ON SOME SOLUTIONS OF A FUNCTIONAL EQUATION RELATED TO THE PARTIAL SUMS OF THE RIEMANN ZETA FUNCTION

Juan Matías Sepulcre Martínez

ABSTRACT. In this paper, we prove that infinite-dimensional vector spaces of α-dense curves are generated by means of the functional equations $f(x) + f(2x) + \cdots + f(nx) = 0$, with $n \geq 2$, which are related to the partial sums of the Riemann zeta function. These curves α-densify a large class of compact sets of the plane for arbitrary small α, extending the known result that this holds for the cases $n = 2, 3$. Finally, we prove the existence of a family of solutions of such functional equation which has the property of quadrature in the compact that densifies, that is, the product of the length of the curve by the nth power of the density approaches the Jordan content of the compact set which the curve densifies.

1. Introduction

For each $n \geq 2$, the functional equation

$$f(x) + f(2x) + \cdots + f(nx) = 0,$$

which represents an important equation as consequence of its connection with the partial sums of the Riemann zeta function, was introduced in the literature by Mora, Cherrault and Ziadi [2] in 2000. Concretely, the cases $n = 2$ and $n = 3$ were used for modeling certain processes related to combustion of hydrogen in a car engine [2, 5].

Some analytic properties of the solutions of the functional equation

$$(1.1) \qquad F(z) + F(2z) + \cdots + F(nz) = 0, \ z \in \mathbb{C},$$

for each $n \geq 2$, were studied in [5]. In fact, it was proved in [5, Theorem 7] that the functions $f_{\beta_n}(z) \equiv z^{\beta_n}$, with $\beta_n = a_n + ib_n$ belonging to the set of the

zeros of

(1.2) $$G_n(z) \equiv 1 + 2^z + \cdots + n^z,$$

are solutions of (1.1).

Observe that $G_n(-z)$ coincides with $\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z}$, the corresponding n^{th} partial sum of the Riemann zeta function. Furthermore, the sets of zeros of the functions $G_n(z)$ and $\zeta_n(z)$ satisfy clearly the relation

$$Z_{G_n(z)} = -Z_{\zeta_n(z)}.$$

A study on the distribution of the zeros of the approximants $G_n(z)$ of the Riemann zeta function for $\mathrm{Re}\, z < -1$ can be seen in [6, 7]. Particularly, for each integer $n \geq 2$, the function (1.2) has infinitely many zeros [5, Prop. 1].

Therefore, it is clear that the functions $g_{\beta_n}(x) \equiv \mathrm{Re}(f_{\beta_n}(x))$ and $h_{\beta_n}(x) \equiv \mathrm{Im}(f_{\beta_n}(x))$ are real solutions of the functional equation

(1.3) $$f(x) + f(2x) + \cdots + f(nx) = 0, \ x > 0,$$

i.e., $g_{\beta_n}(x) = x^{a_n} \cos(b_n \log(x))$ and $h_{\beta_n}(x) = x^{a_n} \sin(b_n \log(x))$ are real solutions of (1.3). Some examples of these functions associated to zeros of $G_n(z)$ with positive and negative real part are shown in Figures 1 and 2, respectively.

In this sense, by taking account the property of α-density [3] which we shall recall further on, it was proved in [2, Prop. 3] that the subspace A formed by the continuous solutions of $f(x) + f(2x) = 0$ contains an infinite set of linearly independent functions that densifies the rectangle $J = [a, b] \times [-1, 1]$, with arbitrary density α for any $[a, b]$, with $a > 0$, and it was extended to the case $n = 3$ [2, Prop. 9] densifying the compact set limited by the curves $y = x^{d_n}$, $y = -x^{d_n}$, $x = a$ and $x = b$, for all $0 < a < b$, where d_n is the real part of γ_n verifying that $z^{\gamma_n} + (2z)^{\gamma_n} + (3z)^{\gamma_n} = 0$. In this paper, firstly, we will extend this result to the general case $n \geq 2$.

On the other hand, related to the concept of α-dense curve, it was introduced in [4] the property of quadrature in cubes of a family of α-dense curves, that is, the product of the length by the density of the curve approaches the volume of the cube that the curve densifies. In the present paper, we will extend this concept to much more general regions and we will assure the existence of a family of solutions of the functional equation (1.3) which has the property of quadrature in the compact that densifies.

Consequently, the main results of this paper can be summarized by Theorem 2.3, Proposition 3.3 and Theorem 4.1.

