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A CHARACTERIZATION THEOREM FOR LIGHTLIKE HYPERSURFACES OF SEMI-RIEMANNIAN MANIFOLDS OF QUASI-CONSTANT CURVATURES

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Abstract. In this paper, we study lightlike hypersurfaces M of semi-Riemannian manifolds $\bar{M}$ of quasi-constant curvatures. Our main result is a characterization theorem for screen homothetic Einstein lightlike hypersurfaces of a Lorentzian manifold of quasi-constant curvature subject such that its curvature vector field ζ is tangent to M.

1. Introduction

B.Y. Chen and K. Yano [2] introduced the notion of a *Riemannian manifold of quasi-constant curvature* as a Riemannian manifold $(\bar{M}, \bar{g})$ endow with the curvature tensor $\bar{R}$ satisfying the following equation:

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,W) &= \alpha\{\bar{g}(Y,Z)\bar{g}(X,W) - \bar{g}(X,Z)\bar{g}(Y,W)\} \\ &+ \beta\{\bar{g}(X,W)\theta(Y)\theta(Z) - \bar{g}(X,Z)\theta(Y)\theta(W) \\ &\quad + \bar{g}(Y,Z)\theta(X)\theta(W) - \bar{g}(Y,W)\theta(X)\theta(Z)\}, \end{aligned} \tag{1.1}$$

for any vector fields X, Y, Z and W of $\bar{M}$, where α and β are smooth functions and θ is a 1-form associated with a non-vanishing smooth unit vector field ζ by

$$\theta(X) = \bar{g}(X, \zeta), \tag{1.2}$$

ζ is called the *curvature vector field* of $\bar{M}$. It is well known that if the curvature tensor $\bar{R}$ is of the form (1.1), then $\bar{M}$ is conformally flat. If $\beta = 0$, then $\bar{M}$ is a space of constant curvature α.

A non-flat Riemannian manifold $\bar{M}$ of dimension $n(> 2)$ is called a *quasi-Einstein* manifold [1] if its Ricci tensor $\bar{Ric}$ satisfies the condition

$$\bar{Ric}(X,Y) = a\,\bar{g}(X,Y) + b\,\phi(X)\phi(Y),$$

where a and b are smooth functions such that $b \neq 0$ and ϕ is a non-vanishing 1-form such that $\bar{g}(X,U) = \phi(X)$ for any vector field X, where U is a unit

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vector field. If $b = 0$, then $\bar{M}$ is an Einstein manifold. It is easily to see that every Riemannian manifold of quasi-constant curvature is quasi-Einstein.

The classification of Einstein lightlike hypersurfaces M in semi-Riemannian manifolds $\bar{M}$ was studied by K.L. Duggal and D.H. Jin [5]. Their main results focused on the geometry of Einstein lightlike hypersurfaces M of a Lorentzian space form $\bar{M}(c)$ of constant curvature c, whose shape operator is conformal to the shape operator of its screen distribution by some non-zero constant φ, which is called the *conformal factor*. Such a M is called *screen homothetic*. The reason for this geometric restriction on M was due to the fact that such a class admits a canonical integrable screen distribution and a symmetric induced Ricci tensor of M. Authors proved a characterization theorem for screen homothetic Einstein lightlike hypersurfaces of a Lorentzian space form as it follow:

Theorem 1.1. *Let M be a screen homothetic Einstein lightlike hypersurface of a Lorentzian space form $\bar{M}^{m+2}(c)$, $m > 2$, such that $Ric = \kappa g$. Then $c = 0$, i.e., $\bar{M}$ is flat manifold, and M is locally a product manifold $\mathcal{C} \times M_1 \times M_2$, where $\mathcal{C}$ is a null curve tangent to the radical distribution, and M_1 and M_2 are leaves of some integerable distributions of M such that*

(1) *If $\kappa \neq 0$, then either M_1 or M_2 is an m-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of κ and the other is a point.*
(2) *If $\kappa = 0$, then M_1 is an $(m-1)$ or an m-dimensional Euclidean space and M_2 is a non-null curve or a point.*

After that, D.H. Jin [6] generalized the above Duggal-Jin's characterization theorem for screen conformal Einstein lightlike hypersurfaces of Lorentzian space forms in which the conformal factor is non-vanishing smooth function φ.

