

## EVOLUTION EQUATIONS ON A RIEMANNIAN MANIFOLD WITH A LOWER RICCI CURVATURE BOUND

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ABSTRACT. We consider the parabolic evolution differential equations such as heat equation and porous-medium equation on a Riemannian manifold  $M$  whose Ricci curvature is bounded below by  $-(n-1)k^2$  and bounded below by 0 on some amount of  $M$ . We derive some bounds of differential quantities for a positive solution and some inequalities which resemble Harnack inequalities.

### 1. Introduction

Parabolic evolution equation on a Riemannian manifold is one of the greatest interests in geometric analysis. Against the case of  $\mathbb{R}^n$ , in the case of a Riemannian manifold, the curvatures come in during the analysis of the equations. For example, a positive solution  $u : \mathbb{R}^n \times (0, T] \rightarrow \mathbb{R}$  of the heat equation

$$\frac{\partial}{\partial t} u = \Delta u, \quad t > 0,$$

is known to be satisfied the following Harnack inequality.

$$u(x_2, t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{n}{2}} u(x_1, t_1) \exp\left(-\frac{\|x_2 - x_1\|^2}{4(t_2 - t_1)}\right), \quad (1)$$

where  $0 < t_1 < t_2$ , and  $x_i \in \mathbb{R}^n$ .

By the celebrated work of Li and Yau([7]), a nice extension of (1) was accomplished to the equations on a complete Riemannian manifold with Ricci curvature bounded below by 0.

Furthermore, there has been important researches on Harnack inequalities for various types of nonlinear parabolic equations on  $\mathbb{R}^n$  (e.g. [2], [3], [4]). For example, Benedetto showed the following intrinsic Harnack type inequality for the degenerate porous medium equation.([3])

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Received December 12, 2013; Accepted January 10, 2014.

2000 *Mathematics Subject Classification.* 53C25.

*Key words and phrases.* heat equation, porous medium equation, Harnack inequality, Ricci curvature.

The present research was conducted by the research fund of Dankook University in 2012.

**Theorem 1.1.** *For an open subset  $\Omega \subset \mathbb{R}^n$ , let  $u \geq 0$  be any local weak solution of the porous medium equation in  $\Omega \times (0, T]$ . Assume that  $u(\bar{x}, \bar{t}) > 0$  for some  $(\bar{x}, \bar{t}) \in \Omega \times (0, T]$ . Then there exist two constants  $C_0, C_1$  depending only upon  $m$  and  $n$  such that*

$$u(\bar{x}, \bar{t}) \leq C_0 \inf_{\bar{x} \in B_R(\bar{x})} u(\bar{x}, \bar{t} + \theta),$$

where

$$\theta = \frac{C_1 R^2}{[u(\bar{x}, \bar{t})]^{m-1}},$$

provided  $B_{2R}(\bar{x}) \times (\bar{t} - \theta, \bar{t} + \theta)$  is contained in  $\Omega \times (0, T]$ .

Also, Auchmuty and Bao studied pointwise bounds of solutions of parabolic equations which could be degenerate by using gradient estimates techniques in [7]([2]). Precisely, they showed that for positive solutions of the porous medium solution

$$\frac{\partial u}{\partial t} = \Delta(u^M), \quad t > 0, \quad x \in \mathbb{R}^n, \quad M > \max\left\{0, 1 - \frac{2}{n}\right\}, \quad M \neq 1,$$

the following estimate holds.

$$f(x_2, t_2) \geq \left(\frac{t_1}{t_2}\right)^\mu \left[ f(x_1, t_1) - \frac{\delta}{4} \frac{|x_2 - x_1|^2}{t_2^\delta - t_1^\delta} \frac{1}{t_1^\mu} \right],$$

where  $f = \left(\frac{M}{M-1}\right) u^{M-1}$ ,  $\mu = \frac{M-1}{M-1+\frac{2}{n}}$ ,  $\delta = 1 - \mu$ .

And these results were extended to the case of a compact Riemannian manifold with Ricci curvature bounded below by 0 as the following by Lee and the author in [5].

**Theorem 1.2.** *Let  $M$  be an  $n$  dimensional Riemannian manifold with non-negative Ricci curvature. Also, let  $u$  be a positive solution of*

$$u_t(x, t) = \Delta u^m(x, t), \quad m > 1, \quad 0 < t \leq T, \quad x \in M$$

and  $B_R(x_0)$  be a totally convex geodesic ball in  $M$ . Then there are two constants  $C_1$  and  $C_2 = C_2(m, n)$ , we have

$$\sup_{B_R(x_0)} u(\cdot, t) \leq C_2 \inf_{B_R(x_0)} u(\cdot, t + \theta),$$

where

$$\theta = \frac{C_1 R^2}{[\inf_{B \times (0, T]} u]^{m-1}}$$

provided  $0 < t - \theta < t + \theta < T$ . Moreover  $C_1$  and  $C_2$  are related by the following.

$$C_2 = 2^{\frac{2}{m-1}} \exp\left(\frac{(m-1)(n+4)}{2C_1 m^2}\right)$$

In this paper, we consider heat equation and porous medium equation on a compact Riemannian manifold  $M$  which possibly has negative Ricci curvature on a certain amount of  $M$ . We derive some inequalities for positive solutions of the equations. The derived inequalities in this article look like Harnack type inequalities, but we remark that the constants in the inequalities essentially depend on the solution.

