Some Optimal Convex Combination Bounds for Arithmetic Mean

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Abstract. In this paper we derive some optimal convex combination bounds related to arithmetic mean. We find the greatest values $\alpha_1$ and $\alpha_2$ and the least values $\beta_1$ and $\beta_2$ such that the double inequalities

\[ \alpha_1 T(a, b) + (1 - \alpha_1) H(a, b) < A(a, b) < \beta_1 T(a, b) + (1 - \beta_1) H(a, b) \]
\[ \alpha_2 T(a, b) + (1 - \alpha_2) G(a, b) < A(a, b) < \beta_2 T(a, b) + (1 - \beta_2) G(a, b) \]

holds for all $a, b > 0$ with $a \neq b$. Here $T(a, b)$, $H(a, b)$, $A(a, b)$ and $G(a, b)$ denote the second Seiffert, harmonic, arithmetic and geometric means of two positive numbers $a$ and $b$, respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the first and second Seiffert means $P(a, b)$ and $T(a, b)$ was introduced by Seiffert [1,2] as follows:

\[ P(a, b) = \frac{a - b}{4 \arctan(\sqrt{a/b}) - \pi} = \frac{a - b}{2 \arcsin \frac{a - b}{a + b}}, \quad T(a, b) = \frac{a - b}{2 \arctan \frac{a - b}{a + b}}. \]

Recently, both means $P$ and $T$ have been the subject of intensive research. In particular, many remarkable inequalities for $P$ and $T$ can be found in the literature [2-6].

Let $A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$ and $H(a, b) = 2ab/(a + b)$ be the arithmetic, geometric and harmonic means of two positive real numbers $a$ and $b$ with $a \neq b$. Then

\[ \min\{a, b\} < H(a, b) < G(a, b) < P(a, b) < A(a, b) < T(a, b) < \max\{a, b\}. \]

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In [7], Seiffert proved
\[ P(a, b) > \frac{3A(a, b)G(a, b)}{A(a, b) + 2G(a, b)} \quad \text{and} \quad P(a, b) > \frac{2}{\pi} A(a, b), \]
for all \( a, b > 0 \) with \( a \neq b \).

In [8], the authors found the greatest value \( \alpha \) and the least value \( \beta \) such that the double inequality
\[ \alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b) \]
holds for all \( a, b > 0 \) with \( a \neq b \).

For other useful inequalities, see [9-20].

The purpose of the present paper is to find the greatest values \( \alpha_1 \) and \( \alpha_2 \) and the least values \( \beta_1 \) and \( \beta_2 \) such that the double inequalities
\[ \alpha_1 T(a, b) + (1 - \alpha_1)H(a, b) < A(a, b) < \beta_1 T(a, b) + (1 - \beta_1)H(a, b) \]
and
\[ \alpha_2 T(a, b) + (1 - \alpha_2)G(a, b) < A(a, b) < \beta_2 T(a, b) + (1 - \beta_2)G(a, b) \]
holds for all \( a, b > 0 \) with \( a \neq b \).

2. Main Results

The first result in this paper is an optimal convex combination bounds of the second Seiffert and harmonic means for arithmetic mean.

**Theorem 2.1.** The double inequality
\[ \alpha_1 T(a, b) + (1 - \alpha_1)H(a, b) < A(a, b) < \beta_1 T(a, b) + (1 - \beta_1)H(a, b) \]
holds for all \( a, b > 0 \) if and only if \( \alpha_1 \leq \frac{3}{4} \) and \( \beta_1 \geq \frac{5}{4} \).

**Proof.** Firstly, we prove that
\[ A(a, b) < \frac{2}{4} T(a, b) + \left(1 - \frac{\pi}{4}\right) H(a, b), \]
(2.1)
\[ A(a, b) > \frac{3}{4} T(a, b) + \frac{1}{4} H(a, b), \]
(2.2)
for all \( a, b > 0 \) with \( a \neq b \).

Without loss of generality, we assume \( a > b \). Let \( t = a/b > 1 \) and \( p \in \left\{ \frac{3}{4}, \frac{\pi}{4}\right\} \).

Then (1.1) leads to
\[ \frac{1}{b} \left\{ A(a, b) - [pT(a, b) + (1 - p)H(a, b)] \right\} = A(t, 1) - [pT(t, 1) + (1 - p)H(t, 1)] = t^2 + 2(2p - 1)t + 1 \frac{1}{2(t + 1)} \arctan \frac{t}{t + 1} f(t), \]
(2.3)
where

\[(2.4) \quad f(t) = \arctan \frac{t - 1}{t + 1} - \frac{p(t^2 - 1)}{t^2 + 2(2p - 1)t + 1}.\]