2. Densification theorem

Definition 2.1. In a metric space (E, d), given a compact subset A, we shall say that a curve $\gamma : I \to E$ densifies A with density $\alpha \geq 0$, or that the curve γ is α-dense in A, if $\gamma(I) \subset A$ and for any $x \in A$ there exists $t \in I$ such that $d(x; \gamma(t)) \leq \alpha$.
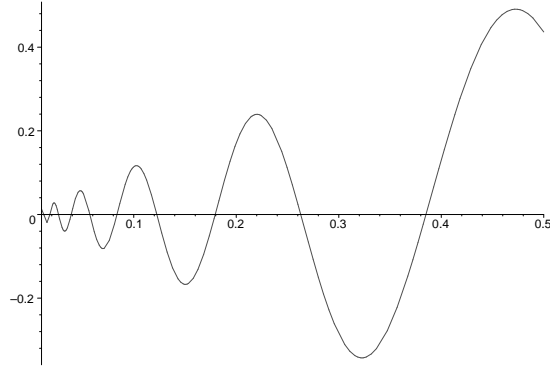
FIGURE 1. $g_{\beta_3}(x)$ associated to $\beta_3 \approx 0.9405921833152 + i\, 8.230709548417$
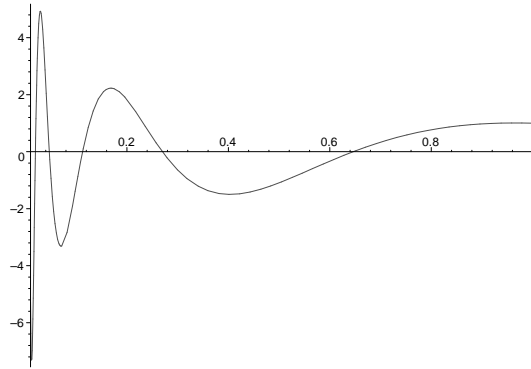
FIGURE 2. $g_{\beta_3}(x)$ associated to $\beta_3 \approx -0.4543970081950 + i\, 3.598171493995$

If a curve γ densifies a compact set A with density α, observe that it also densifies A with density α' for all $\alpha' \geq \alpha$.

Furthermore, observe that if $E = \mathbb{R}^n$ with $n \geq 2$, the case $\alpha = 0$ leads us to the Peano curves provided that the compact subset A has positive Jordan content. These functions γ are also known as space-filling curves (see [8]) and so they can be considered as a special subclass of the α-dense curves for the limit case $\alpha = 0$, that is, they are α-dense curves for all $\alpha \geq 0$.

Definition 2.2. A compact subset A of (E, d) is said to be densifiable if for any $\alpha > 0$ there exists an α-dense curve in A.

Before we consider the main result about the densification of certain compact sets, it is important to establish the following arrangement of the zeros of $G_n(z)$, which are closely related to the functional equation (1.3):

let $\beta_n^{(j)} = a_n^{(j)} + i b_n^{(j)}$ and $\beta_n^{(k)} = a_n^{(k)} + i b_n^{(k)}$ be typical zeros of $G_n(z)$. Then we will say that $\alpha_n^{(j)}$ is previous to $\alpha_n^{(k)}$ (and we take $j < k$) if either $b_n^{(j)}$ is less than $b_n^{(k)}$ or $b_n^{(j)} = b_n^{(k)}$ with $a_n^{(j)} < a_n^{(k)}$, i.e., we order its zeros by means of their imaginary parts. Thus, since $G_n(z)$ is an analytic function and the real part of its zeros is bounded [5], it is clear that $\left| b_n^{(j)} \right|$ tends to infinity when j goes to infinity. On the contrary, in the case of the zeros of $G_n(z)$ were bounded, there would exist a convergent sequence of such zeros and it would be $G_n(z) \equiv 0$ because of the Identity Principle [1, Theorem 1.3.7].

Let V_n denote the vector space of the real continuous solutions of (1.3).

Theorem 2.3. *Let $n \geq 2$. Then for each positive number α, there exists a zero of $G_n(z)$, say $\beta_n = a_n + i b_n$, such that the solutions of (1.3) associated with it, $g_{\beta_n}(x) = x^{a_n} \cos(b_n \log(x))$ and $h_{\beta_n}(x) = x^{a_n} \sin(b_n \log(x))$, have density α in the compact set K_{a_n} determined by $y = x^{a_n}$, $y = -x^{a_n}$, and the straight lines $x = \delta$, $x = \Delta$, with $\Delta > \delta > 0$. Furthermore, the functions $g_{\beta_n}(x)$ and $h_{\beta_n}(x)$ are linearly independent in the space V_n.*

Proof. Let n be an integer greater than 1 and $\beta_n^{(j)} = a_n^{(j)} + i b_n^{(j)}$ be a zero of $G_n(z)$. Since $G_n(\overline{z}) = \overline{G_n(z)}$ for all complex z, without loss of generality, we may assume $b_n^{(j)} > 0$. Observe that the zeros of the associated real function $g_{\beta_n^{(j)}}(x) = x^{a_n^{(j)}} \cos(b_n^{(j)} \log(x)) \in V_n$ satisfy

$$b_n^{(j)} \log(x) = (2k+1)\frac{\pi}{2}, \text{ with } k \in \mathbb{Z},$$

i.e., its zeros are given by

$$(2.1) \qquad x_{n,k}^{(j)} = e^{\frac{2k+1}{2b_n^{(j)}}\pi}, \ k \in \mathbb{Z}.$$