We refer [6] and [1] for detail on the heat equation and porous medium equation respectively.

## 2. Notations and preliminaries

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with a Riemannian metric  $g$ , and denote  $g(\xi, \eta)$  and  $g(\xi, \xi)$  by  $\langle \xi, \eta \rangle_g$  and  $|\xi|_g^2$  respectively. We also use the notation  $\rho$  as the distance function induced from  $g$ , i.e., for  $x_1, x_2 \in M$ ,  $\rho(x_1, x_2)$  is the distance between  $x_1$  and  $x_2$ . Denote  $Rc_g$  by the Ricci tensor of  $(M, g)$ . We say that  $(M, g)$  has a Ricci curvature bounded below by  $-(n-1)k^2$  if  $Rc_g(\xi, \xi) \geq -(n-1)k^2 |\xi|_g^2$  for all tangent vectors  $\xi$ . In this article, we are mainly interested in an  $n$ -dimensional compact Riemannian manifold  $(M, g)$  with the Ricci curvature bounded below by  $-(n-1)k^2$ . Now let  $\nabla_g$  and  $\Delta_g$  be the gradient and the laplacian induced from the Riemannian metric  $g$ . The following is very useful well-known formula in geometric analysis.

**Theorem 2.1.** (Bochner formula)

*For a smooth function  $u : (M, g) \rightarrow \mathbb{R}$ , the following identity holds.*

$$\frac{1}{2} \Delta_g |\nabla u|_g^2 = \langle \nabla_g \Delta_g u, \nabla_g u \rangle_g + |\nabla_g \nabla_g u|_g^2 + Rc_g(\nabla_g u, \nabla_g u)$$

Next, consider a conformal metric  $\tilde{g} = \lambda^2 g$  of  $g$  for a positive constant  $\lambda$ . Then the followings are easily checked to hold.

$$\begin{aligned} \nabla_{\tilde{g}} &= \nabla_g \\ \Delta_{\tilde{g}} &= \lambda^{-2} \Delta_g \\ |\nabla_{\tilde{g}} u|_{\tilde{g}} &= \lambda^{-2} |\nabla_g u|_g \\ Rc_{\tilde{g}}(\nabla_{\tilde{g}} u, \nabla_{\tilde{g}} u) &= \lambda^{-2} Rc_g(\nabla_g u, \nabla_g u) \end{aligned} \tag{2}$$

Since  $\nabla_{\tilde{g}} = \nabla_g$ , throughout the paper we use the notation  $\nabla_{\tilde{g}} = \nabla_g = \nabla$  ambiguously.

Now let

$$L_g = \frac{\partial}{\partial t} - \Delta_g$$

be the heat operator on  $M$ , and  $u : M \times (0, T] \rightarrow \mathbb{R}^+$  be a positive solution of  $L_g(u) = 0$ . Consider a time scaled map  $\tilde{u}(x, t) = u(x, \lambda^{-2}t)$  of  $u$ , then we have the following from (2).

$$L_{\tilde{g}}(\tilde{u}) = \lambda^{-2} L_g(u) = 0 \tag{3}$$

Similarly for  $m > 1$ , let

$$L_g^{(m)} = \frac{\partial}{\partial t} - \Delta_g^{(m)},$$

where  $\Delta_g^{(m)}u = \Delta_g u^m$ , be the porous media operator on  $M$ , and  $u : M \times (0, T] \rightarrow \mathbb{R}^+$  be a positive solution of  $L_g^{(m)}(u) = 0$ . Consider a time scaled map  $\tilde{u}(x, t) = u(x, \lambda^{-2}t)$  of  $u$ , then we also have the following from (2).

$$L_{\tilde{g}}^{(m)}(\tilde{u}) = \lambda^{-2}L_g^{(m)}(u) = 0 \quad (4)$$

### 3. Heat equation

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with the Ricci curvature bounded from below by  $-(n-1)k^2$  for some positive constant  $k$ . For the heat operator  $L_g = \frac{\partial}{\partial t} - \Delta_g$  on  $M$ , let  $u : M \times (0, T] \rightarrow \mathbb{R}^+$  be a positive solution of  $L_g(u) = 0$ . Define  $N_\epsilon = \{(x, t) \in M \times [\epsilon, T] \mid \Delta_g u(x, t) = 0\}$ , and let  $\bar{N}_\epsilon$  be the projection  $N_\epsilon$  onto  $M$ . Suppose there is a compact subset  $W \subset M \setminus \bar{N}_\epsilon$  such that the Ricci curvature is non-negative on  $M \setminus W$ . By using these notations we can get the following theorem.

**Theorem 3.1.** *Suppose that the above conditions hold for a complete Riemannian manifold  $M$ , and let  $m_\epsilon = \min_{W \times [\epsilon, T]} (\Delta_g \ln u)^2$  and  $M_\epsilon = \max_{W \times [\epsilon, T]} |\nabla \ln u|_g^2$ . Then there exists positive  $\alpha = \alpha(\epsilon, n, m_\epsilon, M_\epsilon)$  such that*

$$\Delta_g \ln u \geq -\frac{\alpha}{t}$$

on  $M \times [\epsilon, T]$ .

