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Embodied Approach to the Concept of Vector and its Application[1]

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The current mathematical education calls for a learning environment from the constructionism perspective that actively creates mathematical objects. This research first analyzes JavaMAL's expression 'move' that enables students to express the agent's behavior constructively before they learn vector as a formal concept. Since expression 'move' is based on a coordinate, it naturally corresponds with the expression of vectors used in school mathematics and lets students take an embodied approach to the concept of vector. Furthermore, as a design tool, expression 'move' can be used in various activities that include vector structure. This research studies the educational significance entailed in JavaMAL's expression 'move'.

Keywords: vector education, embodied expression, representation, covariational reasoning, probabilistic process, JavaMAL
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1. INTRODUCTION

These days there is a growing emphasis on the importance of nurturing integrated human talents in various fields in the name of STEAM (Science, Technology, Engineering and Mathematics). In this context, science and mathematics have a higher probability of being converged as convergence between phenomena and the theory explaining the phenomena. However, under the current curriculum, students learn physical phenomena and the mathematical concept which form the basis of the phenomena separately. In most of the cases, mathematical concept is introduced formally without the opportunity of experiencing concrete thinking. Expressing mathematical concepts in an embodied way using metaphor can facilitate students' concrete thinking. Metaphors enable students better understand abstract concepts by connecting them to the concepts already familiar to the students. By allowing students to understand abstract concepts through psychological simulation, advanced metaphors let students even predict the change of a result that accompanies change in the concept. Similarly, there have been studies on teaching the concept of vector from an embodied approach under "Action and effects" perspective. This chapter will analyze education methods under the current curriculum and study the embodied approach necessary for mathematics education starting from the example of vector learning.

1.1. Vector Learning in School Mathematic

Vector is a useful tool in our daily lives, expressing the relationship between power and movement. However, under the current curriculum, students get to use the concept of vector in the relationship between power and movement in science class during middle school and learn the theoretical base of it in mathematics during high school. In other words, students learn the usage of vectors first and then learn the theory of vector. Therefore, students learn vector again as a new concept without experiencing its usefulness in our daily lives.

> ... physical quantities that have both magnitude and direction is called **vector**. When denoting vector, an arrowhead should be attached to the segment AB to express both the magnitude and the direction. The arrowhead points to the direction and the length of the segment AB shows the **magnitude of the vector**. Point A is the **tail** and point B is the **head** of the vector. In mathematical symbols, vector is expressed as $\overrightarrow{AB}$.

Figure 1. Description of the vector in existing textbook (Kye et al, 2009, p. 136)

In existing textbooks, vector is defined as 'the quantities that have both the magnitude and direction' and arrows are used as a metaphor. Arrows are useful in visualizing the definition and calculation of vectors; however, given the nature of the printed texts, they can also bring misunderstanding on the fundamental traits of vector (equivalent of vector) by fixating the dynamic concept of vector as its representative image. Such problem can be resolved by defining vector from an axiomatic approach but it seems that a metaphorical approach of using arrows was adopted in school curriculums to suit the level of students. Students then later build on their incomplete knowledge on vector (equivalence relation) through individual problem-solving activities.