Analogously, the zeros of the real function $h_{\beta_n^{(j)}}(x) = x^{a_n^{(j)}} \sin(b_n^{(j)} \log(x))$ are given by

$$(2.2) \qquad y_{n,k}^{(j)} = e^{\frac{k}{b_n^{(j)}}\pi}, \ k \in \mathbb{Z}.$$

On the other hand, the points (2.2) satisfy

$$g_{\beta_n^{(j)}}(y_{n,k}^{(j)}) = y_{n,k}^{a_n^{(j)}} \cos\left(b_n^{(j)} \log(e^{\frac{k}{b_n^{(j)}}\pi})\right) = y_{n,k}^{a_n^{(j)}} \cos k\pi = \pm y_{n,k}^{a_n^{(j)}},$$

i.e., the function $g_{\beta_n^{(j)}}(x)$ attains its extreme values just at the points (2.2). Reciprocally, the function $h_{\beta_n^{(j)}}(x)$ takes its extreme values at the points (2.1). Moreover,

$$x_{n,k}^{(j)}, \ y_{n,k}^{(j)} \longrightarrow +\infty, \text{ when } k \to +\infty,$$
$$x_{n,k}^{(j)}, \ y_{n,k}^{(j)} \longrightarrow 0, \quad \text{ when } k \to -\infty,$$

for each j.

Hence, for any $\Delta > \delta > 0$, in order to find the real part of a zero of $G_n(z)$, say $\beta_n := a_n + ib_n$, and to prove the arbitrary density of the curves defined by the associated functions $g_{\beta_n^{(j)}}(x)$ and $h_{\beta_n^{(j)}}(x)$ on the compact defined by a_n, K_{a_n}, determined by $y = x^{a_n}$, $y = -x^{a_n}$, and the straight lines $x = \delta$, $x = \Delta$, it is sufficient to take β_n such that the set $\{x_{n,k}, y_{n,k}, \ k \in \mathbb{Z}\}$ is topologically dense in the interval $[\delta, \Delta]$.

Previously, we shall prove that

(2.3)
$$y_{n,k}^{(j)} < x_{n,k}^{(j)} < y_{n,k+1}^{(j)}$$

for any j. Indeed,

$$y_{n,k}^{(j)} < x_{n,k}^{(j)}$$

is equivalent to

$$e^{\frac{k}{b_n^{(j)}}\pi} < e^{\frac{2k+1}{2b_n^{(j)}}\pi},$$

that is

$$\frac{k}{b_n^{(j)}} < \frac{k}{b_n^{(j)}} + \frac{1}{2b_n^{(j)}},$$

which is true because of $b_n^{(j)} > 0$.

On the other hand, observe that

$$x_{n,k}^{(j)} < y_{n,k+1}^{(j)}$$

is equivalent to

$$e^{\frac{2k+1}{2b_n^{(j)}}\pi} < e^{\frac{k+1}{b_n^{(j)}}\pi},$$

and, therefore

$$\frac{1}{2b_n^{(j)}} < \frac{1}{b_n^{(j)}},$$

which is also true, and consequently (2.3) is proved.

Given $l \in \mathbb{N}$, let now $\beta_{n,l}^{(j)}$ be the maximum of the set determined by

$$\{x_{n,k}^{(j)} - y_{n,k}^{(j)}; \ y_{n,k+1}^{(j)} - x_{n,k}^{(j)} : |k| \le l\}.$$

Observe that $\beta_{n,l}^{(j)}$ is strictly positive noticing (2.3). Now, since

$$x_{n,k}^{(j)} - y_{n,k}^{(j)} = e^{\frac{k\pi}{b_n^{(j)}}} \left[e^{\frac{2\pi}{b_n^{(j)}}} - 1 \right]$$

and

$$y_{n,k+1}^{(j)} - x_{n,k}^{(j)} = e^{\frac{k\pi}{b_n^{(j)}}} \left[e^{\frac{\pi}{b_n^{(j)}}} - e^{\frac{\pi}{2b_n^{(j)}}} \right],$$

then

$$\max\{x_{n,k}^{(j)} - y_{n,k}^{(j)} \ : \ |k| \le l\} = x_{n,l}^{(j)} - y_{n,l}^{(j)} = e^{\frac{l\pi}{b_n^{(j)}}} \left[e^{\frac{\pi}{2b_n^{(j)}}} - 1 \right]$$

and

$$\max\{y_{n,k+1}^{(j)} - x_{n,k}^{(j)} \; : \; |k| \le l\} = y_{n,l+1}^{(j)} - x_{n,l}^{(j)} = e^{\frac{l\pi}{b_n^{(j)}}}\left[e^{\frac{\pi}{b_n^{(j)}}} - e^{\frac{\pi}{2b_n^{(j)}}}\right],$$

respectively. Hence,

$$\beta_{n,l}^{(j)} = \max\left\{e^{\frac{l\pi}{b_n^{(j)}}}\left[e^{\frac{\pi}{2b_n^{(j)}}} - 1\right], e^{\frac{l\pi}{b_n^{(j)}}}\left[e^{\frac{\pi}{b_n^{(j)}}} - e^{\frac{\pi}{2b_n^{(j)}}}\right]\right\}.$$