*Proof.* Denote  $F = \ln u$  and  $P = \Delta_g F$ , and we will show that  $P \geq -\frac{\alpha}{t}$  on  $M \times [\epsilon, T]$ . For a positive constant  $\lambda$ , consider the conformal metric  $\tilde{g} = \lambda^2 g$  of  $g$ . Denote  $\tilde{F} = \ln \tilde{u}$  and  $\tilde{P} = \Delta_{\tilde{g}} \tilde{F}$  for a time scaled map  $\tilde{u}(x, t) = u(x, \lambda^{-2}t)$  of  $u$ . Then by (3),  $\tilde{u}$  is a positive solution of  $L_{\tilde{g}}(\tilde{u}) = 0$  on  $[\lambda^2\epsilon, \lambda^2T]$ . Hence from

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{F} &= \frac{\frac{\partial}{\partial t} \tilde{u}}{\tilde{u}} = \frac{\Delta_{\tilde{g}} \tilde{u}}{\tilde{u}} \\ \nabla \tilde{F} &= \frac{\nabla \tilde{u}}{\tilde{u}} \\ \Delta_{\tilde{g}} \tilde{F} &= \frac{\Delta_{\tilde{g}} \tilde{u}}{\tilde{u}} - \frac{|\nabla \tilde{u}|_{\tilde{g}}^2}{\tilde{u}^2} \end{aligned}$$

we have

$$\frac{\partial}{\partial t} \tilde{F} = \Delta_{\tilde{g}} \tilde{F} + |\nabla \tilde{F}|_{\tilde{g}}^2.$$

So

$$\begin{aligned}
\frac{\partial}{\partial t} \tilde{P} &= \Delta_{\tilde{g}} \left( \frac{\partial}{\partial t} \tilde{F} \right) = \Delta_{\tilde{g}} \left( \Delta_{\tilde{g}} \tilde{F} + |\nabla \tilde{F}|_{\tilde{g}}^2 \right) \\
&= \Delta_{\tilde{g}} \tilde{P} + 2 \langle \Delta_{\tilde{g}} \nabla \tilde{F}, \nabla \tilde{F} \rangle_{\tilde{g}} + 2 |\nabla \nabla \tilde{F}|_{\tilde{g}}^2 \\
&= \Delta_{\tilde{g}} \tilde{P} + 2 \langle \nabla \Delta_{\tilde{g}} \tilde{F}, \nabla \tilde{F} \rangle_{\tilde{g}} + 2 Rc_{\tilde{g}} \left( \nabla \tilde{F}, \nabla \tilde{F} \right) + 2 |\nabla \nabla \tilde{F}|_{\tilde{g}}^2 \\
&\geq \Delta_{\tilde{g}} \tilde{P} + 2 \langle \nabla \Delta_{\tilde{g}} \tilde{F}, \nabla \tilde{F} \rangle_{\tilde{g}} + 2 Rc_{\tilde{g}} \left( \nabla \tilde{F}, \nabla \tilde{F} \right) + \frac{2}{n} \tilde{P}^2
\end{aligned}$$

So by (2) we have

$$\lambda^{-2} \frac{\partial}{\partial t} P \geq \lambda^{-2} \Delta_g P + 2 \langle \nabla \Delta_g F, \nabla F \rangle_g + 2 \lambda^{-2} Rc_g (\nabla F, \nabla F) + \frac{2}{n} \lambda^{-4} P^2,$$

and

$$\frac{\partial}{\partial t} P \geq \Delta_g P + 2 \lambda^2 \langle \nabla \Delta_g F, \nabla F \rangle_g + 2 Rc_g (\nabla F, \nabla F) + \frac{2}{n \lambda^2} P^2.$$

Because  $Rc_g (\nabla F, \nabla F) \geq 0$  on  $M \setminus W$ , we have

$$\frac{\partial}{\partial t} P \geq \Delta_g P + 2 \lambda^2 \langle \nabla \Delta_g F, \nabla F \rangle_g + \frac{2}{n \lambda^2} P^2$$

on  $M \setminus W$ . With our notations

$$m_\epsilon = \min_{W \times [\epsilon, T]} P \text{ and } M_\epsilon = \max_{W \times [\epsilon, T]} |\nabla F|_g^2.$$

Since  $m_\epsilon > 0$ , if we take  $\lambda < \sqrt{\frac{m_\epsilon}{2n(n-1)k^2 M_\epsilon}}$  then we have

$$\begin{aligned}
\frac{\partial}{\partial t} P &\geq \Delta_g P + 2 \lambda^2 \langle \nabla \Delta_g F, \nabla F \rangle_g + 2 Rc_g (\nabla F, \nabla F) + \frac{2}{n \lambda^2} P^2 \\
&\geq \Delta_g P + 2 \lambda^2 \langle \nabla \Delta_g F, \nabla F \rangle_g - 2(n-1)k^2 |\nabla F|_g^2 + \frac{2}{n \lambda^2} P^2 \\
&\geq \Delta_g P + 2 \lambda^2 \langle \nabla \Delta_g F, \nabla F \rangle_g + \frac{1}{n \lambda^2} P^2
\end{aligned}$$

on  $W$ . So finally we have

$$\frac{\partial}{\partial t} P \geq \Delta_g P + 2 \lambda^2 \langle \nabla \Delta_g F, \nabla F \rangle_g + \frac{1}{n \lambda^2} P^2$$

on the whole of  $M$ . Consider that the solution of the ordinary initial value differential equation

$$\begin{cases} \frac{d}{dt} K(t) &= \frac{1}{n \lambda^2} (K(t))^2 \\ K(0) &= -\infty \end{cases}$$

is given by  $K(t) = -\frac{n \lambda^2}{t}$ . So we can have the following.

$$\frac{\partial}{\partial t} (P - K) \geq \Delta_g (P - K) + 2 \lambda^2 \langle \nabla (P - K), \nabla F \rangle_g + \frac{1}{n \lambda^2} (P^2 - K^2),$$

Hence by the usual maximum principle argument, we have

$$P \geq -\frac{n \lambda^2}{t}.$$

Taking  $\alpha = n\lambda^2$  completes the proof.  $\square$

We remark that in Theorem 3.1 if the equation  $L_g(u) = 0$  holds on  $[0, T]$  for a positive solution  $u$ , then the dependency on  $\epsilon$  of  $\alpha$  can be removed.