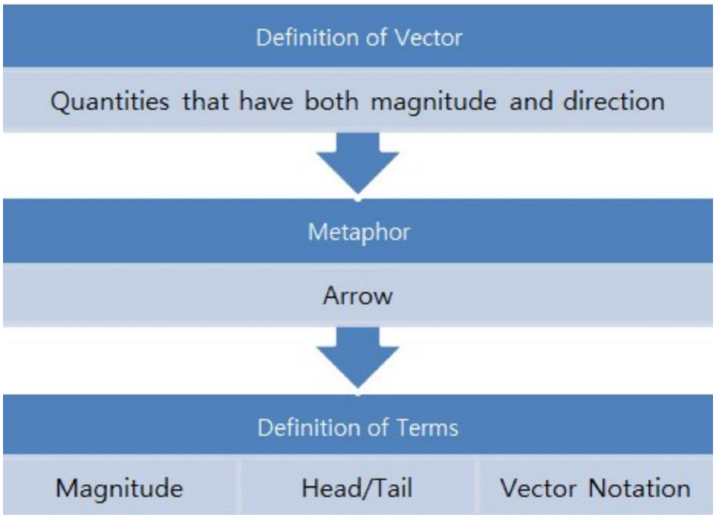


Figure 2. Description of the vector in existing textbook

1.2. Vector Education from an Embodied Approach

There has been extensive research on vector education. In regards to the concept of vector, Watson, Spyrou & Tall (2003) classified the relationship between the embodied approach and the symbolized approach into three worlds as “Embodied world, Symbolic world, and Formal world.” Along with it, the “Action and Effects” approach was introduced to help students better understand the concept of vector.

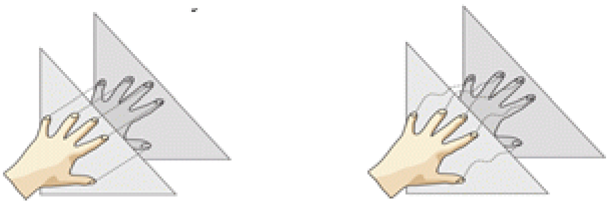


Figure 3. The vector from the “Action and Effects” approach

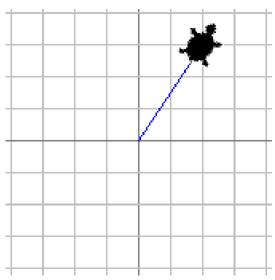
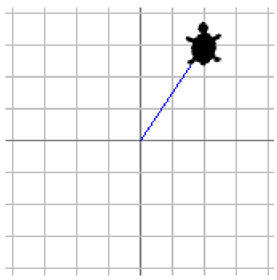
The “Action and Effects” approach regards physical movements that each vector expresses as actions while considering the effects that result from such actions as vector.

Focusing on the head and tail rather than the middle process, the hands shown in the pictures above can be understood as the same (vector) since they produce the same effects despite slightly different location and forms (Watson, Spyrou & Tall, 2003). This approach allows students to naturally understand the equivalence relationship on vector. As we can see from this example, understanding and learning any concepts heavily depend not only on how difficult the concept is but also on how the concept is expressed.

That way, it has been pointed out by many researchers that studying how to express mathematical concept is highly important in mathematics education. In regards to the conditions the expressions should have, Pape (2001) argued that they should be similar to various social activities and should be regarded as a tool for thinking, studying, and justification. It means that the expressions should give students opportunities for learning by designing as students learn for themselves based on embodied activities.

An attempt to embody linguistic expressions into agent's movements can be found in Logo designed by Papert (1980). Logo is intrinsic and local as it perceives the most basic structure that forms plane figures as 'length' and 'angle' that starts from the point where the agent is located. However, there is a need for a new learning environment more suitable to the school curriculum as a coordinate geometry that school mathematics mostly deals with is extrinsic and global. Therefore, Cho, Song & Kim (2007) suggested the expression 'move' to approach the agent's movement through vector in JavaMAL.

Table 1. Visualization of vector in Logo and JavaMAL

Logo		JavaMAL	
	RT –33.6 FD 3.605		move 2,3

By using Logo's expression system and JavaMAL's 'move' expression system, the figure above made approximate visualization of vector that corresponds to the component (2, 3) at the coordinate geometry. Under Logo's traditional expression system of 'forward' and 'rotate,' 'length' and 'angle' served as the source of information whereas under JavaMAL's expression system of 'move,' the increased amount of x and the increased amount of y based on the axis of coordinates serve as the source of information. To express the movement of a turtle that corresponds to 'move 2,3' by Logo's expression system of 'forward' and 'rotate,' it should rotate by using the value of $\arctan\frac{3}{2}$ and move by the value of $\sqrt{2^2+3^2}$ as shown in the figure. As we can see from this example, under the Logo's expression system, complicated calculation involving trigonometric functions or square root is needed to express vector movement. By introducing expression 'move' at JavaMAL, however, learners can easily and simply express the vector movement and construct various mathematical models based on that.

