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AN ITERATIVE METHOD FOR ORTHOGONAL PROJECTIONS OF GENERALIZED INVERSES[†]

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ABSTRACT. This paper describes an iterative method for orthogonal projections $AA^\dagger$ and $A^\dagger A$ of an arbitrary matrix A, where $A^\dagger$ represents the Moore-Penrose inverse. Convergence analysis along with the first and second order error estimates of the method are investigated. Three numerical examples are worked out to show the efficacy of our work. The first example is on a full rank matrix, whereas the other two are on full rank and rank deficient randomly generated matrices. The results obtained by the method are compared with those obtained by another iterative method. The performance measures in terms of mean CPU time (MCT) and the error bounds for computing orthogonal projections are listed in tables. If $Z_k, k = 0, 1, 2, \ldots$ represents the k-th iterate obtained by our method then the sequence of the traces $\{trace(Z_k)\}$ is a monotonically increasing sequence converging to the rank of (A). Also, the sequence of traces $\{trace(I - Z_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*.

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1. Introduction

Let $\mathbb{C}^{m\times n}$ and $\mathbb{C}_r^{m\times n}$ denote the set of all complex $(m \times n)$ matrices and all complex $(m \times n)$ matrices of rank r, respectively. For $A \in \mathbb{C}_r^{m\times n}$, let I, $A^\dagger$ A^*, $R(A)$, $N(A)$ and $rank(A)$ represent the identity matrix of appropriate order, the Moore-Penrose inverse, the conjugate transpose, the range space, the null space and the rank of A, respectively. Penrose [4] has shown that $AA^\dagger$ and $A^\dagger A$ are hermitian idempotents and thus known as orthogonal projections of A. He has further shown that $AA^\dagger$ is a projection on $R(A)$ along $N(A^*)$ and $A^\dagger A$ is

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a projection on $R(A^*)$ along $N(A)$. Many applications of statistics, prediction theory, control analysis and numerical analysis often require computation of generalized inverses and its associated orthogonal projections. Accordingly, it is important both practically and theoretically to find efficient algorithms for computing a Moore-Penrose inverse and its associated orthogonal projections of a given arbitrary matrix. For $A \in \mathbb{C}_r^{n\times n}$, we denote its eigenvalues by

$$\lambda_1(AA^*) \geq \ldots \geq \lambda_r(AA^*) > \lambda_{r+1}(AA^*) = \ldots \lambda_n(AA^*) = 0. \quad (1)$$

The Moore-Penrose inverse has been extensively studied by many researchers [6, 4, 10, 9, 5, 12, 1] and many direct and iterative methods along with their convergence analysis and estimation of errors are proposed in the literature. It is the unique matrix X satisfying the following four Penrose equations.

$$(i)AXA = A, (ii)XAX = X, (iii)(AX)^* = AX, (iv)(XA)^* = XA \quad (2)$$

An iterative method for $k = 0, 1, 2, \ldots$

$$Y_{k+1} = Y_k(2I - AY_k) \quad (3)$$

starting with $Y_0 = \alpha A^*$ generates a sequence $\{Y_k\}$ converging to $A^\dagger$ if α satisfies

$$0 < \alpha < \frac{2}{\lambda_1(A^*A)}$$

This method is a variant of the well known quadratically convergent Schultz method. In [2], its relation to the iterative method

$$X_{k+1} = X_k + \alpha(I - X_kA)A^*$$

for $X_0 = \alpha A^*$ is shown to be $Y_k = X_{2^k-1}$, for $k = 0, 1, 2, \ldots$. Another iterative method for computing the Moore-Penrose inverse based on the Penrose equations $XAX = X$ and $(XA)^* = XA$ given by Petkovic et al [10] starting with $X_0 = \beta A^*$ is

$$X_{k+1} = (I - \beta X_kA)^*X_k + \beta X_k, \ k = 0, 1, 2, \ldots$$

where, β be an appropriate real number. If L is the desired limit matrix and X_k is the k-th estimate of L, then the convergence properties of examined iterative method can be studied with the aid of error matrix $E_k = X_k - L$. If an iterative method is expressible as a simple matrix formula, E_{k+1} is a sum of several terms

- zero order term consisting of a matrix which does not depend on E_k,
- one or more first order matrix terms in which E_k or its conjugate transpose E_k^* appears only once,
- higher-order terms in which E_k or E_k^* appears at least twice

All suitable algorithms have a zero-order term equal to 0. Hence, the first-order term determine the terminal convergence properties.