**Corollary 3.2.** *Under the same conditions as those in Theorem 3.1,*

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left(\frac{t_2}{t_1}\right)^\alpha e^{-\frac{\{\rho(x_1, x_2)\}^2}{4(t_2 - t_1)}}$$

holds for the same  $\alpha$  in Theorem 3.1.

*Proof.* Let  $\gamma : [t_1, t_2] \rightarrow M$  be a minimizing geodesic from  $x_1$  to  $x_2$  with

$$\left|\frac{d\gamma}{dt}\right| = \frac{\rho(x_1, x_2)}{t_2 - t_1}.$$

$$\begin{aligned} F(x_2, t_2) - F(x_1, t_1) &= \int_{t_1}^{t_2} \frac{d}{dt} F(\gamma(t), t) dt \\ &= \int_{t_1}^{t_2} \left( \frac{\partial}{\partial t} F(\gamma(t), t) + \left\langle \nabla F(\gamma(t), t), \frac{d\gamma}{dt}(t) \right\rangle \right) dt \\ &= \int_{t_1}^{t_2} \frac{\partial}{\partial t} \frac{u}{u} - \left( \left| \frac{\nabla u}{u} \right|^2 + \frac{1}{4} \left| \frac{d\gamma}{dt} \right|^2 \right) \\ &= \int_{t_1}^{t_2} \left( P - \frac{1}{4} \left| \frac{d\gamma}{dt} \right|^2 \right) dt \\ &\geq \int_{t_1}^{t_2} \left( -\frac{\alpha}{t} - \frac{\{\rho(x_1, x_2)\}^2}{4(t_2 - t_1)^2} \right) dt \\ &= \ln \left( \frac{t_1}{t_2} \right)^\alpha - \frac{\{\rho(x_1, x_2)\}^2}{4(t_2 - t_1)} \end{aligned}$$

By exponentiation the both sides, we completes the proof.  $\square$

#### 4. Porus medium equation

Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with the Ricci curvature bounded from below by  $-(n-1)k^2$  for some positive constant  $k$ . Let  $u : M \times (0, T] \rightarrow \mathbb{R}^+$  be a positive solution of the porus medium equation  $L_g^{(m)}(u) = \frac{\partial}{\partial t} u - \Delta_g u^m = 0$  ( $m > 1$ ). Define  $N_\epsilon^{(m)} = \{(x, t) \in M \times [\epsilon, T] \mid (m-2)|\nabla u|^2 + u\Delta u = 0\}$ , and  $\bar{N}_\epsilon^{(m)}$  as the projection  $N_\epsilon^{(m)}$  onto  $M$ . Suppose there is a compact subset  $W^{(m)} \subset M \setminus \bar{N}_\epsilon^{(m)}$  such that the Ricci curvature is non-negative on  $M \setminus W^{(m)}$ .

**Theorem 4.1.** *Suppose that the above conditions hold for a complete Riemannian manifold  $M$ . Let  $v = \frac{m}{m-1} u^{m-1}$ , and  $m_\epsilon^{(m)} = \min_{W \times [\epsilon, T]} (\Delta_g v)^2$ ,*

$M_\epsilon^{(m)} = \max_{W \times [\epsilon, T]} |\nabla v|_g^2$ . Then there exists positive  $\beta = \beta(\epsilon, n, m_\epsilon, M_\epsilon)$  such that

$$\Delta_g v \geq -\frac{\beta}{t}$$

on  $M \times [\epsilon, T]$ .

*Proof.* Denote  $Q = \Delta_g v$ , and we show that  $Q \geq -\frac{\beta}{t}$  on  $M \times [\epsilon, T]$ . For a positive constant  $\lambda$ , consider the conformal metric  $\tilde{g} = \lambda^2 g$  of  $g$ . Now let  $\tilde{u}(x, t) = u(x, \lambda^{-2}t)$ ,  $\tilde{v} = \frac{m}{m-1} \tilde{u}^{m-1}$ . Then we have

$$L_{\tilde{g}}^m(\tilde{u}) = \lambda^{-2} L_g^m(u) = 0.$$

First, we can have that the following equation comes from  $\tilde{u}$  is a solution of  $L_{\tilde{g}}^m(\tilde{u}) = 0$ .

$$\frac{\partial}{\partial t} \tilde{u} = \Delta_{\tilde{g}} \tilde{u}^m = m(m-1) \tilde{u}^{m-2} |\nabla \tilde{u}|_{\tilde{g}}^2 + m \tilde{u}^{m-1} \Delta_{\tilde{g}} \tilde{u} \quad (5)$$

Just by the definition of  $\tilde{v}$  and (5), we have the followings.