The expression 'move' maintains an embodied approach which the Logo's expression of 'forward' and 'rotate' entails. It is important to maintain an embodied approach in math education because it gives an opportunity to interpret mathematical concept composed of static expressions into dynamic processes. Let us use the example of function. In learning function, pointwise and covariational aspects as well as holistic shapes should all be considered but the current curriculum only gives emphasis on pointwise aspect. In Korean schools, pointwise aspect is much more emphasized than covariational aspect in our curriculum. Constructing function piecewise through expression 'move' helps students understand function's graph as an agent's local movement and puts the relationship between the two variables at the center. It is an opportunity to realize the formal structures of function from a dynamic perspective.

Now let us examine the concept of vector that the expression 'move' entails from various perspectives and learn quadratic curve that can be realized through such expressions.

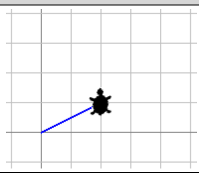
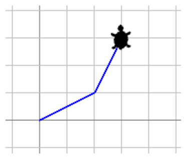
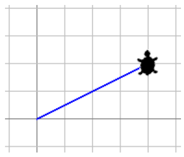
2. THE CONCEPT OF VECTOR ENTAILED IN EXPRESSION 'MOVE' AND LEARNING OF QUADRATIC CURVE

Under the current curriculum, vector and quadratic curve are introduced together at 'Geometry and Vector' but the two concepts are taught without any consideration for special connections between the two concepts. However, when a quadratic curve is embodied through the agent's movements on every point, such movement can be substituted as vector concept. Accordingly, expression 'move' shows us the possibility of approaching quadratic curves from a vector perspective. This section will examine the vector concept entailed in the expression 'move' and prove that quadratic curves can be analyzed through expression 'move' from a local and embodied approach. In particular, we will mainly focus on circular and elliptical graphs and move on to the quadratic curve of $y = x^2$ under the covariational relationship of function.

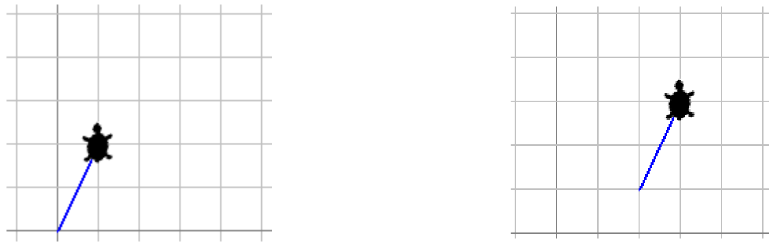
2.1. The Concept of Vector Entailed in Expression 'move'

As JavaMAL's expression 'move' gradually facilitates vector learning through an embodied approach, researchers have continued to explain vector from the embodied perspective. Watson, Spyrou & Tall (2003) classified various learning types of the vector concept into three types such as the embodied, symbolic and formal world and explained how individual students gradually learn the concept of vector in each world. In the embodied world, students perceive the concept as actions taken based on sense and behaviors and express it as geometric forms or three-dimensional spatial behaviors. In the symbolic world, the embodied behaviors are expressed with symbols. Behaviors that count, add and classify them into groups are expressed with mathematical symbols of addition, multiplication, and division. In the formal world, expressions are not based on concrete materials or behaviors but are derived from axiom and definitions. Activities of thinking over appropriate movements that correspond to vector and basic calculations of vector (adding, multiplying by scalar) and writing them with expression 'move' can be an exercise connecting the embodied and the symbolic world.

Table 2. 'move' expression in the embodied world and the symbolic world

Classification	Embodied world		Symbolic world	
			JavaMAL Rep.	Matrix Rep.
Vector	Turtle moving (a) blocks rightward and (b) blocks upward		move a,b	$\begin{pmatrix}a\\b\end{pmatrix}$
Adding Vector	Turtle moving one by one following the two vectors' expression		move a,b move c,d	$\begin{pmatrix}a\\b\end{pmatrix}+\begin{pmatrix}c\\d\end{pmatrix}$
Multiplying Vector by Scalar	Turtle repeatedly moving (a) blocks rightward and (b) blocks upward the number of given times		REPEAT k {move a,b}	$k\begin{pmatrix}a\\b\end{pmatrix}$

The previously mentioned approach of viewing vector as effect helps students to naturally understand the equivalence relation on vector from the embodied approach. Such educational effect can also be found in the expressions of JavaMAL. The movement of $\vec{x} = (1,2)$ that starts from the origin can be expressed as move 1,2 and the same movement that starts from (2,1) is expressed as tt 2,1; move 1,2.