The objective of this paper is to generalize the above characterization theorem for screen homothetic Einstein lightlike hypersurfaces of a Lorentzian manifold of quasi-constant curvature. We prove a characterization theorem for screen homothetic lightlike hypersurfaces M of a Lorentzian manifold $\bar{M}$ of quasi-constant curvature subject such that the curvature vector field ζ of $\bar{M}$, defined by (1.2), is tangent to M.

2. Lightlike hypersurface

It is well-known [3] that the normal bundle $TM^{\perp}$ of the lightlike hypersurfaces (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a subbundle of the tangent bundle TM and coincides with the radical distribution $Rad(TM) = TM \cap TM^{\perp}$. Thus there exists a non-degenerate complementary vector bundle $S(TM)$ of $Rad(TM)$ in TM, which is called a *screen distribution*, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \tag{2.1}$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. It is well-known [3] that, for any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique lightlike vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0, \quad \forall\, X \in \Gamma\left(S(TM)|_{\mathcal{U}}\right).$$

Then the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follows:

$$T\bar{M} = TM \,\oplus\, tr(TM) = \{Rad(TM) \,\oplus\, tr(TM)\} \,\oplus_{orth}\, S(TM). \tag{2.2}$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively.

In the sequel, we take X, Y, Z, $W \in \Gamma(TM)$, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$ with respect to the decomposition (2.1). Then the local Gauss and Weingartan formulas for M and $S(TM)$ are given respectively by

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, && (2.3)\\
\bar{\nabla}_X N &= -A_{N} X + \tau(X)N; && (2.4)\\
\nabla_X PY &= \nabla^*_X PY + C(X, PY)\xi, && (2.5)\\
\nabla_X \xi &= -A^*_{\xi} X - \tau(X)\xi, && (2.6)
\end{aligned}$$

where ∇ and ∇^* are the linear connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_{N} and A^*_{ξ} are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on TM. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric. From the fact $B(X,\, Y) = g(\bar{\nabla}_X Y,\, \xi)$, we know that B is independent of the choice of the screen distribution $S(TM)$ and

$$B(X, \xi) = 0. \tag{2.7}$$

The induced connection ∇ of M is not metric and satisfies

$$\begin{aligned}
(\nabla_X g)(Y, Z) &= B(X, Y)\, \eta(Z) + B(X, Z)\, \eta(Y), && (2.8)\\
\eta(X) &= \bar{g}(X, N).
\end{aligned}$$

But the induced connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms B and C are related to their shape operators by

$$\begin{aligned}
B(X, Y) &= g(A^*_{\xi} X, Y), \qquad & \bar{g}(A^*_{\xi} X, N) = 0, && (2.9)\\
C(X, PY) &= g(A_{N} X, PY), \qquad & \bar{g}(A_{N} X, N) = 0. && (2.10)
\end{aligned}$$

From (2.9), A^*_{ξ} is $S(TM)$-valued and self-adjoint on TM such that

$$A^*_{\xi}\xi = 0. \tag{2.11}$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the connections $\bar{\nabla}$, ∇ and ∇^* respectively. Using the Gauss-Weingarten formulas for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$ such that

$$\bar{g}(\bar{R}(X,Y)Z,\, PW) = g(R(X,Y)Z,\, PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \tag{2.12}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y), \tag{2.13}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, N) = \bar{g}(R(X,Y)Z,\, N), \tag{2.14}$$

$$\bar{g}(\bar{R}(X,Y)\xi,\, N) = g(A^*_\xi X, A_N Y) - g(A^*_\xi Y, A_N X) - 2d\tau(X,Y), \tag{2.15}$$

$$g(R(X,Y)PZ,\, PW) = g(R^*(X,Y)PZ,\, PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \tag{2.16}$$

$$\bar{g}(R(X,Y)PZ,\, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). \tag{2.17}$$