There is very little work done on the computation of orthogonal projections as their computation is a very difficult task. They are important in applied fields of nature science, such as solution to various systems of linear equation, eigenvalue problems, the linear least square problems and in determining the rank and the nullity of rectangular matrices. Golub and Kahan [7] have observed that the correct determination of rank of A is a critical factor in these methods, even more so in the direct methods for computing $A^\dagger$. Ben-Israel and Cohen [3] developed the following iterative method for computing $AA^\dagger$ based on $\{Y_k\}$ obtained from (3). For $k = 0, 1, 2, \ldots$ define

$$M_{k+1} \quad = \quad 2M_k - M_k^2 \tag{4}$$

for $M_0 = \gamma AA^*$, where, $M_k = AY_k$ and γ satisfies

$$0 < \gamma < \frac{2}{\lambda_1(A^*A)}.$$

They have also established that the sequence of traces $\{trace(M_k)\}$ is a monotonically increasing sequence converging to the rank(A) and the sequence of traces $\{trace(I - M_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*.

This paper describes an iterative method for orthogonal projections $AA^\dagger$ and $A^\dagger A$ of an arbitrary matrix A, where $A^\dagger$ represents the Moore-Penrose inverse. Convergence analysis along with the first and second order error estimates of the method are investigated. Three numerical examples are worked out to show the efficacy of our work. The first example is on a full rank matrix, whereas the other two are on full rank and rank deficient randomly generated matrices. The results obtained by the method are compared with those obtained by another iterative method. The performance measures in terms of mean CPU time (MCT) and the error bounds for computing orthogonal projections are listed in tables. If $Z_k, k = 0, 1, 2, \ldots$ represents the k-th iterate obtained by our method then the sequence of the traces $\{trace(Z_k)\}$ is a monotonically increasing sequence converging to the rank of (A). Also, the sequence of traces $\{trace(I - Z_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*, where I and A^* denote the identity matrix of appropriate order and conjugate transpose of A.

This paper is organized in five Sections. The first Section is the introduction. In Section 2, the iterative method for computing the orthogonal projections of a generalized inverse of an arbitrary complex matrix A is described. A convergence theorem is established along with the first and second order error terms in Section 3. It is also shown that the sequence of traces $\{trace(Z_k)\}$ is a monotonically increasing sequence converging to the rank(A) and the sequence of traces $\{trace(I - Z_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*. In Section 4, three numerical examples are worked out to show the efficacy of our work. One example is on full rank matrix and the other two are on generated randomly full rank and rank deficient matrices. The results obtained by the method are compared with those obtained by another iterative method. The performance measures in terms of mean CPU time (MCT) and the error bounds for computing orthogonal projections are listed in tables. Finally, conclusions are included in Section 5.

2. An iterative method for $AA^\dagger$

In this section, we shall extend the iterative method of [10] to compute the orthogonal projections of the generalized inverses. For this purpose, we first describe the iteration given in [10] for computing $A^\dagger$ and its convergence properties. Assume that $A \in \mathbb{C}^{m\times n}$ then $X = A^\dagger \in \mathbb{C}^{n\times m}$. Using (ii) and (iv) of equation (2), we get

$$X^* = (XAX)^* = X^*(XA)^* = X^*XA$$

Hence, for arbitrary $\beta \in R$, this gives

$$X^* = X^* - \beta(X^*XA - X^*) = X^*(I - \beta XA) + \beta X^*$$

or

$$X = (I - \beta XA)^* X + \beta X$$

This leads to the following iterative method given in [10]

$$X_{k+1} = (I - \beta X_k A)^* X_k + \beta X_k,\ k \geq 0$$

with $X_0 = \beta A^*$ for an appropriate real number β. This can also be written as

$$X_{k+1} = (1 + \beta)X_k - \beta X_k A X_k,\ k \geq 0 \tag{5}$$

Now, to get our iterative method for computing $AA^\dagger$, we pre-multiply (5) with $X_0 = \beta A^*$ by A and take $Z_k = AX_k$. This gives

$$Z_0 = \beta AA^*$$

$$\begin{aligned} AX_{k+1} &= (1+\beta)AX_k - \beta AX_k AX_k \\ Z_{k+1} &= (1+\beta)Z_k - \beta Z_k^2 \end{aligned} \tag{6}$$

for some appropriate real number β. The following Lemmas will be useful for establishing the convergence analysis of (6) with $Z_0 = \beta AA^*$ in section 3.