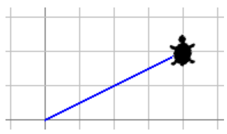
move 1,2 tt 2,1; move 1,2

Figure 4. Identical two vectors with the different initial points

In each case, the expression for $\vec{x} = (1,2)$ is identical to move 1,2 and it is not difficult to figure out that it implements the identical movements of the agent regardless of the starting point. It allows students to perceive vector with the same direction and magnitude identically. It is ultimately the same method of viewing vector as effect and can be a chance to exercise equivalence relation on vector.

JavaMAL expresses vector as the movements of an agent represented by a turtle. To express such movements, however, the process of abstraction is needed, which is the core part of the JavaMAL's expression. Students tend to focus only on the interpretation of the simulation's result in an existing learning environment as it emphasizes only the results that had been unilaterally suggested by a computer. Students end up skipping the process of abstraction and find it difficult to understand the mathematical relationship inherent in the simulation. JavaMAL suggested 'move $\Delta x, \Delta y$' as a way to express vector that corresponds to (x, y). That expression is a tool that allows abstraction of the concepts to students just like 'forward' and 'rotate' in Logo. The vector that corresponds to (x, y) is abstracted to move $\Delta x, \Delta y$ and the agent automatically makes the movement accordingly. During this process, students will be able to not only understand the concept of vector in a more specific and concrete way but also experience computational thinking. As a result, students are able to learn that the following understandings expressed at different levels are the same one.

Table 3. Computational thinking in the 'move' expression

Concept	Computational thinking	
	abstraction	automation
Vector $\vec{x}$, $\vec{x} = (1,2)$	move a,b	
Symbolic world		Embodied world

While making commands 'move', students participating in the above activity could familiarize themselves with the concept of vector by using them as a design tool. They go beyond understanding vector as an abstract concept by engaging in activities of designing mathematical objects. Therefore, the activity above could be understood as an example of 'learning by designing'. The learning environment of JavaMAL that allows students to make vector more specifically is beneficial to students as they can experience vector much earlier than the time when the mathematical concept of vector is formed. Furthermore, examining and observing how the agent modifies its movement in accordance with the changing conditions help students better understand the concept of vector between the perspective embodied through 'what if' learning and mathematical symbols.

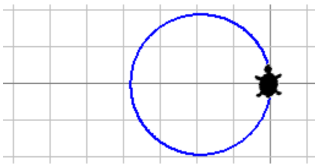
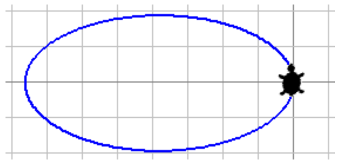
2.2. Learning Quadratic Curves with the Expression 'move'

Through the expressions of 'forward' and 'rotate' in Logo, we can observe a curve formed along with the changes in a variable. Similarly, properties that have been relatively little known can be figured out by using symbols (Papert, 1980). Through the expression 'move' at JavaMAL, we can also observe a curve while changing $\Delta\boldsymbol{x}, \Delta\boldsymbol{y}$. It is also very useful in understanding the traits of a curve changing in the direction of an axis. More specifically, let us analyze the movements of a circle. The expression 'move' at JavaMAL is repeated in such a way that each $\Delta\boldsymbol{x}, \Delta\boldsymbol{y}$ maintains a proper relation and the turtle rotates by a constant angle. For instance, in case a circle is expressed as a 360-sided polygon, therefore the movement of the following 360 vectors should be made consecutively (Kim, 2008).

$$(-\sin 1^\circ, \cos 1^\circ),\ (-\sin 2^\circ, \cos 2^\circ),\ (-\sin 3^\circ, \cos 3^\circ), \cdots,\ (-\sin 360^\circ, \cos 360^\circ)$$

Those expressions are useful when predicting and observing how a curve will change when the movement toward x axis multiplies by constant times. In the movement above, if we increase all the $\Delta\boldsymbol{x}$ by two times, we can easily predict that it will change into an ellipse whose ratios of both the left and right sides are all increased by two times. Multiplying $\Delta\boldsymbol{x}$ by constant times through JavaMAL's expression 'move' can be done by simple manipulation as the following.