The *Ricci tensor* $\bar{Ric}$ of $\bar{M}$ is defined by

$$\bar{Ric}(X,Y) = trace\{Z \to \bar{R}(Z,X)Y\},$$

for any $X,\, Y \in \Gamma(T\bar{M})$. Let $\dim \bar{M} = m+2$. Locally, $\bar{Ric}$ is given by

$$\bar{Ric}(X,Y) = \sum_{i=1}^{m+2} \epsilon_i\, \bar{g}(\bar{R}(E_i,X)Y,\, E_i), \tag{2.18}$$

where $\{E_1,\, \dots,\, E_{m+2}\}$ is an orthonormal frame field of $T\bar{M}$.

3. Screen homothetic lightlike hypersurfaces

Now we consider an induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M, where $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$ and let $E = \{\xi, N, W_a\}$ be the corresponding frame field on $\bar{M}$. By using (2.18), we get

$$\bar{Ric}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, \bar{g}(\bar{R}(W_a,X)Y,\, W_a) + \bar{g}(\bar{R}(\xi,X)Y,\, N) + \bar{g}(\bar{R}(N,X)Y,\, \xi), \quad \forall\, X,\, Y \in \Gamma(T\bar{M}), \tag{3.1}$$

where $\epsilon_a (= \pm 1)$ denotes the causal character of respective vector field W_a. Let $R^{(0,\,2)}$ denote the induced Ricci type tensor of type (0, 2) on M given by

$$R^{(0,\,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}\,, \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.2}$$

Using the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M, we obtain

$$R^{(0,\,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a,X)Y,\, W_a) + \bar{g}(R(\xi,X)Y,\, N). \tag{3.3}$$

Substituting (2.12) and (2.14) in (3.1) an using (2.9) and (2.10), we obtain

$$R^{(0,\,2)}(X,Y) = \bar{Ric}(X,Y) + B(X,Y)trA_N - g(A_N X, A^*_\xi Y) - \bar{g}(R(\xi,Y)X,\, N), \quad \forall\, X,\, Y \in \Gamma(TM). \tag{3.4}$$

This shows that $R^{(0,2)}$ is not symmetric. The tensor field $R^{(0,2)}$ is called its *induced Ricci tensor* [4], and denote it by *Ric*, if it is symmetric. If $R^{(0,2)}$ is an induced Ricci tensor *Ric* of M and $Ric = \kappa g$, then M is called an *Einstein manifold*. In this case, if $m > 1$, then we show that κ is a constant.

Using (2.15), (3.4) and the first Bianchi's identity, we obtain

$$R^{(0,2)}(X, Y) - R^{(0,2)}(Y, X) = 2d\tau(X, Y), \quad \forall\, X, Y \in \Gamma(TM).$$

From this equation, we have the following theorem:

Theorem 3.1. [3, 4] *Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then the Ricci type tensor $R^{(0,2)}$ is symmetric if and only if the 1-form τ is closed, i.e., $d\tau = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$.*

Remark 1. In case $d\tau = 0$, by the cohomology theory there exist a smooth function l such that $\tau = dl$. Thus we get $\tau(X) = X(l)$. If we take $\tilde{\xi} = \gamma\xi$, then we have $\tau(X) = \tilde{\tau}(X) + X(\ln\gamma)$. Setting $\gamma = \exp(l)$ in this equation, we get $\tilde{\tau}(X) = 0$. We call the pair $\{\xi, N\}$ such that the corresponding 1-form τ vanishes the *canonical null pair* of M. Although $S(TM)$ is not unique and the lightlike geometry depends on its choice but it is canonically isomorphic to the factor vector bundle $S(TM)^\sharp = TM/Rad(TM)$ due to Kupeli [8]. Thus all $S(TM)$ are mutually isomorphic. In the sequel, we deal with only lightlike hypersurfaces M equipped with the canonical null pair $\{\xi, N\}$.