Finally, taking into account

$$e^{\frac{l\pi}{b_n^{(j)}}}\left[e^{\frac{\pi}{b_n^{(j)}}} - e^{\frac{\pi}{2b_n^{(j)}}}\right] > e^{\frac{l\pi}{b_n^{(j)}}}\left[e^{\frac{\pi}{2b_n^{(j)}}} - 1\right],$$

we deduce

$$\beta_{n,l}^{(j)} = e^{\frac{l\pi}{b_n^{(j)}}}\left[e^{\frac{\pi}{b_n^{(j)}}} - e^{\frac{\pi}{2b_n^{(j)}}}\right].$$

Now, given an arbitrary $\alpha > 0$, we will determine $\beta_n = a_n + ib_n$, a zero of $G_n(z)$, with b_n sufficiently large such that

$$e^{\frac{\pi}{b_n}} - e^{\frac{\pi}{2b_n}} < \alpha e^{\frac{-l\pi}{b_n}},$$

which is possible noticing $b_n^{(j)} \to \infty$. Indeed, $e^{\frac{\pi}{b_n^{(j)}}} - e^{\frac{\pi}{2b_n^{(j)}}} \longrightarrow 0$ and $\alpha e^{\frac{-l\pi}{b_n^{(j)}}} \to \alpha$.

Therefore, with this β_n, it follows that

$$\alpha_{n,l} = e^{\frac{l\pi}{b_n}}\left[e^{\frac{\pi}{b_n}} - e^{\frac{\pi}{2b_n}}\right] < e^{\frac{l\pi}{b_n}}\left[\alpha e^{-\frac{l\pi}{b_n}}\right] = \alpha.$$

Consequently, by choosing $l \in \mathbb{N}$ sufficiently large such that $y_{n,-l} < \delta$ and $y_{n,l+1} > \Delta$, we deduce that the functions $g_{\beta_n}(x) = x^{a_n}\cos(b_n\log(x))$ and $h_{\beta_n}(x) = x^{a_n}\sin(b_n\log(x))$ have density α on K_{a_n}.

Finally, it only remain to prove that, given $\beta_n^{(j)}$ a zero of $G_n(z)$ with positive imaginary part, the functions $g_{\beta_n^{(j)}}(x)$ and $h_{\beta_n^{(j)}}(x)$ are linearly independent. But, this property is deduced from [5, Theorem 7] where the reader can find a proof concerning the fact that the functions $f_{\beta_n^{(j)}}(z) \equiv z^{\beta_n^{(j)}}$ are linearly independent and, consequently, the theorem follows. $\square$

As can be seen above, a surprising connection between α-dense curves and functional equations emerges when we try to solve the functional equation (1.3). In Figure 3, we show an example of the compact set for the case $a_n > 0$, that is, the real part of the zero of $G_n(z)$ whose existence is assured by Theorem 2.3 is strictly positive.

Naturally, a particular case of the previous theorem is obtained when we take $n = 2$. In that case, all zeros of $G_2(z)$ are imaginary, hence a particular set of compact sets in $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ for which there exists a functional equation that contains infinitely many solutions that α-densify them, are the rectangles.
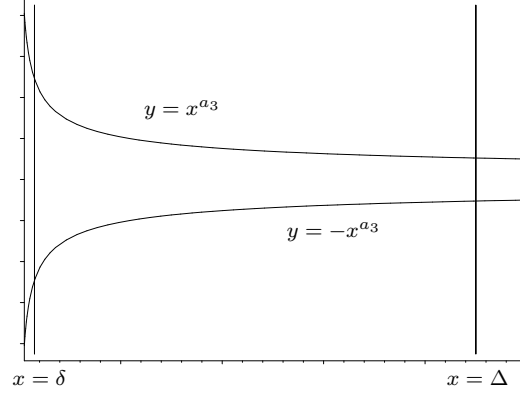
FIGURE 3. K_{a_3} associated to $a_3 \approx -0.4543970081950$

3. The property of quadrature in the case $n = 2$

As we showed above, the property of quadrature in cubes of a family of α-dense curves was introduced in [4]. This concept can be extended to much more general regions of the following manner:

Definition 3.1. Let K be a densifiable set of R^{n+1} with positive Jordan content $J_{n+1}(K)$. A family $\{\gamma_\alpha : \alpha \in \Gamma\}$ of rectifiable α-dense curves in K is said to have the property of quadrature in K if

$$\lim_{\alpha \to 0} L(\gamma_\alpha)\alpha^n = J_{n+1}(K),$$

where $L(\gamma_\alpha)$ is the length of each γ_α.