Table 4. Expressing circle and ellipse in turtle's movement

Expressing circle in turtle's movement	Expressing ellipse in turtle's movement
for *x*=1 to 360 move –Sin(*x*),Cos(*x*) next	for x=1 to 360 move –2*Sin(*x*),Cos(*x*) next
	

If we calculate the movement above in the geometric coordinate, it becomes an ellipse whose length of a minor axis equals the diameter of the circle and the length of the major axis is two times the diameter of the circle.

By analyzing the agent's commands, JavaMAL allows us to figure out in advance the properties that can be proved by calculation. It is in line with the Logo learning which assumes that relatively little known properties can be clarified by using symbols just as we can observe the formation of a curve along with the changes in a variable. JavaMAL's expression 'move' is similar to the Logo's expression 'forward, rotate' as it can narrate a curve from the differential geometric perspective. But it is different from Logo as it can be easily connected to the coordinate geometry based on the axis of coordinates. Abelson (1980) pointed out that turtle geometry is much more intrinsic, local and procedural than coordinate geometry. JavaMAL's expression 'move' is similar to the turtle geometry and coordinate geometry. It displays local behaviors and procedures for forming a circle by only using the next movement from the agent's present location during the calculation shown above that produces the shape of a circle. Nonetheless, it lies between the turtle geometry and coordinate geometry as it requires an axis of coordinates, which is not related and essential to a circle. Following is the characteristics of the turtle geometry and coordinate geometry pointed out by Abelson (1980) along with the position of the JavaMAL.

Turtle geometry	for *m*=1 to 360 RT 1, FD 1 next	Intrinsic Local Procedural
JavaMAL	for *x* =1 to 360 move –Sin(*x*),Cos(*x*) next	
Coordinate geometry	$x^2 + y^2 = r^2$	Extrinsic Global Emphasis on Equations

Figure 5. Turtle geometry, JavaMAL, and Coordinate geometry

Kim (2008) argues that the command in the turtle geometry is dynamic expressions with time whereas the coordinate geometry used in school mathematics is a static expression. Kim continues to point out that historically research on curves that have dynamic geometric significance has been evolved into static algebraic expressions in the process of pursuing mathematical certainty. Since the works of expressing figures are closely related to dynamic situations in our real life, research on the expression that connects the coordinate geometry and dynamic situations is important. In this respect, JavaMAL offers an opportunity to enhance our understanding of curves as they suggest various ways to express curves.

3. LEARNING ACTIVITIES FOR GRAPHS AND PROBABILITIES WITH THE EXPRESSION ‘MOVE’

In school mathematics, curve graphs and probabilities were taught mostly by definition. However, given their dynamic meaning, bottom-up methods should also be considered. Looking at curve graphs locally through expression ‘move’ allows us to compose curve graphs embodied in dynamic vector movements. If we introduce the concept of limit, it naturally moves to the concept of tangent line and can even be linked to the research of differential and integral calculus of curves. Moreover, the expression ‘move’ in the fields of probabilities/ statistics not only enables a bunch of randomly produced data to be expressed easily but also provides an environment to analyze the result. This section will examine various learning activities that can be composed by using the expression 'move'.

3.1. Constructing Graphs from the Perspective of Rate-of-Change

Under the current curriculum, function is taught too much from the perspective of correspondence. In particular, function's graphs are treated as a set of points satisfying an algebraic expression. This kind of approach makes students difficult to understand a covariational relation between the two variables, from which the concept of function originally evolved. Students fail to gain an integrated understanding on a dynamic graph in science class which evolved from an object's movement and a function's graph in mathematics and later find the differential and integral calculus very challenging. That is why we need to teach the concept of function from the perspective of change as well. (Carlson, Oehrtman & Engelke, 2010; Woo, 2013).