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{v} &= m \tilde{u}^{m-2} \tilde{u}_t = m^2 \tilde{u}^{2m-4} \left( (m-1) |\nabla \tilde{u}|_{\tilde{g}}^2 + \tilde{u} \Delta_{\tilde{g}} \tilde{u} \right) \\ \nabla \tilde{v} &= m \tilde{u}^{m-2} \nabla \tilde{u} \\ \Delta_{\tilde{g}} \tilde{v} &= m \tilde{u}^{m-3} \left( (m-2) |\nabla \tilde{u}|_{\tilde{g}}^2 + \tilde{u} \Delta_{\tilde{g}} \tilde{u} \right) \end{aligned}$$

From the above three equations, we can easily check that

$$\frac{\partial}{\partial t} \tilde{v} = (m-1) \tilde{v} \Delta_{\tilde{g}} \tilde{v} + |\nabla \tilde{v}|_{\tilde{g}}^2. \quad (6)$$

Now let  $\tilde{Q} = \Delta_{\tilde{g}} \tilde{v}$ , then

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{Q} &= \Delta_{\tilde{g}} \left( \frac{\partial}{\partial t} \tilde{v} \right) = (m-1) \Delta_{\tilde{g}} (\tilde{v} \Delta_{\tilde{g}} \tilde{v}) + \Delta_{\tilde{g}} |\nabla \tilde{v}|_{\tilde{g}}^2 \\ &= (m-1) \{ (\Delta_{\tilde{g}} \tilde{v})^2 + \tilde{v} \Delta_{\tilde{g}} (\Delta_{\tilde{g}} \tilde{v}) + 2 \langle \nabla \tilde{v}, \nabla \Delta_{\tilde{g}} \tilde{v} \rangle \} + \Delta_{\tilde{g}} |\nabla \tilde{v}|_{\tilde{g}}^2 \\ &= (m-1) \{ \tilde{Q}^2 + \tilde{v} \Delta_{\tilde{g}} \tilde{Q} + 2 \langle \nabla \tilde{v}, \nabla \tilde{Q} \rangle \} \\ &\quad + 2 \{ \langle \nabla \tilde{Q}, \nabla \tilde{v} \rangle + |\nabla \nabla \tilde{v}|_{\tilde{g}}^2 + Rc_{\tilde{g}}(\nabla \tilde{v}, \nabla \tilde{v}) \} \\ &\geq (m-1) \tilde{v} \Delta_{\tilde{g}} \tilde{Q} + 2m \langle \nabla \tilde{v}, \nabla_{\tilde{g}} \tilde{Q} \rangle + \left( m-1 + \frac{2}{n} \right) \tilde{Q}^2 + 2Rc_{\tilde{g}}(\nabla \tilde{v}, \nabla \tilde{v}). \end{aligned}$$

So we have

$$\begin{aligned} \lambda^{-2} \frac{\partial}{\partial t} Q &\geq (m-1) \lambda^{-2} v \Delta_g Q + 2m \langle \nabla v, \nabla_g Q \rangle + \left( m-1 + \frac{2}{n} \right) \lambda^{-4} Q^2 \\ &\quad + 2\lambda^{-2} Rc_g(\nabla v, \nabla v), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} Q &\geq (m-1)v\Delta_g Q + 2m\lambda^2 \langle \nabla v, \nabla_g Q \rangle + \left(m-1 + \frac{2}{n}\right) \lambda^{-2} Q^2 \\ &\quad + 2Rc_g(\nabla v, \nabla v). \end{aligned}$$

Because  $Rc_g(\nabla F, \nabla F) \geq 0$  on  $M \setminus W^{(m)}$ , we have

$$\frac{\partial}{\partial t} Q \geq (m-1)v\Delta_g Q + 2m\lambda^2 \langle \nabla v, \nabla_g Q \rangle + \left(m-1 + \frac{2}{n}\right) \lambda^{-2} Q^2$$

on  $M \setminus W^{(m)}$ . With our notations

$$m_\epsilon^{(m)} = \min_{W^{(m)} \times [\epsilon, T]} Q \text{ and } M_\epsilon^{(m)} = \max_{W^{(m)} \times [\epsilon, T]} |\nabla v|_g^2.$$

Since  $m_\epsilon^{(m)} > 0$ , if we take  $\lambda < \sqrt{\frac{(m-1+\frac{2}{n})m_\epsilon^{(m)}}{2n(n-1)k^2 M_\epsilon^{(m)}}}$  then we have

$$\frac{\partial}{\partial t} Q \geq (m-1)v\Delta_g Q + 2m\lambda^2 \langle \nabla v, \nabla_g Q \rangle + \frac{1}{2} \left(m-1 + \frac{2}{n}\right) \lambda^{-2} Q^2$$

on  $W^{(m)}$ . So finally we have

$$\frac{\partial}{\partial t} Q \geq (m-1)v\Delta_g Q + 2m\lambda^2 \langle \nabla v, \nabla_g Q \rangle + \frac{1}{2} \left(m-1 + \frac{2}{n}\right) \lambda^{-2} Q^2$$

on the whole of  $M$ . By the similar maximum principle argument as in the proof of Theorem 3.1, we have

$$Q \geq -\frac{2\lambda^2}{\left(m-1 + \frac{2}{n}\right)t}.$$

Taking  $\beta = \frac{2\lambda^2}{m-1+\frac{2}{n}}$  completes the proof.  $\square$

Let  $\nu = (m-1)\beta$ , then from (6) and Theorem 4.1 we have

$$\frac{\partial}{\partial t} v + \frac{\nu}{t} v - |\nabla v|_g^2 \geq 0,$$

by using this, we can get the following inequality for  $V(x, t) = t^\nu v(x, t)$ .