Lemma 2.1. *$AA^\dagger Z_k = Z_k$, for all $k \geq 0$.*

Proof. Using mathematical induction, for $k = 0$, we get

$$AA^\dagger Z_0 = \beta AA^\dagger AA^* = \beta AA^* = Z_0$$

Let it holds for some k, i.e., $AA^\dagger Z_k = Z_k$. It is easy to show that it also holds for $k + 1$, since,

$$\begin{aligned} AA^\dagger Z_{k+1} &= (1+\beta)AA^\dagger Z_k - \beta AA^\dagger Z_k Z_k \\ &= (1+\beta)Z_k - \beta Z_k^2 \\ &= Z_{k+1} \end{aligned}$$

□

Lemma 2.2. *$Z_k AA^\dagger = Z_k$, for all $k \geq 0$.*

Proof. Using mathematical induction, for $k = 0$, we get

$$\begin{aligned} Z_0AA^\dagger &= \beta AA^*AA^\dagger = \beta AA^*(AA^\dagger)^* = \beta AA^*(A^\dagger)^*A^* \\ &= \beta A(AA^\dagger A)^* = \beta AA^* = Z_0 \end{aligned}$$

Let it holds for some k, i.e., $Z_kAA^\dagger = Z_k$. It is easy to show that it also holds for $k+1$, since,

$$\begin{aligned} Z_{k+1}AA^\dagger &= (1+\beta)Z_kAA^\dagger - \beta Z_kZ_kAA^\dagger \\ &= (1+\beta)Z_k - \beta Z_k^2 \\ &= Z_{k+1} \end{aligned}$$

□

3. Convergence analysis

In this section, we shall establish the convergence analysis of the iterative method (6) with $Z_0 = \beta AA^*$ described in Section 2 for computing $AA^\dagger$.

Theorem 3.1. *Iterative method (6) with $Z_0 = \beta AA^*$ converges to $Z = AA^\dagger$ under the assumption*

$$\| (\beta AA^* - Z) \| < 1, \ \ 0 < \beta \le 1.$$

For $\beta < 1$, the method has a linear convergence, while for $\beta = 1$, its convergence is quadratic. The first-order and the second order terms corresponding to the error estimation of (6) are equal to

$$error_1 = (1-\beta)E_k, \ \ error_2 = -\beta E_k^2 \tag{7}$$

respectively.

Proof. We shall first prove that

$$\| Z_k - Z \| \to 0$$

as $k \to \infty$. From (6), we get

$$\begin{aligned} Z_{k+1} - Z &= Z_{k+1} - AA^\dagger \\ &= (1+\beta)Z_k - \beta Z_k^2 - AA^\dagger \\ &= AA^\dagger Z_k + \beta Z_kAA^\dagger - \beta Z_k^2 - AA^\dagger \\ &= -(\beta Z_k - AA^\dagger)(Z_k - AA^\dagger) \\ &= -(\beta Z_k - Z)(Z_k - Z) \end{aligned}$$

Taking norm on both sides, this gives

$$\|Z_{k+1} - Z\| \le \|\beta Z_k - Z\| \|Z_k - Z\|$$

Using

$$\beta Z_k - Z = \beta(Z_k - Z) - (1-\beta)Z$$

we obtain

$$Z_{k+1} - Z \quad = \quad -\beta(Z_k - Z)^2 + (1-\beta)(Z_k - Z)$$

The sequence of error matrices $\{E_k\}$ satisfies

$$E_{k+1} = -\beta(E_k)^2 + (1-\beta)E_k \tag{8}$$

Now, we will show that $s_k = \|E_k\| \to 0$ as $k \to \infty$. By using mathematical induction we shall first prove that $s_k < 1$ for all $k = 0, 1, \ldots$. It holds true for $k = 0$, since $s_0 = \| (Z_0 - Z) \| < 1$. Assume that it holds true for some k, i.e., $s_k < 1$. To show that it also holds true for $k+1$, we take norm on (8) to get

$$s_{k+1} \leq \beta s_k^2 + (1-\beta)s_k < \beta s_k + (1-\beta)s_k = s_k < 1 \tag{9}$$

