IDENTITIES WITH ADDITIVE MAPPINGS
IN SEMIPRIME RINGS

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Abstract. The aim of this paper is to prove the next result. Let $n > 1$ be an integer and let $R$ be a $n!$-torsion free semiprime ring. Suppose that $f : R \to R$ is an additive mapping satisfying the relation $[f(x), x^n] = 0$ for all $x \in R$. Then $f$ is commuting on $R$.

1. Introduction and the main theorem

Throughout, $R$ will represent an associative ring with a center $Z(R)$. Let $n > 1$ be an integer. A ring $R$ is $n$-torsion free if $nx = 0, x \in R$, implies $x = 0$. The Lie product (or a commutator) of elements $x, y \in R$ will be denoted by $[x, y]$ (i.e., $[x, y] = xy - yx$). Recall that a ring $R$ is prime if $aRb = \{0\}$, $a, b \in R$, implies that either $a = 0$ or $b = 0$. Furthermore, a ring $R$ is called semiprime if $aRa = \{0\}, a \in R$, implies $a = 0$. We will denote by $C$ and $Q$ the extended centroid and the maximal right ring of quotients of a semiprime ring $R$, respectively. For the explanation of the extended centroid as well as the maximal right ring of quotients of a semiprime ring we refer the reader to [4]. As usual, the socle of a ring $R$ will be denoted by $soc(R)$.

An additive mapping $D : R \to R$ called a derivation on $R$ if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. An additive mapping $f : R \to R$ is called centralizing on $R$ if $[f(x), x] \in Z(R)$ holds for all $x \in R$. In a special case, when $[f(x), x] = 0$ for all $x \in R$, the mapping $f$ is said to be commuting on $R$. A classical result of Posner [21] (Posner’s second theorem) states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Posner’s second theorem in general cannot be proved for semiprime rings as shows the following example. Let $R_1$ and $R_2$ be prime rings with $R_1$ commutative and set $R = R_1 \oplus R_2$. Further, let $D_1 : R_1 \to R_1$ be a nonzero derivation. A mapping $D : R \to R$ defined by
$D((r_1, r_2)) = (D_1(r_1), 0)$ is then a nonzero commuting derivation. It is also easy to show that if $D : R \to R$ is a commuting derivation on a semiprime ring $R$, then $D$ maps $R$ into $Z(R)$ (see, for example, the end of the proof of Theorem 2.1 in [25]). Furthermore, Brešar [7] proved that every additive commuting mapping of a prime ring $R$ is of the form $x \mapsto \lambda x + \zeta(x)$, where $\lambda$ is an element of the extended centroid $C$ and $\zeta : R \to C$ is an additive mapping. For results concerning commuting mappings, centralizing mappings and related problems we refer the reader to [1,5–13,18,22–28] where further references can be found.

In [18] Vukman and the first named author generalized the result proved by Brešar and Hvala for prime rings [9].

**Theorem 1** ([18, Theorem 2]). Let $R$ be a 2-torsion free semiprime ring. Suppose that an additive mapping $f : R \to R$ satisfies the relation

$$[f(x), x^2] = 0$$

for all $x \in R$. Then $f$ is commuting on $R$.

This result motivated us to prove our main theorem.

**Main Theorem.** Let $n > 1$ be a fixed integer and $R$ a $n!$-torsion free semiprime ring. Suppose that an additive mapping $f : R \to R$ satisfies the relation

$$[f(x), x^n] = 0$$

for all $x \in R$. Then $f$ is commuting on $R$.

Let us point out that the above theorem might be of some interest from the functional analysis point of view as well since $C^*$-algebras (moreover, semisimple Banach algebras) are semiprime.

2. Proof of the main theorem

Let $n > 1$ be a fixed integer. Before proving our main theorem, let us fix some notation and write two results (Lemma 1 and Proposition 1) which we will need in the following. Let $m > 1$ be an integer and $F$ an arbitrary field. Then $M_m(F)$ denotes the algebra of all $m \times m$ matrices over the field $F$. Recall that $Z(M_m(F)) = FI$, where $I \in M_m(F)$ is the identity matrix. By $E_{ij} \in M_m(F)$, $1 \leq i, j \leq m$, we will denote the matrix with $(i, j)$-entry equal to one and all the others equal to zero.

**Lemma 1.** Let $R = M_m(F)$, $m > 1$, and $A \in R$. Suppose that

$$[A, X^n] = 0$$

for all $X \in R$. Then $A \in FI$.

**Proof.** Let $P$ be an idempotent matrix in $M_m(F)$. Setting $X = P$ in (2) and multiplying left side by $(I - P)$, we see that $(I - P)AP = 0$ for any idempotent matrix $P$. Thus, $A$ is a diagonal matrix. Note that $UAU^{-1}$ must
be diagonal for each invertible element $U \in M_m(\mathbb{F})$, since $[U A U^{-1}, X^n] = 0$ for all $X \in M_m(\mathbb{F})$. Write $A = \sum_{i=1}^m \alpha_i E_{ii}$, where $\alpha_i \in \mathbb{F}$. Then, for each $j > 1$ the $(1,j)$-entry of $(I + E_{1j})A(I + E_{1j})^{-1}$ equals 0. That is, $\alpha_j = \alpha_1$ for $j > 1$. Hence, $A \in FI$, as desired.