To emphasize the covariational relation of function, we should let students construct a graph while adjusting the changes of the two variables from the point where the graph starts rather than letting students put several dots that satisfy an algebraic expression and construct a graph by connecting the dots. In this respect expression 'move $\Delta x, \Delta y$' can be used for students to construct a dynamic graph in a computer environment.

Woo (2013) let 9th grade students to construct graphs that are convex downward, convex upward and straight by using the expression 'move $\Delta x, \Delta y$' under the following five conditions:

(1) Δx should be maintained steady at 10,
(2) Δy should be maintained steady at 10,
(3) Δx increases by 5,
(4) Δy increases by 5, and
(5) Freely constructed.

As a result, students started to recognize that the shape of a graph is determined by the ratio of Δx and Δy, which is the change in the pattern of rate-of-change. It means that students could interpret the graph's change no longer through the covariational perspective alone but through the changing patterns of y in accordance with the change in x.

Furthermore, students can learn that the trajectory of an object that makes a parabolic movement turns into a quadratic function as they dynamically construct graphs through the expression 'move $\Delta x, \Delta y$.'

Convex Upward		Convex Downward	

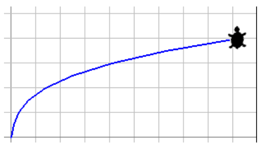
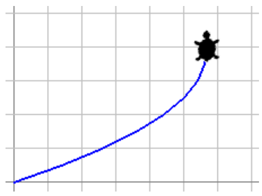
move 1, 5
move 2, 5
move 4, 5
move 7, 5
move 11, 5
move 16, 5
move 22, 5
move 29, 5
		move 14, 5
		move 12, 5
		move 10, 5
		move 8, 5
		move 6, 5
		move 4, 5
		move 2, 5
		move 1, 5
			

Figure 6. Dynamic construction of convex upward and convex downward graphs

For better understanding, let us take an example of a situation where a turtle is swimming across the flowing river. More specifically, let us suppose that the river is steadily moving from left to right by 1 for every 1 second. Suppose the turtle moves upward at a speed that increases by 2 for every second; for instance, it initially moves by 1 for the first second and then moves by 3 for the next second. This example suggested a parabolic movement in the context of a turtle's movement that makes a uniform motion in the horizontal direction and a uniformly accelerated motion due to gravity in the vertical direction. By using the expression 'move $\Delta x, \Delta y$', students can express the path of a turtle like Figure 7. After n seconds, the location of the turtle is changed into the point which moved rightward by n and upward by n^2, thereby satisfying the quadratic function $y = x^2$. Therefore, the path of the turtle becomes a quadratic graph. The expression 'move $\Delta x, \Delta y$' can be understood as move

$$\frac{dx}{dt}, \frac{dy}{dt}$$

and students end up expressing the movement of a moving object by informally using speed vector. Moreover, given

$$\frac{dx}{dt} = 1,$$

if we see the x axis as the time t axis, the activity could be understood as construction of a quadratic function by using instantaneous rate-of-change or the slope of the tangent line.

move 1, 1;
move 1, 3;
move 1, 5;
move 1, 7;
move 1, 9;

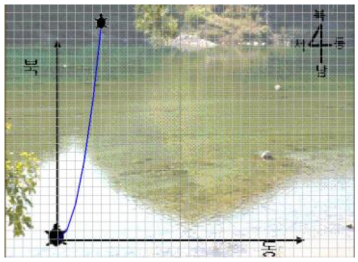


Figure 7. Dynamic construction of the graph of quadratic function with the move $\Delta x, \Delta y$

Constructing graphs through 'move $\Delta x, \Delta y$' is different from the way students draw graphs under the current school curriculum by connecting the dots that satisfy an algebraic expression. Through these activities, students are able to understand the kinematic characteristics of function such as instantaneous rate-of-change in differential. Moreover they are also able to experience the convergence between a trajectory of a moving object in physics and the graph of function in mathematics.

As an expression that embodies the concept of vector, expression 'move $\Delta x, \Delta y$' can also be used as a tool for students to understand the graph of function from a dynamic perspective such as speed vector, instantaneous rate-of-change, and the slope of a tangent line. It also provides an opportunity to perceive function in an integrated way as students see function not only from the perspective of correspondence but also from the perspective of change.