Then the following proposition easily follows.

Proposition 3.2. *Let K be a densifiable set of R^{n+1} with positive Jordan content $J_{n+1}(K)$ and $\{\gamma_\alpha : \alpha \in \Gamma\}$ a family of rectifiable α-dense curves in K satisfying*

$$0 < \lim_{\alpha \to 0} L(\gamma_\alpha)\alpha^n \le J_{n+1}(K).$$

Thus the family $\{\gamma_\beta : \beta = \delta\alpha, \ \alpha \in \Gamma\}$, with

$$\delta = \left(\frac{J_{n+1}(K)}{\lim\limits_{\alpha \to 0} L(\gamma_\alpha)\alpha^n} \right)^{\frac{1}{n}},$$

has the property of quadrature in K.

In this section we assure the existence of a family of continuous solutions of the functional equation for the case $n = 2$ which has the property of quadrature in the compact set that densifies.

Proposition 3.3. *The subspace A formed by the continuous solutions of*

$$(3.1) \qquad\qquad\qquad f(x) + f(2x) = 0$$

contains an infinite set of functions which have the property of quadrature in the compact that densifies.

Proof. As we said above, in the case $n = 2$ all zeros of $G_2(z)$ are imaginary and the compact sets which appear in the densification theorem are only rectangles. Moreover, in connection with this theorem, let p, q be integer numbers with $p < q$ and $2^p - 2^q \geq 1$. It can be verified [2, Prop. 3] that the solutions

$$g_k(x) = \cos\left(b_k \log(x)\right)$$

and

$$h_k(x) = \sin\left(b_k \log(x)\right),$$

with $b_k = \frac{(2k+1)\pi}{\log 2}$, $k \in \mathbb{Z}$, of the equation (3.1) have density α in the compact region $K_0 := [2^p, 2^q] \times [-1, 1]$ if $k \geq m$, with m satisfying

$$(3.2) \qquad\qquad\qquad 1 - 2^{-(3/2)/(2m+1)} < \alpha 2^{-q}.$$

Observe that the area of this compact region is

$$(3.3) \qquad\qquad\qquad J_2(K_0) = (2^q - 2^p) \cdot 2,$$

which is greater than or equal to 2. Taking now, for example, the solution $g_k(x) = \cos\left(b_k \log(x)\right)$, the length of its graphic on the interval $[2^p, 2^q]$ is given by

$$(3.4) \qquad\qquad \mathrm{L}(g_k(x)) = \int_{2^p}^{2^q} \sqrt{1 + \frac{b_k^2}{x^2}\sin^2\left(b_k \log x\right)}\ dx.$$

Then, with the change of variables $b_k \log x = t\ \left(dx = \frac{e^{t/b_k}}{b_k}dt\right)$, the integral (3.4) is transformed into

$$\int_{b_k \log 2^p}^{b_k \log 2^q} \sqrt{1 + \frac{b_k^2}{e^{2t/b_k}}\sin^2 t}\ \cdot\ \frac{e^{t/b_k}}{b_k}\ dt = \int_{b_k \log 2^p}^{b_k \log 2^q} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t}\ dt$$

$$= \sum_{j=p(2k+1)}^{q(2k+1)-1} I_k^{(j)},$$

with

$$(3.5) \qquad I_k^{(j)} = \int_{b_k \log y_{k,j}}^{b_k \log y_{k,j+1}} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t}\ dt \quad\text{and}\quad y_{k,j} = 2^{\frac{j}{2k+1}}.$$

Noticing that

$$b_k \log y_{k,j} = b_k \log 2^{\frac{j}{2k+1}} = \frac{(2k+1)\pi}{\log 2}\log 2^{\frac{j}{2k+1}} = j\pi,$$

the expression (3.5) of the integrals that define $I_k^{(j)}$ is

$$I_k^{(j)} = \int_{j\pi}^{(j+1)\pi} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t} \, dt.$$

Now, since there is a finite number of integrals (specifically $(q-p)(2k+1)$ terms), we can state that there exist j_1, j_2 such that

$$(3.6) \qquad I_k^{(j_1)} = \min\{I_k^{(j)}, \ j = p(2k+1), p(2k+1)+1, \ldots, q(2k+1)-1\},$$

$$(3.7) \qquad I_k^{(j_2)} = \max\{I_k^{(j)}, \ j = p(2k+1), p(2k+1)+1, \ldots, q(2k+1)-1\}.$$