Thus, it holds true for all $k = 0, 1, 2, \ldots$. Hence, $s_k \geq 0$ is a decreasing and bounded sequence. This implies that s_k converges to s as $k \to \infty$. This gives $0 \leq s < 1$. Taking limit as $k \to \infty$ on (9) we get

$$s \leq \beta s^2 + (1-\beta)s$$

this implies that either $s = 0$ or $s \geq 1$ and hence we conclude that $s = 0$. This completes the proof that $s_k \to 0$ when $k \to \infty$. Thus, $Z_k \to Z$ when $k \to \infty$. This proves the first part of the theorem. Putting $Z_k = E_k - Z$ in (6), it is not difficult to verify that the error matrix E_{k+1} given by

$$E_{k+1} = (1+\beta)E_k - \beta Z E_k - \beta E_k Z - \beta E_k^2$$

implies

$$\begin{aligned} error_1 &= (1+\beta)E_k - \beta Z E_k - \beta E_k Z \\ error_2 &= -\beta E_k^2 \end{aligned}$$

Using Lemma 2.1 and Lemma 2.2, this gives

$$\begin{aligned} error_1 &= (1+\beta)(Z_k - Z) - \beta Z(Z_k - Z) - \beta(Z_k - Z)Z \\ &= (1-\beta)(Z_k - Z) \\ &= (1-\beta)E_k \end{aligned}$$

Clearly, $error_1$ vanishes if and only if $\beta = 1$, while $error_2$ is always non-zero. Hence, the method has linear convergence for $\beta \neq 1$ and quadratic for $\beta = 1$. This completes the proof of the theorem. □

The convergence condition can easily be verified by getting the equivalent condition which does not involve the unknown Z. To obtain this, we use the following Lemma.

Lemma 3.2 ([8]). . *Let $M \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$ be given. There is at least one matrix norm $\| \,.\, \|$ such that*

$$\rho(M) \leq \| M \| \leq \rho(M) + \epsilon$$

where $\rho(M) = \max\{| \lambda_1(M) |, \ldots, | \lambda_n(M) |\}$ denotes the spectral radius of M.

This leads to the convergence condition $\| (\beta AA^* - Z) \| < 1$ equivalent to $\rho(\beta AA^* - Z) < 1$. The following Lemma shows one property of the spectral radius function ρ.

Lemma 3.3 ([11]). . *If $P \in \mathbb{C}^{n\times n}$ and $S \in \mathbb{C}^{n\times n}$ are such that $P = P^2$ and $PS = SP$ then*

$$\rho(PS) \leq \rho(S)$$

Lemma 3.4. *Let the eigenvalues of matrix AA^* satisfy (1). The $\rho(\beta AA^* - Z) < 1$ is satisfied if*

$$\max_{1\leq i\leq r} | 1 - \beta\lambda_i(AA^*) | < 1$$

Proof. Let $P = Z$ and $S = \beta AA^* - I$. since $P^2 = P$ and

$$\begin{aligned} PS &= \beta AA^\dagger AA^* - AA^\dagger = \beta AA^* - AA^\dagger = \beta A(AA^\dagger A)^* - AA^\dagger \\ &= \beta AA^*(AA^\dagger)^* - AA^\dagger = \beta AA^* AA^\dagger - AA^\dagger = SP, \end{aligned}$$

From Lemma 3.2, we get

$$\rho(\beta AA^* - Z) \leq \rho(\beta AA^* - I) = \max_{1\leq i\leq r} | 1 - \beta\lambda_i(AA^*) | < 1$$

Thus, $\beta\lambda_i(AA^*) - 1$, for $i = 1, 2, \ldots, r$ are the eigenvalues of the matrix $\beta\lambda_i AA^* - I$. □

Theorem 3.5. *The iterative method given by (6) with $Z_0 = \beta AA^*$ satisfies*

$$\lim_{k\to+\infty} \frac{s_{k+1}}{s_k} = \lim_{k\to+\infty} \frac{d_{k+1}}{d_k} = 1 - \beta$$

where, $d_k = \|E_{k+1} - E_k\|$

Proof. From (8), we get

$$E_{k+1} = -\beta(E_k)^2 + (1 - \beta)E_k$$

This gives

$$1 - \beta - \beta\frac{\| E_k \|^2}{\| E_k \|} \leq \frac{\| E_{k+1} \|}{\| E_k \|} \leq 1 - \beta + \beta\frac{\| E_k \|^2}{\| E_k \|} \tag{10}$$