Let M be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. We may assume that the curvature vector field ζ of $\bar{M}$ is a spacelike unit tangent vector field of M. In this case, if ζ belongs to $Rad(TM)$, then $\zeta = e\xi$, where $e = \theta(N) \neq 0$. From this fact, we have $1 = \bar{g}(\zeta,\zeta) = e^2 g(\xi,\xi) = 0$. It is a contradiction. This enables one to choose a screen distribution $S(TM)$ which contains ζ due to (2.1). This implies that *if ζ is tangent to M, then it belongs to $S(TM)$* which we assume in this paper.

Definition 1. A lightlike hypersurface M of a semi-Riemannian manifold $\bar{M}$ is *screen conformal* [4, 5, 6] if the shape operators A_N and A^*_ξ are related by $A_N = \varphi A^*_\xi$, or equivalently, the second fundamental forms B and C satisfy

$$C(X, PY) = \varphi B(X, Y), \tag{3.5}$$

where φ is a non-vanishing smooth function on a coordinate neighborhood $\mathcal{U}$ in M. If φ is a non-zero constant, then we say that M is *screen homothetic*.

Example 1. Let $(R^7_2, \bar{g}_0)$ be a 7-dimensional semi-Euclidean space of index 2 with signature $(-,-,+,+,+,+,+)$ of the canonical basis

$$\{\partial x_1, \partial x_2, \cdots, \partial x_6, \partial x_7 = \zeta\}.$$

Consider a lightlike hypersurface M of R^7_2, defined by

$$X(u_1, u_2, u_3, u_4, u_5, t) = (u_1 + u_2 + u_3, u_1, u_2, u_3, u_4, u_5, t),$$

whose radical distribution $Rad(TM)$ is spanned by

$$\xi = \partial_1 - \partial_2 + \partial_3 + \partial_4.$$

We consider a complementary vector bundle F^* of $TM^\perp$ in $S(TM)^\perp$ and take $V^* = \partial_1 - \partial_2 \in \Gamma(F^*)$, $V^* \neq 0$, such that $\bar{g}_0(\xi, V^*) \neq 0$. Then the transversal vector bundle is given by $tr(TM) = Span\{N\}$, where

$$N = \frac{1}{\bar{g}_0(\xi, V^*)}\left\{V^* - \frac{\bar{g}_0(V^*, V^*)}{2\bar{g}_0(\xi, V^*)}\xi\right\} = -\frac{1}{4}(\partial_1 - \partial_2 - \partial_3 - \partial_4).$$

It follows that the corresponding screen distribution $S(TM)$ is spanned by

$$\{W_1 = \partial_1 + \partial_2, \quad W_2 = \partial_3 - \partial_4, \quad W_3 = \partial_5, \quad W_4 = \partial_6, \quad W_5 = \partial_7 = \zeta\}.$$

Taking the covariant derivative to N along R^7_2, we get

$$\bar{\nabla}_X N = \frac{1}{4}\bar{\nabla}_X \xi, \quad \text{since } \bar{\nabla}_X V^* = 0.$$

Using Gauss and Weingarten formulae, we obtain

$$-A_N X + \tau(X)N = -\frac{1}{4}(A^*_\xi X + \tau(X)\xi).$$

Taking the scaler product with ξ and N to this, we get $\tau(X) = 0$, which gives

$$A_N X = \frac{1}{4}A^*_\xi X, \quad \forall\, X \in \Gamma(TM).$$

Thus M is a screen homothetic lightlike hypersurface of conformal factor $\varphi = \frac{1}{4}$.

Theorem 3.2. *Let M be a screen conformal lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. If ζ is tangent to M, then the tensor field $R^{(0,\,2)}$ is an induced symmetric Ricci tensor of M.*

Proof. Replacing W by N to (1.1) and using the fact $\theta(N) = 0$, we have

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,\, N) &= \alpha\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \\ &+ \beta\{\theta(Y)\eta(X) - \theta(X)\eta(Y)\}\theta(Z). \end{aligned} \tag{3.6}$$

Replacing Z by ξ to (3.6) and using $\theta(\xi) = 0$, we have $\bar{g}(\bar{R}(X,Y)\xi,\, N) = 0$. Comparing this result with (2.15) and using the fact $A_N = \varphi A^*_\xi$, we show that $d\tau = 0$. Thus, by Theorem 3.1, we have our assertion. □