Consider an arbitrary $\alpha > 0$, by taking $\beta = \frac{1}{(q-p)(2k+1)}$, with k satisfying the condition (3.2), i.e., $1 - 2^{-(3/2)/(2k+1)} < \alpha 2^{-q}$, clearly $\beta \to 0$ when $k \to \infty$, and by defining

$$L_\beta := \beta \cdot \mathrm{L}(g_k(x)),$$

it is clearly verified that

$$I_k^{(j_1)} \leq L_\beta \leq I_k^{(j_2)},$$

with

$$(3.8) \qquad I_k^{(j_1)} = \int_{j_1\pi}^{(j_1+1)\pi} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t} \, dt,$$

$$(3.9) \qquad I_k^{(j_2)} = \int_{j_2\pi}^{(j_2+1)\pi} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t} \, dt,$$

and j_1, j_2 determined in (3.6) and (3.7) respectively.

Applying the change of variables $u = t - j_1\pi$, the integral (3.8) becomes

$$I_k^{(j_1)} = \int_{j_1\pi}^{(j_1+1)\pi} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t} \, dt = \int_0^\pi \sqrt{\frac{e^{2(u+j_1\pi)/b_k}}{b_k^2} + \sin^2 u} \, du.$$

Analogously, substituting $u = t - j_2\pi$ in the integral (3.9), since $\sin^2 u = \sin^2(u + j\pi)$ for all $j \in \mathbb{Z}$, it holds that

$$I_k^{(j_2)} = \int_{j_2\pi}^{(j_2+1)\pi} \sqrt{\frac{e^{2t/b_k}}{b_k^2} + \sin^2 t} \, dt = \int_0^\pi \sqrt{\frac{e^{2(u+j_2\pi)/b_k}}{b_k^2} + \sin^2 u} \, du.$$

Now, taking limits, according to the convergence theorems, it follows that

$$\lim_{k\to\infty} I_k^{(j_1)} = \lim_{k\to\infty} I_k^{(j_2)} = \int_0^\pi \sqrt{\sin^2 u} \, du = 2.$$

Consequently,

$$\lim_{\beta\to 0} L_\beta = \lim_{\beta\to 0} \beta \mathrm{L}(g_k(x)) = 2.$$

To conclude, noticing (3.3), it follows that $J_2(K_0) \geq 2$ and, by applying the method described by Proposition 5, that is, taking $\alpha' = (2^q - 2^p)\beta$, since

$$\lim_{\alpha' \to 0} L(g_k(x)) \cdot \alpha' = (2^q - 2^p) \lim_{\beta \to 0} L(g_k(x)) \cdot \beta = (2^q - 2^p) \cdot 2,$$

it is verified that

$$\lim_{\alpha' \to 0} L(g_k(x)) \cdot \alpha' = J_2(K_0),$$

which proves the property of quadrature in the case $n = 2$. $\square$

4. The property of quadrature in the general case

Using the method described above, we will generalize the property of quadrature to each $n \geq 2$.

Theorem 4.1. *The subspace V_n formed by the continuous solutions of (1.3) contains an infinite set of functions which have the property of quadrature in the compact that densifies.*

Proof. According to Theorem 2.3, given an arbitrary $\alpha > 0$, we can find a zero of (1.2), $\beta_n = a_n + ib_n$, such that the functions $g_{\beta_n}(x) = x^{a_n} \cos(b_n \log(x))$ and $h_{\beta_n}(x) = x^{a_n} \sin(b_n \log(x))$, contained in V_n, have density α in the compact K_{a_n} determined by $y = x^{a_n}$, $y = -x^{a_n}$, and the straight lines $x = e^{p\pi}$, $x = e^{q\pi}$, for some $p, q \in \mathbb{Z}$, with $p < q$ and b_n sufficiently large such that

$$(4.1) \qquad\qquad e^{\frac{\pi}{b_n}} - e^{\frac{\pi}{2b_n}} < \alpha e^{\frac{-l\pi}{b_n}}$$

for some $l \in \mathbb{N}$.

In this case, the area of K_{a_n} is given by

$$J_2(K_{a_n}) = 2 \int_{e^{p\pi}}^{e^{q\pi}} x^{a_n} \, dx = \begin{cases} \dfrac{2}{a_n + 1}\left[e^{(a_n+1)p\pi} - e^{(a_n+1)q\pi} \right], & \text{if } a_n \neq -1 \\ 2(q - p)\pi, & \text{if } a_n = -1. \end{cases}$$