Taking limit as $k \to +\infty$ on (10), we get $\frac{s_{k+1}}{s_k} \to 1 - \beta$, since $\| E_k \| \to 0$ from Theorem 3.1. Similarly, using $Z_{k+1} - Z_k = E_{k+1} - E_k$, we get

$$d_k = \| E_{k+1} - E_k \| = \| -\beta(E_k)^2 + (1 - \beta)E_k - E_k \| = \beta \| E_k + E_k^2 \|$$

This leads to

$$\lim_{k\to+\infty}\frac{d_k}{s_k}=\lim_{k\to+\infty}\frac{d_k}{\parallel E_k \parallel}=\beta$$

Also,

$$\lim_{k\to\infty}\frac{d_{k+1}/s_{k+1}}{d_k/s_k}=1$$

implies

$$\lim_{k\to+\infty}\frac{d_{k+1}}{d_k}=\lim_{k\to\infty}(\frac{d_{k+1}/s_{k+1}}{d_k/s_k}.\frac{s_{k+1}}{s_k})=1-\beta$$

□

The following Lemma shows one additional property of the sequence $\{Z_k\}$.

Lemma 3.6. *The sequence $\{Z_k\}$ generated by (6) with $Z_0 = \beta AA^*$ satisfies $R(Z_k) = R(AA^*)$ and $N(Z_k) = N(AA^*)$ for $k \geq 0$.*

Proof. This Lemma can be proved by mathematical induction. It trivially holds for $k = 0$. Let $y \in N(Z_k)$. From (6), we have

$$Z_{k+1}y = (1+\beta)Z_k y - \beta Z_k Z_k y = 0$$

This gives $y \in N(Z_{k+1})$ and $N(Z_k) \subseteq N(Z_{k+1})$. Proceeding similarly, it can be shown that $R(Z_k) \supseteq R(Z_{k+1})$. From mathematical induction, this gives $N(Z_k) \supseteq N(Z_0) = N(AA^*)$ and $R(Z_k) \subseteq R(Z_0) = R(AA^*)$. To prove equality, let us consider $N = \cup_{k\in\mathbf{N}_0} N(Z_k)$. If $y \in N$ then $y \in N(Z_{k_0})$ for some $k_0\in \mathbf{N}_0$, where $\mathbf{N}_0$ be the set of natural numbers. This leads to $Z_k y = 0$ for all $k \geq k_0$. Using Theorem 3.1, this gives

$$Zy = \lim_{k\to+\infty} Z_k y = 0$$

Thus, $y \in N(Z) = N(AA^\dagger) = N(A^*)$. This implies $N \subseteq N(A^*)$. From

$$N(Z_k) \subseteq N \subseteq N(A^*) \subseteq N(AA^*) \subseteq N(Z_k)$$

we get $N(Z_k) = N(AA^*)$. Also from

$$dimR(Z_k) = m - dimN(Z_k) = m - dimN(AA^*) = dimR(AA^*)$$

and $R(Z_k) \subseteq R(AA^*)$, we get $R(Z_k) = R(AA^*)$. □

4. Numerical examples

In this section, we shall work out three numerical examples to show the efficacy of our work. All these examples are run on an Intel core 2 Duo processor running at 2.80 GHz and using MATLAB 7.4 (R2009b). The first example is on a full rank matrix, whereas the other two are on full rank and rank deficient randomly generated matrices. The results obtained by our method are compared with those obtained by another method for computing the orthogonal projections. The performance measures in terms of mean CPU time (MCT) and the error bounds for computing orthogonal projections are listed in tables. The randomly generated matrices are tested 50 times. Figures are plotted for $\{trace(Z_k)\}$ and $\{trace(I - Z_k)\}$, where $Z_k, k = 0, 1, 2, \ldots$ represents the k-th iterate obtained by our method, with x-axis representing the number of iterations and y-axis representing these sequences. The termination criterion used is $\max\{\|Z_{k+1} - Z_k\|_F\} \leq \epsilon$ where, $\|.\|_F$ stands for the Frobenius-norm of a matrix, The value of the parameter ϵ is taken as equal to 10^{-7}.