$$\begin{aligned}
V(x_2, t_2) - V(x_1, t_1) &= \int_{t_1}^{t_2} \frac{d}{dt} V(\gamma(t), t) dt \\
&= \int_{t_1}^{t_2} \left[ \frac{\partial}{\partial t} V(\gamma(t), t) + \left\langle \nabla V(\gamma(t), t), \frac{\partial}{\partial t} \gamma(t) \right\rangle_g \right] dt \\
&= \int_{t_1}^{t_2} \left[ \nu t^{\nu-1} v + t^\nu \frac{\partial}{\partial t} v + \left\langle t^\nu \nabla v, \frac{\partial}{\partial t} \gamma \right\rangle_g \right] dt \\
&\geq \int_{t_1}^{t_2} \left[ \nu t^{\nu-1} v + t^\nu \frac{\partial}{\partial t} v - t^\nu \left( |\nabla v|_g^2 + \frac{1}{4} \left| \frac{\partial}{\partial t} \gamma \right|_g^2 \right) \right] dt \\
&\geq -\frac{1}{4} \int_{t_1}^{t_2} t^\nu \left| \frac{\partial}{\partial t} \gamma \right|_g^2 dt,
\end{aligned}$$

In this formula, we can estimate  $V(x_2, t_2) - V(x_1, t_1)$  by obtaining the possible minimum value of  $\int_{t_1}^{t_2} t^\nu |\dot{\gamma}|^2 dt$  among the paths  $\{\gamma | \gamma(t_1) = x_1, \gamma(t_2) = x_2\}$ . And the curve can be found in the following lemma whose proof is contained in [5].

**Lemma 4.2.** [5] *For  $t_1 < t_2$ , let  $\gamma$  be a smooth curve in  $M$  such that  $\gamma(t_1) = x_1, \gamma(t_2) = x_2$ . Then*

$$\int_{t_1}^{t_2} t^\nu \left| \frac{\partial}{\partial t} \gamma \right|_g^2 dt \geq \frac{1-\nu}{t_2^{1-\nu} - t_1^{1-\nu}} \{\rho(x_1, x_2)\}^2.$$

Furthermore if  $\gamma(t) = \alpha \left( \frac{1}{1-\nu} t^{1-\nu} \right)$  for a shortest geodesic  $\alpha$  from  $x_1$  to  $x_2$ , then the equality holds.

We can regard  $\gamma(t) = \alpha \left( \frac{1}{1-\mu} t^{1-\mu} \right)$  as an optimal path in the sense that the integral  $\int_{t_1}^{t_2} t^\mu |\dot{\gamma}|_g^2 dt$  is minimal, and this fact plays an important role in this article. Note that  $\gamma(t)$  has the same root of a geodesic. From the above observations and Theorem 4.1, we have the following theorem.

**Theorem 4.3.** *Suppose all of the assumptions in Theorem 4.1 hold for a complete Riemannian manifold  $M$ . Then for  $x_1, x_2 \in M$  and  $\epsilon < t_1 < t_2 < T$ ,*

$$v(x_2, t_2) \geq \left( \frac{t_1}{t_2} \right)^\nu \left[ v(x_1, t_1) - \frac{\delta}{4t_1^\nu} \frac{\rho(x_1, x_2)^2}{t_2^\delta - t_1^\delta} \right],$$

where  $v = \frac{m}{m-1} u^{m-1}$ ,  $\nu = (m-1)\beta$ ,  $\delta = 1-\nu$ .

*Proof.* Let  $V(x, t) = t^\nu v(x, t)$ . Then by the argument after Theorem 4.1 and by Lemma 4.2, we have

$$V(x_2, t_2) - V(x_1, t_1) = t_2^\nu v(x_2, t_2) - t_1^\nu v(x_1, t_1) \geq -\frac{1}{4} \frac{1-\nu}{t_2^{1-\nu} - t_1^{1-\nu}} \{\rho(x_1, x_2)\}^2,$$

and it completes the proof.  $\square$

Now we consider another gradient estimates for a different gradient quantity.

**Theorem 4.4.** [5] *Let  $u$  is a solution of the porous medium equation  $L_g^{(m)}(u) = 0$ . Define  $H = \ln u$ , and  $R = u^{m-1}\Delta_g H + m(u^{m-1} - 1)|\nabla H|_g^2$ . Then*

$$\begin{aligned} \frac{\partial}{\partial t} H &= mR + m^2|\nabla H|_g^2, \\ \frac{\partial}{\partial t} R &= mu^{m-1}\Delta_g R + 2m^2u^{m-1}\langle \nabla R, \nabla H \rangle_g \\ &\quad + m(m-1)R^2 - m^3(m-1)|\nabla H|_g^4 \\ &\quad + 2m^2u^{m-1}|\nabla \nabla H|_g^2 + 2m^2u^{m-1}Rc_g(\nabla H, \nabla H). \end{aligned}$$

In Theorem 4.4, the term  $-m^3(m-1)|\nabla H|_g^4$  is an obstacle to get a proper lower bound for  $\frac{\partial}{\partial t} H$ . In order to avoid this problem, we consider following another dilation-scaled map of  $u$ . For a positive constant  $\omega$ , let  $s = \omega^{-1}t$  and define

$$\tilde{u}(x, s) = \omega^{\frac{1}{m-1}} \tilde{u}(x, \omega s) = \omega^{\frac{1}{m-1}} \tilde{u}(x, t),$$

where  $\tilde{u}(x, t) = u(x, \lambda^{-2}t)$ . Then

$$\begin{aligned} \tilde{u}_s &= \omega^m u_t(x, t), \\ \Delta_g \tilde{u}^m &= \omega^m \Delta_g u(x, t)^m. \end{aligned}$$

It can be readily checked that  $\tilde{u}_s = \Delta_g \tilde{u}^m$ , hence  $\tilde{u}$  is also a solution of the porous medium equation. By using this argument we can have the following theorem.