Take $p, q \in \mathbb{Z}$ so that $J_2(K_{a_n}) \geq 2$ and consider $g_{\beta_n}(x) = x^{a_n} \cos(b_n \log(x))$. The length of the graphic of $g_{\beta_n}(x)$ in the interval $[e^{p\pi}, e^{q\pi}]$ is

$$(4.2) \quad \mathrm{L}(g_{\beta_n}(x)) = \int_{e^{p\pi}}^{e^{q\pi}} \sqrt{1 + x^{2(a_n-1)}[a_n \cos(b_n \log x) - b_n \sin(b_n \log x)]^2} \, dx$$

and, by putting $b_n \log x = t \left(dx = \frac{e^{t/b_n}}{b_n} dt \right)$, the expression (4.2) is transformed into

$$(4.3) \qquad \begin{aligned} &\int_{b_n \log e^{p\pi}}^{b_n \log e^{q\pi}} \sqrt{1 + e^{2t(a_n-1)/b_n}[a_n \cos t - b_n \sin t]^2} \, \frac{e^{t/b_n}}{b_n} \, dt \\ &= \int_{b_n \log e^{p\pi}}^{b_n \log e^{q\pi}} \sqrt{\frac{e^{2t/b_n}}{b_n^2} + \frac{e^{2a_n t/b_n}}{b_n^2}[a_n \cos t - b_n \sin t]^2} \, dt. \end{aligned}$$

Now, noticing that $b_n = b'_n + c_n$, with $b'_n = [b_n]$ the integer part of b_n, and $c_n \in [0, 1)$, it can be deduced that $b_n \to \infty$ if and only if $b'_n \to \infty$, and so (4.3) can be expressed as

$$(4.4) \qquad \sum_{k=pb'_n}^{qb'_n - 1} I_n^{(k)},$$

with

$$(4.5) \qquad I_n^{(k)} = \int_{b_n \log y_{n,k}}^{b_n \log y_{n,k+1}} \sqrt{\frac{e^{2t/b_n}}{b_n^2} + \frac{e^{2a_n t/b_n}}{b_n^2} [a_n \cos t - b_n \sin t]^2} \; dt,$$

and $y_{n,k} = e^{\frac{k\pi}{b'_n}}$. Finally, since

$$b_n \log y_{n,k} = b_n \log e^{\frac{k\pi}{b'_n}} = \frac{k\pi b_n}{b'_n} = \frac{k\pi(b'_n + c_n)}{b'_n} = k\pi + \frac{c_n k\pi}{b'_n},$$

then (4.5) is transformed into

$$I_n^{(k)} = \int_{k\pi + \frac{c_n k\pi}{b'_n}}^{(k+1)\pi + \frac{c_n(k+1)\pi}{b'_n}} \sqrt{\frac{e^{2t/b_n}}{b_n^2} + \frac{e^{2a_n t/b_n}}{b_n^2} [a_n \cos t - b_n \sin t]^2} \; dt.$$

Observe that, since there is a finite number of terms in (4.4) (specifically $(q - p)b'_n$ terms), there exist k_1, k_2 such that

$$(4.6) \qquad I_n^{(k_1)} = \min\{I_n^{(k)}, \; k = pb'_n, pb'_n + 1, \dots, qb'_n - 1\},$$

$$(4.7) \qquad I_n^{(k_2)} = \max\{I_n^{(k)}, \; k = pb'_n, pb'_n + 1, \dots, qb'_n - 1\}.$$

Now, taking $\beta = \frac{1}{(q-p)b'_n}$, with b'_n verifying (4.1), i.e., $e^{\frac{\pi}{b'_n}} - e^{\frac{\pi}{2b'_n}} < \alpha e^{\frac{-l\pi}{b'_n}}$ for some $l \in \mathbb{N}$, it is verified that $\beta \to 0$ when $b'_n \to \infty$, therefore $\beta \to 0$ when $b_n \to \infty$. Moreover, by defining

$$L_\beta := \beta \cdot \mathrm{L}(g_{\beta_n}(x)),$$

it holds that

$$I_n^{(k_1)} \le \mathrm{L}_\beta \le I_n^{(k_2)},$$

where $I_n^{(k_1)}$, $I_n^{(k_2)}$, determined in (4.6) and (4.7) respectively, are of the form (4.5).

Now, with the change of variables $u = t - \left(k_1\pi + \frac{c_n k_1 \pi}{b'_n} \right)$ for the integral $I_n^{(k_1)}$, it holds that

$$(4.8) \quad I_n^{(k_1)} = \int_0^{\pi + \frac{c_n \pi}{b'_n}} \sqrt{\frac{e^{2(u + k_1\pi + \frac{c_n k_1 \pi}{b'_n})/b_n}}{b_n^2} + \frac{e^{2a_n(u + k_1\pi + \frac{c_n k_1 \pi}{b'_n})/b_n}}{b_n^2}} \; .$$

$$\overline{\left[a_n \cos \left(u + k_1\pi + \frac{c_n k_1 \pi}{b'_n} \right) - b_n \sin \left(u + k_1\pi + \frac{c_n k_1 \pi}{b'_n} \right) \right]^2} \, du.$$