Example 4.1. Consider the matrix A of order (5×4) of $rank(A) = 4$ and $N(A^*) = 1$ given by

$$A = \begin{pmatrix} 0.2794 & 0.1676 & 0.0645 & 0.2326 \\ 0.0065 & 0.2365 & 0.2274 & 0.1261 \\ 0.2271 & 0.1430 & 0.1009 & 0.2867 \\ 0.1265 & 0.1015 & 0.1806 & 0.2846 \\ 0.2773 & 0.0632 & 0.0503 & 0.1979 \end{pmatrix}$$

The eigenvalues of the matrix AA^* are

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (0.0020, 0.0146, 0.0832, 0.6152, 0)$$

The convergence criteria for (6) for eigenvalues of AA^* is given by

$$\max_{1 \leq i \leq 4} \mid 1 - \beta\lambda_i(AA^*) \mid = 0.9988 < 1.$$

Thus, the sequence of iterates $\{Z_k\}$ generated by (6) converge to the orthogonal projection $AA^\dagger$ given by

$$AA^\dagger = \begin{pmatrix} 0.6382 & 0.0855 & 0.3784 & -0.2344 & 0.1596 \\ 0.0855 & 0.9798 & -0.0895 & 0.0554 & -0.0377 \\ 0.3784 & -0.0895 & 0.6042 & 0.2451 & -0.1669 \\ -0.2344 & 0.0554 & 0.2451 & 0.8482 & 0.1033 \\ 0.1596 & -0.0377 & -0.1669 & 0.1033 & 0.9296 \end{pmatrix}$$

The Mean CPU time (MCT) and the error bounds of our method and the method (4) for computing the orthogonal projection are listed in TABLE 1. The $trace(Z_k)$ and $trace(I - Z_k)$ are plotted with the number of iterations in FIGURE 1 and FIGURE 2. As expected, the sequence $\{trace(Z_k)\}$ is a monotonically increasing sequence converging to the rank(A) and the sequence of $\{trace(I - Z_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*.

Example 4.2. Consider a (30×30) matrix A whose elements are generated randomly from $[-0.2, 0.2]$ with rank$(A) = 30$ and N$(A^*) = 0$. The Mean CPU time (MCT) and the error bounds of our method and the method (4) for computing the orthogonal projection are listed in TABLE 2. The $trace(Z_k)$ and $trace(I - Z_k)$ are plotted with the number of iterations in FIGURE 3 and FIGURE 4. As expected, the sequence $\{trace(Z_k)\}$ is a monotonically increasing sequence converging to the rank(A) and the sequence of $\{trace(I - Z_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*.

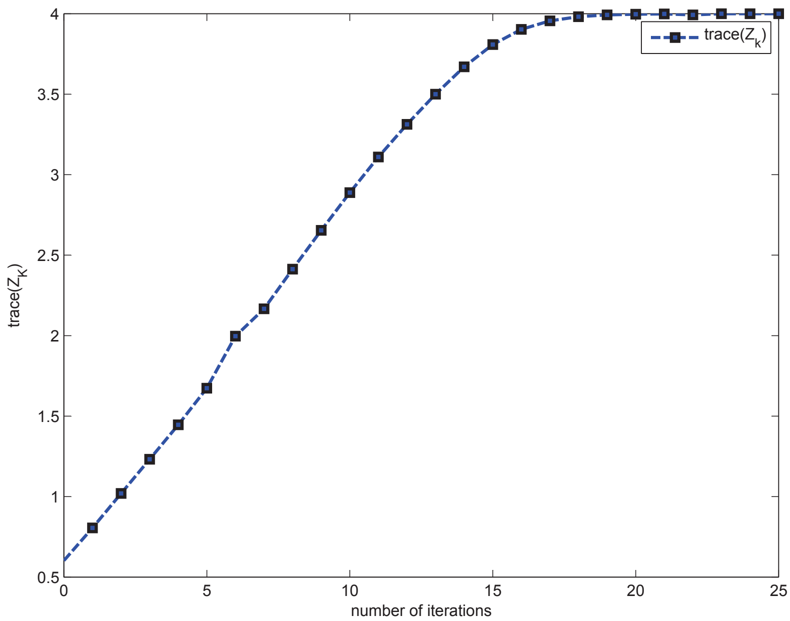

FIGURE 1. $trace(Z_k)$ for Example 1

TABLE 1. Mean CPU time (MCT) and Error bounds for different values of β

Methods	β	MCT	$\|Z_k - Z_{k-1}\|_F$	$\|AA^\dagger - Z_k\|_F$
method (4)	0.8921	3.7594e-3	3.0168e-13	1.0861e-4
our method		3.7619e-3	9.9334e-12	1.0861e-4
method (4)	0.9220	3.5585e-3	1.1462e-13	1.0861e-4
our method		3.5899e-3	2.8413e-12	1.0861e-4
method (4)	0.9832	3.1495e-3	4.5827e-13	1.0861e-4
our method		3.1522e-3	3.9863e-12	1.0861e-4