3.2. Construction of a Probability Process

As 'move $\Delta x, \Delta y$' is an expression that embodies the concept of vector, it moves the agent with a certain magnitude and direction. As previously mentioned, the expression can be used for an informal learning in a computer environment to teach the basic concept and characteristics of vector, dynamic construction of graphs and the concept of differential. We will now look at examples how the expression 'move $\Delta x, \Delta y$' is used for learning probability/statistics and explore its educational significance.

When random variables

$$X_1, X_2, X_3, \cdots$$

are mutually independent and take the value of $+1$ or -1 with the probability of

$$p \in [0,1] \text{ and } 1-p$$

respectively, we call stochastic sequence $\{S_n\}$, defined as

$$S_0 = 0,\ S_n = \sum_{k=1}^{n} X_k \ (n \geq 1),$$

as one-dimensional simple random walk. There are many ways to visualize such simple random walk. Let us take the example of visualizing it through vector. We should first correspond 1 and -1 to quadratic vectors $\vec{a} = (1, 1)$ and $\vec{b} = (1, -1)$ respectively. Then on the two-dimensional plane draw a path of a small particle that moves by $\vec{a}, \vec{b}$ which corresponds to $X_1, X_2, X_3, \cdots, X_n$ and it ends up being the same with drawing a graph by connecting the dots of $\{(k, S_n) \mid k = 0, 1, \cdots, n\}$ on the two-dimensional plane.

Figure 8 shows is an example of visualizing the case of $S_9 = 1$ by this method.

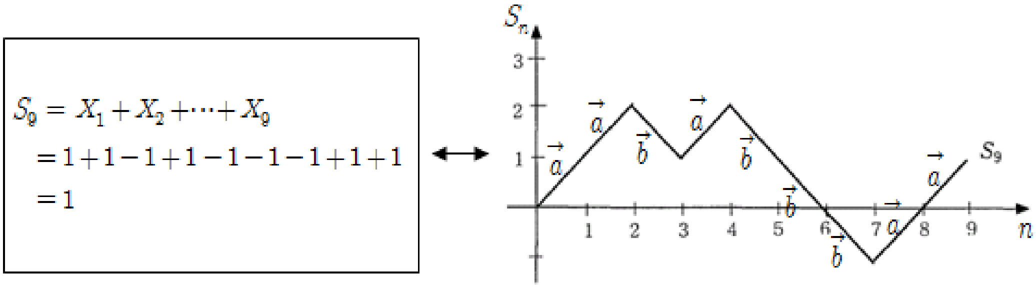


Figure 8. Visualization of simple random walk through vector

Random walk can be easily visualized in the JavaMAL Microworld environment by considering the turtle agent as a particle and probabilistically applying the expression 'move $\Delta x, \Delta y$' that corresponds with vector. Figure 9 shows an example of expressing symmetric simple random walk of $p = \frac{1}{2}$ at JavaMAL. a and b have been substituted by the expression 'move $\Delta x, \Delta y$' and X='a, b' means to select and implement a or b with the probability of 50%.

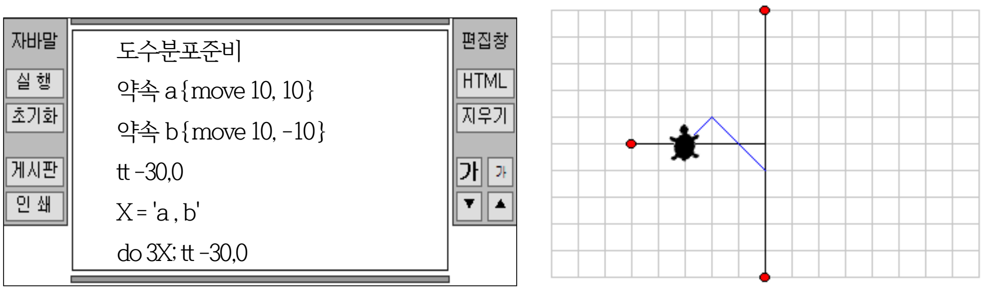


Figure 9. Visualization of random walk with the '$\boldsymbol{move}\ \Delta \boldsymbol{x}, \Delta \boldsymbol{y}$'

By applying repetitive command, learners can visualize probabilistic random walk repeated hundreds or thousands of times. Furthermore, learners can construct a bar graph as they move the turtle by the expression 'move $\Delta x, \Delta y$' depending on the accumulated frequency of arriving at each point. Figure 10 shows the result of the random walk experiment repeated 300 times when $n = 50$ under the JavaMAL environment and an empirical distribution of its random variable S_{50}.