Simple computations lead to

\[(2.5) \quad \lim_{t \to 1^+} f(t) = 0, \quad \lim_{t \to +\infty} f(t) = \frac{\pi}{4} - p.\]

\[(2.6) \quad f'(t) = \frac{(-4p^2 + 2p + 1)t^4 + 4(p - 1)t^3 + 2(4p^2 - 6p + 3)t^2 + 4(p - 1)t + (-4p^2 + 2p + 1)}{(1 + t^2)(t^2 + 2(2p - 1)t + 1)^2},\]

where

\[(2.7) \quad g(t) = (-4p^2 + 2p + 1)t^2 - 2(4p^2 - 4p + 1)t + (-4p^2 + 2p + 1).\]

Now we distinguish between two cases:

**case 1** \( p = \frac{3}{4} \). In this case,

\[(2.8) \quad g(t) = \frac{1}{4}(t^2 - 2t + 1) = \frac{1}{4}(t - 1)^2 > 0, \text{ for } t > 1.\]

Therefore, inequality (2.2) follows from (2.3)-(2.7). Notice that in this case, the second equality in (2.5) becomes

\[\lim_{t \to +\infty} f(t) = \frac{\pi}{4} - \frac{3}{4} > 0.\]

**case 2** \( p = \frac{7}{4} \). From (2.7) we have

\[(2.9) \quad \lim_{t \to 1^+} g(t) = 4p(3 - 4p) = \pi(3 - \pi) < 0, \quad \lim_{t \to +\infty} g(t) = +\infty,\]

\[(2.10) \quad g'(t) = 2(-4p^2 + 2p + 1)t - 2(4p^2 - 4p + 1),\]

\[(2.11) \quad \lim_{t \to 1^+} g'(t) = \pi(3 - \pi) < 0, \quad \lim_{t \to +\infty} g'(t) = +\infty,\]

\[(2.12) \quad g''(t) = 2(-4p^2 + 2p + 1) = \frac{1}{2}(-\pi^2 + 2\pi + 4) > 0,\]
From (2.12) we clearly see that \( g'(t) \) is increasing for \( t > 1 \), which together with (2.11) implies that there exists \( \lambda_1 > 1 \) such that \( g'(t) < 0 \) for \( t \in (1, \lambda_1) \) and \( g'(t) > 0 \) for \( t \in (\lambda_1, +\infty) \). Hence \( g(t) \) is strictly decreasing for \( t \in (1, \lambda_1) \) and strictly increasing for \( t \in (\lambda_1, +\infty) \). (2.9) implies that there exists \( \lambda_2 > 1 \) such that \( g(t) < 0 \) for \( t \in (1, \lambda_2) \) and \( g(t) > 0 \) for \( t \in (\lambda_2, +\infty) \). This result together with (2.6) implies that \( f(t) \) is strictly decreasing for \( t \in (1, \lambda_2) \) and strictly increasing for \( t \in (\lambda_2, +\infty) \). Notice that if \( p = \pi/4 \), then the second equality in (2.5) becomes

\[
\lim_{t \to +\infty} f(t) = 0.
\]

Thus \( f(t) < 0 \) for all \( t > 1 \). Inequality (2.1) follows.

Secondly, we prove that \( \frac{3}{4}T(a, b) + \frac{1}{4}H(a, b) \) is the best possible lower convex combination bound of the second Seiffert and harmonic means for arithmetic mean.

If \( \alpha_1 > \frac{3}{4} \), then (2.7) (with \( \alpha_1 \) in place of \( p \)) leads to

\[
\lim_{t \to 1^+} g(t) = 4\alpha_1(3 - 4\alpha_1) < 0.
\]

From this result and the continuity of \( g(t) \) we clearly see that there exists \( \delta = \delta(\alpha_1) > 0 \) such that \( g(t) < 0 \) for \( t \in (1, 1 + \delta) \). Then (2.6) implies \( f'(t) < 0 \) for \( t \in (1, 1 + \delta) \). Thus \( f(t) \) is decreasing for \( t \in (1, 1 + \delta) \). Since (2.5), then \( f(t) < 0 \) for \( t \in (1, 1 + \delta) \), which is equivalent to, by (2.3), that

\[
A(t, 1) < \alpha_1 T(t, 1) + (1 - \alpha_1) H(t, 1),
\]

for \( t \in (1, 1 + \delta) \).

Finally, we prove that \( \frac{3}{4}T(a, b) + (1 - \frac{3}{4})H(a, b) \) is the best possible upper convex combination bound of the second Seiffert and harmonic means for arithmetic mean.

If \( \beta_1 < \frac{3}{4} \), then from (1.1) one has

\[
\lim_{t \to +\infty} \beta_1 T(t, 1) + (1 - \beta_1) H(t, 1) = \lim_{t \to +\infty} \frac{\beta_1(t^2 - 1) + 4(1 - \beta_1)t \arctan \frac{t - 1}{t + 1}}{(t + 1)^2 \arctan \frac{t - 1}{t + 1}} = \frac{4\beta_1}{\pi} < 1.
\]

Inequality (2.13) implies that for any \( \beta_1 < \frac{3}{4} \) there exists \( X = X(\beta_1) > 1 \) such that

\[
\beta_1 T(t, 1) + (1 - \beta_1) H(t, 1) < A(t, 1)
\]

for \( t \in (X, +\infty) \).