TABLE 2. Mean CPU time (MCT) and Error bounds

Methods	MCT	$\|Z_k - Z_{k-1}\|_F$	$\|AA^\dagger - Z_k\|_F$
method (4)	7.2632e-2	3.2052e-12	3.2060e-12
our method	8.8860e-2	4.6687e-11	6.0421e-12

Example 4.3. Consider a rank deficient (100×50) matrix A whose elements are generated randomly from $[-0.2, 0.2]$ with $\text{rank}(A) = 50$ and $\text{N}(A^*) = 50$. The Mean CPU time (MCT) and the error bounds of our method and the method (4) for computing the orthogonal projection are listed in TABLE 3. The $trace(Z_k)$

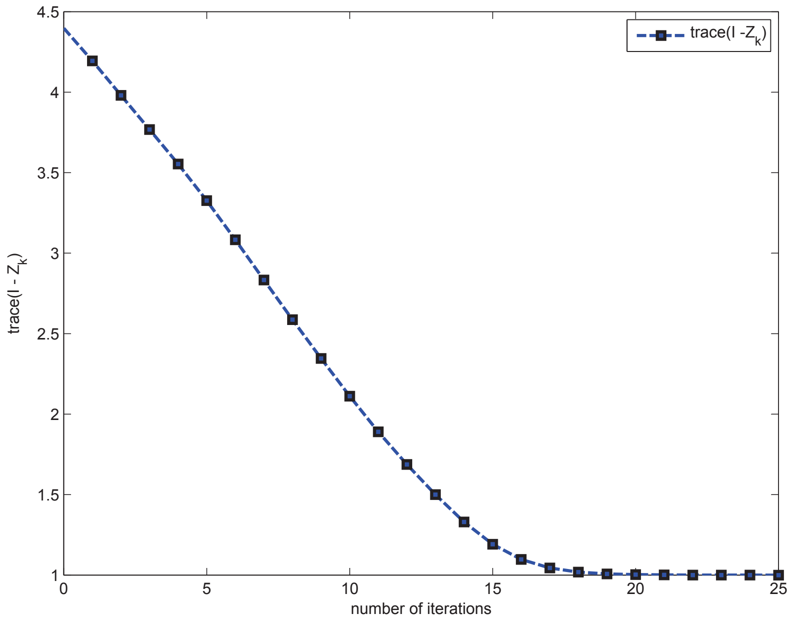

FIGURE 2. $trace(I - Z_k)$ for Example 1

and $trace(I - Z_k)$ are plotted with the number of iterations in FIGURE 5 and FIGURE 6. As expected, the sequence $\{trace(Z_k)\}$ is a monotonically increasing sequence converging to the rank(A) and the sequence of $\{trace(I - Z_k)\}$ is a monotonically decreasing sequence converging to the nullity of A^*.

TABLE 3. Mean CPU time (MCT) and Error bounds

Methods	MCT	$\|Z_k - Z_{k-1}\|_F$	$\|AA^\dagger - Z_k\|_F$
method (4)	8.5859e-3	1.1547e-9	1.1547e-9
our method	6.8860e-3	9.4627e-9	3.6502e-8

5. Conclusions

An iterative method for computing orthogonal projections of an arbitrary matrix A is developed. Convergence analysis along with the first and second order error estimates of the method are investigated. Three numerical examples are worked out to show the efficacy of our work. The first example is on a full rank matrix, whereas the other two are on full rank and rank deficient randomly generated matrices. The results obtained by the method are compared with those obtained by another iterative method. The performance measures in