Random walk which has been visualized through the expression 'move $\Delta x, \Delta y$' can be interpreted into random experiments in various contexts. For example, Figure 10(a) can be understood as the result of conducting experiment 300 times of throwing a coin 50 times which has the same probability of showing head and tail. The path constructed by the agent's command to move corresponds to the result of an experiment composed of head and tail of a coin. And an empirical distribution of the random variable X, the number of times the head appeared when throwing a coin by 50 times, can be expressed into a bar graph as shown in the Figure 10(b).

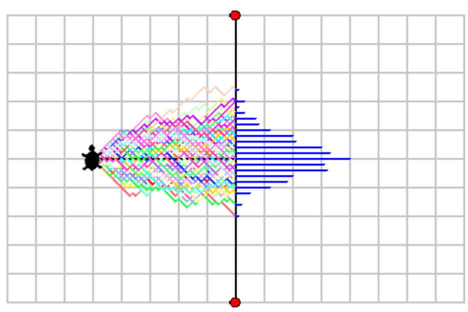
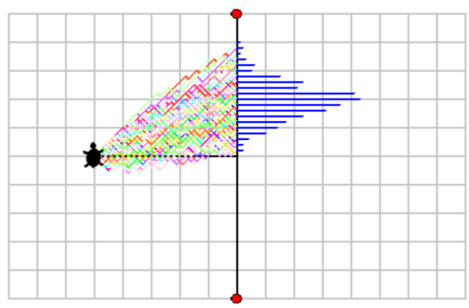
(a) $n = 50,\ p = q = 0.8$	(b) $n = 50,\ p = 0.7,\ q = 0.3$
	

Figure 10. The experiment of random walk over 300 times and the distribution of S_{50}

Conversely, in the JavaMAL environment, learners can interpret the situation of probability into the situation of random walk by using the expression 'move $\Delta x, \Delta y$' and construct a visual model they need. For instance, the game situation whereby 20KRW is gained when the head of a coin appears while 50KRW is lost when the tail appears can be expressed by randomly selecting 'move $1{,}20$' and 'move $1, -50$'. As such, 'move $\Delta x, \Delta y$' can be a very useful tool for visualizing the situation of probability. In particular, the visual model constructed by 'move $\Delta x, \Delta y$' allows learners to construct the knowledge by themselves as they can have direct access to the process and principles that constructed the model.

4. CONCLUSION

This paper explored on the expression 'move' embedded in the vector structure. By using the expression 'move,' students could move the agent in a way embedded in the vector concept. As students move body-syntonic agents, they could be naturally embodied with the concept of vector and equivalence relation. Therefore, the expression 'move' could be understood as 'the embodied expression embodying the concept of vector'. However, the expression 'move' has not been artificially introduced just for teaching vector. By using the expression 'move,' students can engage in diverse activities constructing graphs from the perspective of differential's rate-of-change while visualizing and studying various probabilistic processes. Through these activities students can be naturally embodied with the concept of vector and study particular fields such as differential and integral calculus and probabilities. The expression 'move' is a symbol that abstracted the concept of vector but it is also an expression that made the concept of vector concrete through the agent. Moreover, the automation process is included in the process whereby the agent's movement becomes concrete in the learning environment. Therefore, learners can not only acquire the mathematical concept through the computer activities using 'move' but also experience abstraction and automation of computational thinking. Brown (1993) pointed out that new concepts are learnt mostly through change and differences; therefore, research can be constantly inspired by asking 'what if' questions like 'what if we change this?'. It is expected that the demand for learning environment will rise where learners can engage in the abstraction of concept through design activities during which they express and modify concepts for themselves and follow-up research needs to be constantly conducted.

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