**Theorem 4.5.** *Suppose all of the assumptions in Theorem 4.1 are satisfied for a complete manifold  $M$ , and let  $B_a(x_0)$  be a totally convex geodesic ball in  $M$ . Also let  $m_\epsilon^{(m)} = \min_{W \times [\epsilon, T]} R^2$ ,  $M_\epsilon^{(m)} = \max_{W \times [\epsilon, T]} |\nabla H|_g^2$ , where  $H$  and  $R$  are the same as in Theorem 4.4. Then there are two constants  $C_1 = C_1(m, n)$  and  $C_2 = C_2(m, n)$  such that*

$$\sup_{B_a(x_0)} u(\cdot, t) \leq C_2 \inf_{B_a(x_0)} u(\cdot, t + \theta),$$

where

$$\theta = \frac{C_1 \omega a^2}{\lambda^2}$$

provided  $0 < t - \theta < t + \theta < T$ . Moreover  $C_1$  and  $C_2$  are related by the following.

$$C_2 = 2^{-\frac{4}{m-1}} \exp\left(-\frac{m-1}{4C_1 m^2}\right).$$

*Proof.* For  $H = \ln u$ , and  $R = u^{m-1}\Delta_g H + m(u^{m-1}-1)|\nabla H|_g^2$ , by Theorem 4.4,

$$\begin{aligned}\frac{\partial}{\partial t} H &= mR + m^2|\nabla H|_g^2, \\ \frac{\partial}{\partial t} R &= mu^{m-1}\Delta_g R + 2m^2u^{m-1}\langle \nabla R, \nabla H \rangle_g \\ &\quad + m(m-1)R^2 - m^3(m-1)|\nabla H|_g^4 \\ &\quad + 2m^2u^{m-1}|\nabla \nabla H|_g^2 + 2m^2u^{m-1}Rc_g(\nabla H, \nabla H).\end{aligned}$$

By taking  $\omega = \frac{2(n+4)}{(\inf u)^{m-1}}$ , where  $\inf u = \inf_{B_a \times [\epsilon, T]} u(x, t)$ , we have an extra condition,  $\tilde{u}^{m-1} > 2(n+4)$ . Since  $\tilde{u}$  is also a solution of the porous medium equation, we can get

$$\begin{aligned}& \frac{1}{2}m(m-1)\tilde{R}^2 - m^3(m-1)|\nabla \tilde{H}|_g^4 + 2m^2\tilde{u}^{m-1}|\nabla \nabla \tilde{H}|_g^2 \\ & \geq \frac{1}{2}m(m-1)\left((\tilde{u}^{m-1}\Delta_{\tilde{g}}\tilde{H})^2 + 2m\tilde{u}^{m-1}(\tilde{u}^{m-1}-1)(\Delta_{\tilde{g}}\tilde{H})|\nabla \tilde{H}|_g^2\right. \\ & \quad \left.+ m^2(\tilde{u}^{m-1}-1)^2|\nabla \tilde{H}|_g^4\right) - m^3(m-1)|\nabla \tilde{H}|_g^4 + \frac{2}{n}m^2\tilde{u}^{m-1}(\Delta \tilde{H})^2 \\ & \geq 0.\end{aligned}$$

So, we have

$$\begin{aligned}\frac{\partial}{\partial s} \tilde{R} & \geq m\tilde{u}\Delta_{\tilde{g}}\tilde{R} + 2m^2\tilde{u}\langle \nabla \tilde{R}, \nabla \tilde{H} \rangle_{\tilde{g}} + \frac{1}{2}m(m-1)\tilde{R}^2 \\ & \quad + 2m^2\tilde{u}^{m-1}Rc_{\tilde{g}}(\nabla \tilde{H}, \nabla \tilde{H})\end{aligned}$$

Since  $Rc_g(\nabla L, \nabla L) \geq 0$  on  $B_a(x_0) \cap (M \setminus W^{(m)})$ , we have

$$\frac{\partial}{\partial s} \tilde{R} \geq m\tilde{u}\Delta_{\tilde{g}}\tilde{R} + 2m^2\tilde{u}\langle \nabla \tilde{R}, \nabla \tilde{L} \rangle + \frac{1}{2}m(m-1)\tilde{R}^2,$$

on  $B_a(x_0) \cap (M \setminus W^{(m)})$ . On the other hand, on  $B_a(x_0) \cap W^{(m)}$ , we have

$$\begin{aligned}\frac{\partial}{\partial s} \tilde{R} & \geq m\tilde{u}\Delta_{\tilde{g}}\tilde{R} + 2m^2\tilde{u}\langle \nabla \tilde{R}, \nabla \tilde{H} \rangle_{\tilde{g}} + \frac{1}{2}m(m-1)\tilde{R}^2 \\ & \quad - 2m^2\tilde{u}^{m-1}(n-1)k^2\lambda^2|\nabla \tilde{H}|_g^2.\end{aligned}$$