Theorem 3.3. *Let M be a screen homothetic lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$ of quasi-constant curvature. If ζ is tangent to M, then the functions α and β vanish identically. Thus $\bar{M}$ is a flat manifold.*

Proof. Using (1.1), (2.14) and (3.6), we have

$$\bar{Ric}(X,Y) = \{(m+1)\alpha + \beta\}g(X,Y) + m\beta\,\theta(X)\theta(Y), \tag{3.7}$$

$$\bar{g}(R(\xi,Y)X,\, N) = \alpha g(X,Y) + \beta\theta(X)\theta(Y). \tag{3.8}$$

Substituting the last two equations into (3.4), we have

$$\begin{aligned} R^{(0,2)}(X,Y) &= \{m\alpha+\beta\}g(X,Y)+\beta(m-1)\theta(X)\theta(Y) \\ &+ B(X,Y)trA_N - g(A_N X, A^*_\xi Y). \end{aligned} \tag{3.9}$$

As $d\tau = 0$, we can take a canonical null pair such that $\tau = 0$ due to Remark 1. Replacing W by ξ to (1.1) and using (2.13) and the fact $\theta(\xi) = 0$, we have

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = 0. \tag{3.10}$$

Assume that M is screen homothetic. Substituting (3.5) into (2.17) and using (3.10), we get $\bar{g}(R(X,Y)PZ,N) = 0$. From this result and the fact $\bar{g}(R(X,Y)\xi,N) = 0$, we show that, for all $Z \in \Gamma(TM)$,

$$\bar{g}(R(X,Y)Z,N) = 0.$$

Replacing X by ξ and Z by X to this and comparing with (3.8), we have

$$\beta\theta(X)\theta(Y) = -\alpha g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{3.11}$$

Taking $X = Y = \zeta$ to (3.11), we get $\beta = -\alpha$. Substituting (3.11) into (3.9) and using the fact $\beta = -\alpha$, we obtain

$$Ric(X,Y) = \varphi\{B(X,Y)trA^*_\xi - g(A^*_\xi X, A^*_\xi Y)\}. \tag{3.12}$$

Substituting (3.11) into (1.1) and using (2.12), (2.13) and (3.5), we have

$$\begin{aligned} R(X,Y)Z &= \alpha\{g(X,Z)Y - g(Y,Z)X\} \\ &+ \varphi\{B(Y,Z)A^*_\xi X - B(X,Z)A^*_\xi Y\}. \end{aligned} \tag{3.13}$$

Substituting (3.13) and $\bar{g}(R(\xi,Y)X, N) = 0$ into (3.3), we also have

$$Ric(X,Y) = -(m-1)\alpha\, g(X,Y) + \varphi\{B(X,Y)trA^*_\xi - g(A^*_\xi X, A^*_\xi Y)\}. \tag{3.14}$$

Comparing (3.12) and (3.14), we obtain $\alpha = 0$ as $m > 1$. As $\beta = -\alpha$, we also have $\beta = 0$. Thus $\bar{M}$ is a flat manifold. □

By the characterization theorem of Duggal-Jin [5] (Theorem 1.1 in this paper), we have the following result:

Theorem 3.4. *Let M be a screen homothetic Einstein lightlike hypersurface of a Lorentzian manifold $\bar{M}^{m+2}$, $m > 2$, of quasi-constant curvature such that $Ric = \kappa g$. If the curvature vector field ζ of $\bar{M}$ is tangent to M, then $\bar{M}$ is flat manifold and M is locally a product manifold $\mathcal{C} \times M_1 \times M_2$, where $\mathcal{C}$ is a null curve tangent to the radical distribution, and M_1 and M_2 are leaves of some integerable distributions of M such that*

1. *If $\kappa \neq 0$, then either M_1 or M_2 is an m-dimensional totally umbilical Einstein Riemannian space form which is isometric to a sphere or a hyperbolic space according to the sign of κ and the other is a point.*
2. *If $\kappa = 0$, then M_1 is an $(m-1)$ or an m-dimensional Euclidean space and M_2 is a non-null curve or a point.*

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