quadrangular pyramid number in Figure 10. However, the n-th step quadrangular pyramid number S_n is the sum of consecutive quadrangular numbers starting from 1 to n, and the n-th step quadrangular number is n^2. Therefore, the n-th step quadrangular pyramid number S_n is an aggregate of the consecutive square numbers starting from 1 to n. Therefore, from the n-th step quadrangular pyramid number S_n we can get an aggregate of consecutive sequence numbers from 1 to n. As we can see from this, figurate numbers not only form mathematical relationship among figurate numbers but also they enable us to make aggregate formulas of consecutive natural numbers, odd numbers, square numbers, and cubes. Therefore, students can understand the sequence patterns better by studying the patterns of figurate numbers than by hard and dull formulas.

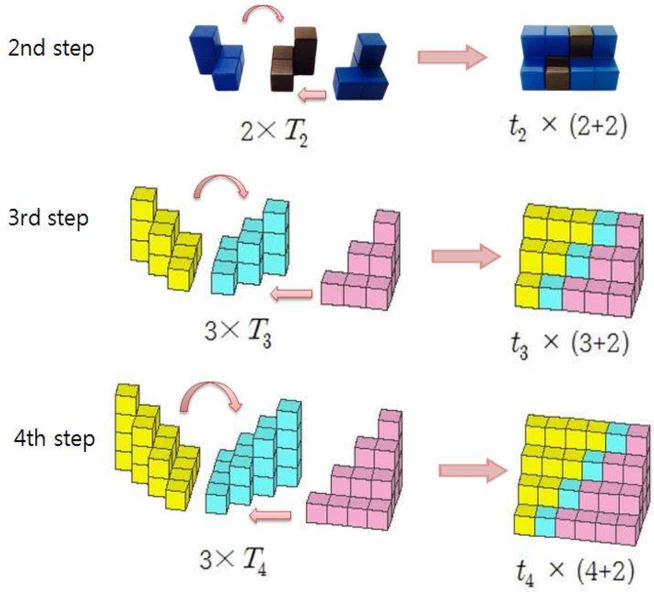
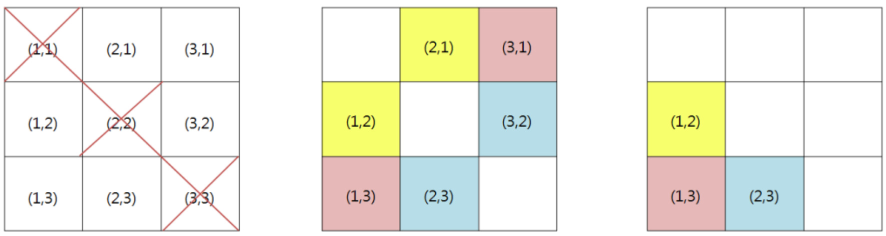


Figure 11. Sum of 3 triangular pyramid numbers

3.3. Combinatorial Approach of Figurate Numbers

It is also possible to understand the patterns of figurate numbers such as triangular numbers and triangular pyramid numbers from a combinatorial approach. For instance, let us select two numbers among natural numbers from 1 to 3 in an ordered pair (x, y) allowing repetition. It is a repeated permutation ${}_3\pi_2$ in which two out of three can be selected repeatedly and therefore makes 3x3 ordered pair matrix like Figure 12(a). In case of permutation ${}_3P_2$ where repetition is not allowed, it will be the number of cases excluding the diagonal matrix like Figure 12(b). However, since the value of the lower triangle (x, y) corresponds to the value of the upper triangle (y, x) in Figure 12(b), the number of combinations ${}_3C_2$, which do not regard orders, becomes a quadratic triangular number like Figure 12(c).

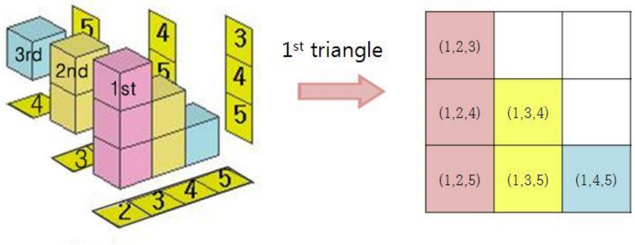


(a) ${}_3\pi_2$ (b) ${}_3P_2$ (c) ${}_3C_2$

Figure 12. Combinatorial approach of 3rd triangular pyramid number

In other words, the quadratic triangular number equals the number of combinations of selecting two out of three. Likewise, cubic triangular number equals the number of combinations of selecting two out of four. If we generalize this, the n-th triangular number is the number of combinations of selecting two out of (n+1).