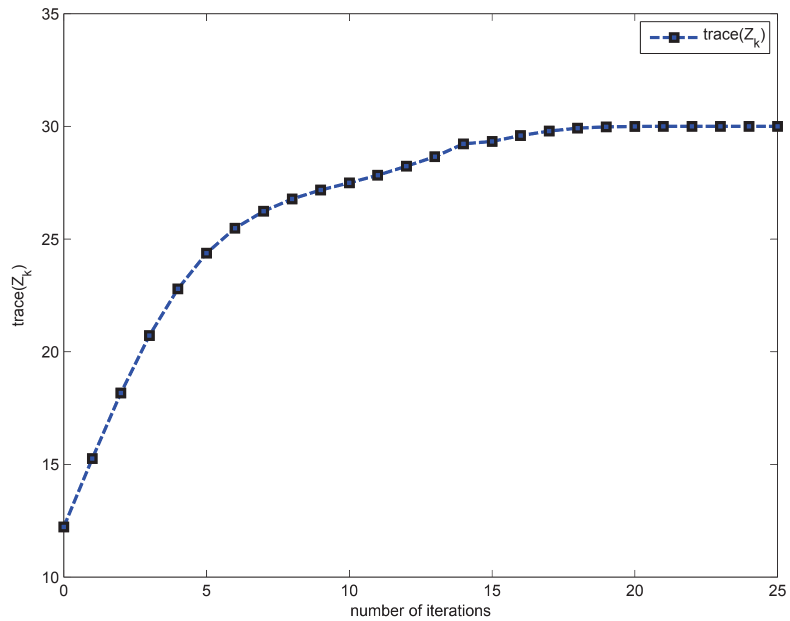

FIGURE 3. $trace(Z_k)$ for Example 2

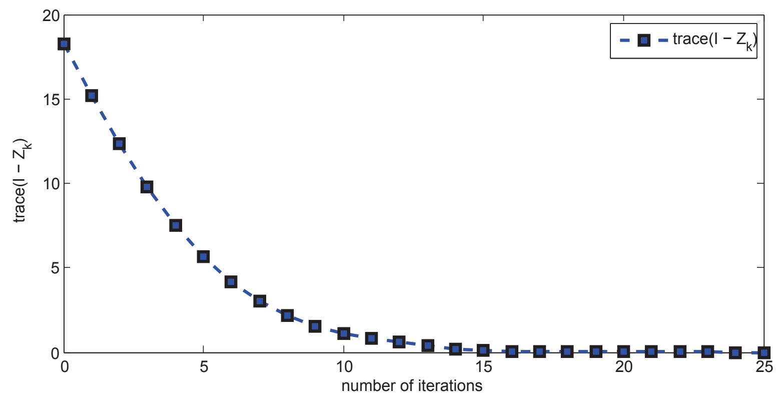

FIGURE 4. $trace(I - Z_k)$ for Example 2

terms of mean CPU time (MCT) and the error bounds for computing orthogonal projections are listed in tables. It is also observed that the sequence of traces $\{trace(Z_k)\}$ is monotonically increasing and converges to the rank of (A) where as, the sequence of traces $\{trace(I-Z_k)\}$ is monotonically decreasing and converges to the nullity of A^*.

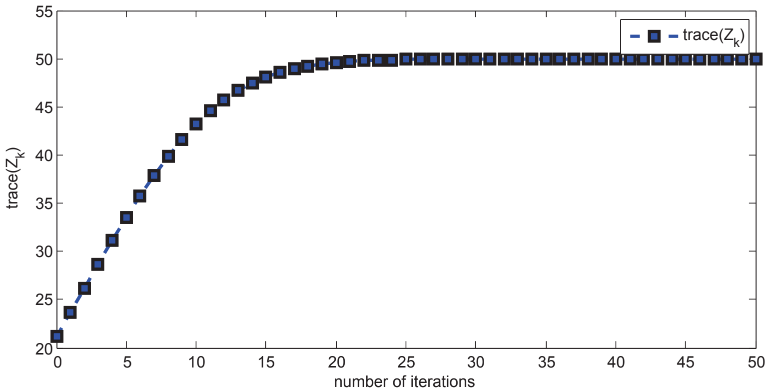

FIGURE 5. $trace(Z_k)$ for Example 3

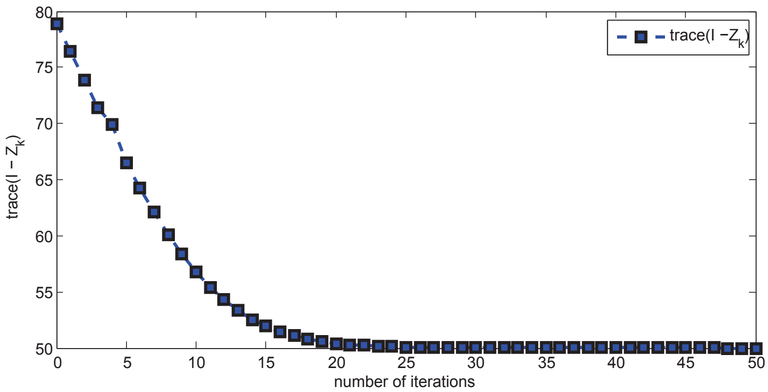

FIGURE 6. $trace(I - Z_k)$ for Example 3

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