If we take

$$\lambda < \sqrt{\frac{(m-1)m_\epsilon^{(m)}}{8m(n-1)\omega(u^\epsilon)^{m-1}k^2M_\epsilon^{(m)}}},$$

the inequality

$$\frac{\partial}{\partial s} \tilde{R} \geq m\tilde{u}\Delta_{\tilde{g}}\tilde{R} + 2m^2\tilde{u}\langle \nabla \tilde{R}, \nabla \tilde{H} \rangle_{\tilde{g}} + \frac{1}{4}m(m-1)\tilde{R}^2$$

holds on  $B_a(x_0) \cap W^{(m)}$ , hence on the whole of  $B_a(x_0)$ . The maximum principle gives the following inequality.

$$\tilde{R} \geq -\frac{4}{m(m-1)s}.$$

So for all  $x_1, x_2 \in B_a(x_0)$ , let  $\gamma$  be a curve with  $\gamma(s_1) = x_1$  and  $\gamma(s_2) = x_2$ . Then we have

$$\begin{aligned} \tilde{H}(x_2, s_2) - \tilde{H}(x_1, s_1) &= \int_{s_1}^{s_2} \frac{d}{ds} \left[ \tilde{H}(\gamma(s), s) \right] ds \\ &= \int_{s_1}^{s_2} \frac{\partial}{\partial s} \tilde{H}(\gamma(s), s) + \left\langle \nabla \tilde{H}(\gamma(s), s), \frac{d\gamma}{ds} \right\rangle_{\tilde{g}} ds \\ &\geq \int_{s_1}^{s_2} \left( m\tilde{R} + m^2 \left| \nabla \tilde{H} \right|_{\tilde{g}}^2 \right) (\gamma(s), s) \\ &\quad - \left| \nabla \tilde{H}(\gamma(s), s) \right| \left| \frac{d\gamma}{ds} \right|_{\tilde{g}} ds \\ &\geq \int_{s_1}^{s_2} -\frac{4}{(m-1)s} - \frac{1}{4m^2} \left| \frac{d\gamma}{ds} \right|_{\tilde{g}}^2 ds \\ &= -\frac{4}{m-1} \ln \left( \frac{s_2}{s_1} \right) - \frac{1}{4m^2} \frac{\rho(x_1, x_2)^2}{(s_2 - s_1)}. \end{aligned}$$

Thus we have

$$\frac{\tilde{u}(x_2, s_2)}{\tilde{u}(x_1, s_1)} \geq \left( \frac{s_2}{s_1} \right)^{-\frac{4}{m-1}} \exp \left( -\frac{m-1}{4m^2} \frac{\rho(x_1, x_2)^2}{s_2 - s_1} \right)$$

on  $B_a(x_0) \times \left[ \frac{\lambda^2 \epsilon}{\omega}, \frac{\lambda^2 \epsilon}{T} \right]$ . Hence we have

$$\frac{\tilde{u}(x_2, t_2)}{\tilde{u}(x_1, t_1)} \geq \left( \frac{t_2}{t_1} \right)^{-\frac{4}{m-1}} \exp \left( -\frac{(m-1)\omega\rho(x_1, x_2)^2}{4m^2(t_2 - t_1)} \right)$$

on  $B_a(x_0) \times [\lambda^2 \epsilon, \lambda^2 T]$  and

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq \left( \frac{t_2}{t_1} \right)^{-\frac{4}{m-1}} \exp \left( -\frac{(m-1)\omega\rho(x_1, x_2)^2}{4m^2\lambda^2(t_2 - t_1)} \right)$$

on  $B_a(x_0) \times [\epsilon, T]$ .

Since  $t_2 = t_1 + \theta$  and  $t_1 - \theta > 0$ , then  $\frac{t_2}{t_1} < 2$ , hence we have

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq 2^{-\frac{4}{m-1}} \exp \left( -\frac{(m-1)\omega\rho(x_1, x_2)^2}{4m^2\lambda^2\theta} \right),$$

and since  $\theta = \frac{C_1\omega a^2}{\lambda^2}$ , we have

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \geq 2^{-\frac{4}{m-1}} \exp \left( -\frac{m-1}{4C_1m^2} \right).$$

Thus we have

$$u(x_1, t_1) \leq C_2 u(x_2, t_2),$$

where  $C_2 = 2^{-\frac{4}{m-1}} \exp\left(-\frac{m-1}{4C_1 m^2}\right)$ .  $\square$

If the manifold  $M$  is compact, then we can have the following corollary.

**Corollary 4.6.** *Suppose all of the assumptions in Theorem 4.5 are satisfied for a compact manifold  $M$ . Then there are two constants  $C_1 = C_1(m, n)$  and  $C_2 = C_2(m, n)$ , satisfying*

$$\sup_M u(\cdot, t) \leq C_2 \inf_M u(\cdot, t + \theta),$$

where

$$\theta = \frac{C_1 \omega \{diam(M)\}^2}{\lambda^2}$$

provided  $0 < t - \theta < t + \theta < T$ . Moreover  $C_1$  and  $C_2$  are related by the following.

$$C_2 = 2^{-\frac{4}{m-1}} \exp\left(-\frac{m-1}{4C_1 m^2}\right).$$

*Proof.* The proof is straightforward from Theorem 4.5.  $\square$

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