**Proposition 1.** Let $R$ be a non-commutative prime ring and $a \in R$ such that

$$[a, x^n] = 0$$

for all $x \in R$. Then $a \in Z(R)$.

**Proof.** Suppose on the contrary that $a \notin Z(R)$. Then

$$f(X) = [a, X^n]$$

is a nontrivial generalized polynomial identity (in the following referred as GPI) for $R$. Using [14], $f(X)$ is also a GPI for $Q$. Denote by $F$ either the algebraic closure of $C$ or $C$ itself according to the cases when $C$ is either infinite or finite dimensional, respectively. Then, using a standard argument (e.g., see [19, Proposition]), $f(X)$ is also a GPI for $Q \oplus C F$. Since $Q \oplus C F$ is a centrally closed prime $F$-algebra [15, Theorem 2.5 and Theorem 3.5], by replacing $R$ and $C$ with $Q \oplus C F$ and $F$, respectively, we may assume that $R$ is centrally closed and $C$ is either finite dimensional or algebraically closed. In a view of Martindale’s theorem [20], $R$ is a primitive ring having a non-zero socle with $C$ as its associated division ring.

Since $a \notin C$, we have $[a, x] \neq 0$ for some $x \in soc(R)$. By Litoff’s theorem [16], there exists an idempotent $e \in soc(R)$ such that $x, ax, xa \in eRe$. Note that $e(exe)e$ is a GPI for $R$. Thus, $[(exe)_e]_e$ is a GPI for $eRe$. Since $eRe \cong M_m(C)$ for some $m \geq 1$, $exe$ is central in $eRe$ by Lemma 1. It follows that there exists $a \in C$ such that $ce = eae$. Hence, $ex = eae = exae = xae = xa$. So $[a, x] = 0$, a contradiction. Therefore, $a \in Z(R)$, as desired.

**Remark.** Let us point out that in Proposition 1 we have no restriction on the characteristic of a non-commutative ring $R$. But if $R$ is 2-nil-torsion free, then the above proposition is a direct consequence of Theorem 2.1 in [25] (see also Theorem 3 in [17] for the generalization). Namely, if we define an inner derivation $D : R \to R$ by $D(x) = [a, x]$, then $D(x^n) = [a, x^n]$. Therefore, if $[a, x^n] = 0$, then $D(x^n)x + xD(x^n) = 0$ for all $x \in R$ and, by [25, Theorem 2.1], $D(x) = [a, x] = 0$ for all $x \in R$. Thus, $a \in Z(R)$.

Now we are ready to prove our main theorem. In the proof we will use some ideas similar to those used in [28].

**Proof of Main Theorem.** By semiprimeness of $R$, there exists a family of prime ideals $\{P_\alpha : \alpha \in I\}$ such that $\cap_{\alpha \in I} P_\alpha = \{0\}$. Without loss of generality, we may assume that prime rings $R/P_\alpha, \alpha \in I$, are 2-torsion free (see [2, p. 459]).

Now, let us fix an arbitrary $\alpha \in I$. It is sufficient to show that $[f(x), x] \in P_\alpha$ for all $x \in R$. Denote by $C$ the extended centroid of a prime ring $R/P_\alpha$ and
by $A$ the central closure of $R/P_\alpha$. One can consider $A$ as a vector space over the field $C$ which can be regarded as a subspace of $A$. Thus, there exists a subspace $B$ of $A$ such that $A = B + C$. Let $\pi$ be the canonical projection of $A$ onto $B$. For $x \in R$ we shall write $\overline{x}$ for the coset $x + P_\alpha \in R/P_\alpha$. Replacing $x$ by $x + p$ in (1) we obtain

$$[f(p), x^n] \in P_\alpha$$

for all $x \in R$ and $p \in P_\alpha$. Therefore, $[f(p), \overline{x^n}] = 0$ for all $x \in R$. Using Proposition 1, it follows that $\overline{f(p)}$ lies in the center of $R/P_\alpha$, which means that $[\overline{f(p)}, \overline{x^n}] = 0$ for all $x \in R, p \in P_\alpha$. In particular, we have $\pi \overline{f(p)} = 0$. This yields that the mapping $\overline{f} : R/P_\alpha \rightarrow A$, $\overline{f}(\overline{x}) = \pi \overline{f(x)}$, is well defined. It is easy to verify that $\overline{f}$ is additive and satisfies $[\overline{f}(\overline{x}), \overline{x^n}] = 0$ for all $x \in R$. Using [3, Theorem 1.1] it follows that $[\overline{f}(\overline{x}), \overline{x^n}] = 0$ which in turn implies $[f(x), x] \in P_\alpha$. The proof is completed.$\square$

In [8], Brešar proved that there are no nonzero skew-commuting additive mappings on a 2-torsion free semiprime ring $R$. In other words, if $R$ is a 2-torsion free semiprime ring and $f : R \rightarrow R$ an additive mapping such that $f(x)x + xf(x) = 0$ for all $x \in R$, then $f = 0$. Motivated by this result, we conclude our paper with the following conjecture.

**Conjecture.** Let $n \geq 1$ be some fixed integer and let $R$ be a semiprime ring with suitable torsion restrictions. Suppose that an additive mapping $f : R \rightarrow R$ satisfies the relation

$$f(x)x^n + x^n f(x) = 0$$

for all $x \in R$. Then $f = 0$.

In the case $n = 1$, the above conjecture has been proved by Brešar in [8].

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