Let us expand the combinatorial approach on the two-dimensional plane figurate numbers into the three-dimensional figurate numbers. In two-dimension, we selected two numbers that correspond to the second-dimensional coordinate of (x, y). Similarly, in three-dimension, we have to select three numbers that correspond to the three-dimensional coordinate (x, y, z). Therefore, the n-th triangular pyramid number T_n is the number of combinations of selecting three out of (n+2). For instance, let us approach the 3rd triangular pyramid number S_3 using combination. Let us suppose (x, y, z) is the ordered pair for three consecutive numbers out of natural numbers from one to five, arranging them from smaller to larger numbers. As we can see from Figure 13, x refer to the x-th triangle, y refers to the y-th pillar and z refers to the z-th height in ordered pairs of (x, y, z). Figure 15(b) shows the number of combinations that correspond to the 1st triangle at the 3^{rd} triangular pyramid number S_3.



(a) 3rd triangular pyramid number (b) 1st triangle

Figure 13. Combinatorics approach of 3rd triangular pyramid number

4. CLOSING

Figurate numbers entail a large number of mathematical contexts and there have been a lot of studies to use and apply them in education. However, most of them failed to use the spatial characteristics of the figurate numbers and focused on calculation and logical thinking. With the introduction of the Free Semester System and 3D printers, this research designed a learning environment and its relevant mathematical contexts where students can design for themselves three dimensional figures composed of cubes and print them out. Programs suggested in this research have the following educational significance.

First, there has been an increasing emphasis on coding education these days, but it has been merely about teaching computational language rather than fundamental computational thinking and failed to be connected to the curriculum. The turtle representation system proposed in the present study can be a tool of coding education easily accessible even to beginners in that it is a language system comprehensible to computers and makes immediate visual feedback possible at the same time. In addition, it can be a model of computational thinking education that connects coding and the subject of mathematics in that, through activities in which figurate numbers are constructed through simple coding that uses a turtle metaphor, mathematical concepts such as substitution, recursion, generalization, and variables can be learned. Figurate numbers entail a large number of mathematical contexts and there have been a lot of studies to use and apply them in education. However, most of them failed to use the spatial characteristics of the figurate numbers and focused on calculation and logical thinking. The present study designed a program that connects figurate numbers to coding, 3D printers, and career path exploration, thus allowing learning that is meaningful to learners to occur in a richer context.

Second, while computational thinking connected to subjects such as mathematics and science has been emphasized in recent years, there has been practical difficulty with the application of such convergence education because, in the general semester system, progress must be covered within pre-established curricula, mid-term and final examinations must be taken, and subjects are strictly distinguished. On the contrary, the free semester system aims at transcending the distinction among subjects and the burden of progress and examinations, encouraging discussions and project learning that connect diverse fields, and linking such activities to career path education. The program suggested by this research provides meaningful mathematical experience to learners through 'learning by design.' Moreover, in that the experience of using coding and a 3D printer can provide a useful career path experience to those who wish to study engineering or to work in computer programming or design in the future, the program can be said to fulfill the spirit of the free semester system. Under the Free Semester System which will be implemented nationwide from 2016, mathematics education should aim to encourage students to use mathematics beyond the scope of answering problems in the textbooks and use them in the process of convergent problem solving. It will ultimately enable students to realize the meaning of learning mathematics and better explore their future career as they find answers to 'why should we learn mathematics,' and 'how is math applied in different sectors?' Moreover, mathematics education should be more than doing calculations with pens and paper, and should provide much more fruitful and meaningful experience by converging them with technologies used in various sectors. Activities of studying figurate numbers with 3D printer have been introduced in this research and will be actually implemented from the next semester in some research schools. In the next research paper, the outcomes and effects of applying the programs suggested in this research will be presented.

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