Analogously, substituting $u = t - \left(k_2\pi + \frac{c_n k_2 \pi}{b'_n} \right)$, the integral $I_n^{(k_2)}$ becomes

(4.9)

$$I_n^{(k_2)} = \int_0^{\pi + \frac{c_n \pi}{b'_n}} \sqrt{\frac{e^{2(u+k_2\pi+\frac{c_n k_2\pi}{b'_n})/b_n}}{b_n^2} + \frac{e^{2a_n(u+k_2\pi+\frac{c_n k_2\pi}{b'_n})/b_n}}{b_n^2}} \cdot$$

$$\overline{\left[a_n \cos \left(u + k_2\pi + \frac{c_n k_2 \pi}{b'_n} \right) - b_n \sin \left(u + k_2\pi + \frac{c_n k_2 \pi}{b'_n} \right) \right]^2} \, du.$$

Taking limits in the different factors that form (4.8) and (4.9), it is verified that

$$\lim_{b'_n \to \infty} \frac{e^{2(u+k\pi+\frac{c_n k\pi}{b'_n})/b_n}}{b_n^2} = 0,$$

and

$$\lim_{b'_n \to \infty} \frac{e^{2a_n(u+k\pi+\frac{c_n k\pi}{b'_n})/b_n}}{b_n^2} \left[a_n \cos \left(u + k\pi + \frac{c_n k\pi}{b'_n} \right) - b_n \sin \left(u + k\pi + \frac{c_n k\pi}{b'_n} \right) \right]^2$$

$$= \sin^2 (u + k\pi) = \sin^2 u.$$

Hence,

$$\lim_{b'_n \to \infty} I_n^{(k_1)} = \lim_{b'_n \to \infty} I_n^{(k_2)} = \int_0^\pi \sin u \, du = 2,$$

and consequently

$$\lim_{\beta \to 0} L_\beta = \lim_{\beta \to 0} \beta \cdot \mathrm{L}(g_{\beta_n}(x)) = 2.$$

To conclude, since $J_2(K_{a_n}) \geq 2$, by applying the method given by Proposition 5, that is, taking

$$\alpha' = \begin{cases} \dfrac{\left[e^{(a_n+1)p\pi} - e^{(a_n+1)q\pi} \right]}{a_n + 1} \beta, & \text{si } a_n \neq -1 \\ (q-p)\pi\beta, & \text{si } a_n = -1 \end{cases},$$

we have $\lim\limits_{\alpha' \to 0} L(g_{\beta_n}(x)) \cdot \alpha' = J_2(K_{a_n})$. Indeed, observe that $\lim\limits_{\alpha' \to 0} L(g_{\beta_n}(x)) \cdot \alpha'$ is equal to

$$\begin{cases} \dfrac{\left[e^{(a_n+1)p\pi} - e^{(a_n+1)q\pi} \right]}{a_n + 1} \lim\limits_{\beta \to 0} L(g_{\beta_n}(x))\beta = J_2(K_{a_n}), & \text{if } a_n \neq -1 \\ (q-p)\pi \lim\limits_{\beta \to 0} L(g_{\beta_n}x))\beta = J_2(K_{a_n}), & \text{if } a_n = -1. \end{cases}$$

Clearly, this method can be also applied to the function

$$h_{\beta_n}(x) = x^{a_n} \sin (b_n \log x)$$

and so the result holds. $\qquad\square$

References

[1] R. B. Ash, *Complex Variables*, Academic Press, New York, 1971.

[2] G. Mora, *A note on the functional equation $F(z) + F(2z) + \cdots + F(nz) = 0$*, J. Math. Anal. Appl. **340** (2008), no. 1, 466–475.

[3] G. Mora and Y. Cherrault, *Characterization and generation of α-dense curves*, Comput. Math. Appl. **33** (1997), no. 9, 83–91.

[4] _____, *A new approach to the reduction of multiple integrals to simple ones using Chebyshev's kernels*, Kybernetes **37** (2008), no. 1, 104–119.

[5] G. Mora, Y. Cherrault, and A. Ziadi, *Functional equations generating space-densifying curves*, Comput. Math. Appl. **39** (2000), no. 9-10, 45–55.

[6] G. Mora and J. M. Sepulcre, *On the distribution of zeros of a sequence of entire functions approaching the Riemann zeta function*, J. Math. Anal. Appl. **350** (2009), no. 1, 409–415.

[7] _____, *The critical strips on the sums $1 + 2^z + \cdots + n^z$*, Abstr. Appl. Anal. **2011** (2011), Art. ID 909674, 15 pp. doi:10.1155/2011/909674.

[8] H. Sagan, *Space-Filling Curves*, Springer-Verlag, New York, 1994.

Department of Mathematical Analysis
University of Alicante
03080-Alicante, Spain
E-mail address: JM.Sepulcre@ua.es