The second result in this paper is an optimal convex combination bounds of the second Seiffert and geometric means for arithmetic Mean.

**Theorem 2.2.** The double inequality \( \alpha_2 T(a, b) + (1 - \alpha_2) G(a, b) < A(a, b) < \beta_2 T(a, b) + (1 - \beta_2) G(a, b) \) holds for all \( a, b > 0 \) with \( a \neq b \) if and only if \( \alpha_2 \leq \frac{3}{5} \) and \( \beta_2 \geq \frac{3}{4} \).
Proof. Firstly, we prove that

(2.14) \[ A(a, b) < \frac{\pi}{4} T(a, b) + \left(1 - \frac{\pi}{4}\right) G(a, b) \]

(2.15) \[ A(a, b) > \frac{3}{5} T(a, b) + \frac{2}{5} G(a, b) \]

for all \(a, b > 0\) with \(a \neq b\). Without loss of generality, we assume \(a > b\). Let \(t = \sqrt{a/b} > 1\) and \(p \in \left\{\frac{3}{2}, \frac{\pi}{4}\right\}\). Then (1.1) leads to

(2.16) \[ \frac{1}{b} \left\{A(a, b) - [pT(a, b) + (1 - p)G(a, b)]\right\} = \frac{A(t^2, 1) - [pT(t^2, 1) + (1 - p)G(t^2, 1)]}{t^2 + 2(p - 1)t + 1} f(t), \]

where

(2.17) \[ f(t) = \arctan \frac{t^2 - 1}{t^2 + 1} - \frac{p(t^2 - 1)}{t^2 + 2(p - 1)t + 1}. \]

Simple computations lead to

(2.18) \[ \lim_{t \to 1^+} f(t) = 0, \quad \lim_{t \to +\infty} f(t) = \frac{\pi}{4} - p, \]

(2.19) \[ f'(t) = \frac{h(t)}{(1 + t^4)(t^2 + 2(p - 1)t + 1)^2} = \frac{(t - 1)^2 g(t)}{(1 + t^4)(t^2 + 2(p - 1)t + 1)^2}, \]

where

\[ h(t) = 2p(-p + 1)t^6 + 2(-2p^2 + 1)t^5 + 2(-p^2 + 5p - 4)t^4 + 4(2p^2 - 4p + 3)t^3 \\
+ 2(-p^2 + 5p - 4)t^2 + 2(-2p + 1)t + 2p(-p + 1) \]

and

(2.20) \[ g(t) = 2p(-p + 1)t^4 + 2(-2p^2 + 1)t^3 + 4(-2p^2 + 2p - 1)t^2 + 2(-2p^2 + 1)t + 2p(-p + 1). \]

It is easy to see that

(2.21) \[ \lim_{t \to 1^+} g(t) = 4p(3 - 5p), \quad \lim_{t \to +\infty} g(t) = +\infty, \]

(2.22) \[ g'(t) = 8p(1 - p)t^3 + 6(-2p^2 + 1)t^2 + 8(-2p^2 + 2p - 1)t + 2(-2p^2 + 1), \]
\((2.23)\) \[ \lim_{t \to 1^+} g'(t) = 8p(3 - 5p), \quad \lim_{t \to +\infty} g(t) = +\infty, \]

\((2.24)\) \[ g''(t) = 24p(1 - p)t^2 + 12(-2p^2 + 1)t + 8(-2p^2 + 2p - 1), \]

\((2.25)\) \[ \lim_{t \to 1^+} g''(t) = 4(-16p^2 + 10p + 1), \quad \lim_{t \to +\infty} g''(t) = +\infty, \]

\((2.26)\) \[ g'''(t) = 48p(1 - p)t + 12(-2p^2 + 1). \]

Now we distinguish between two cases.

**case 1** \( p = \frac{3}{5} \). It follows from \((2.21), (2.23), (2.25)\) and \((2.26)\) that

\((2.27)\) \[ \lim_{t \to 1^+} g(t) = 0, \quad \lim_{t \to +\infty} g(t) = +\infty, \]

\((2.28)\) \[ \lim_{t \to 1^+} g'(t) = 0, \quad \lim_{t \to +\infty} g'(t) = +\infty, \]

\((2.29)\) \[ \lim_{t \to 1^+} g''(t) = \frac{124}{25} > 0, \quad \lim_{t \to +\infty} g''(t) = +\infty, \]

\((2.30)\) \[ g'''(t) = \frac{12}{25}(24t + 7) > 0, \]

From \((2.30)\) we clearly see that \( g'''(t) \) is strictly increasing for \( t > 1 \), which together with \((2.29)\) implies that \( g''(t) > 0 \) for all \( t > 1 \). Thus \( g'(t) \) is strictly increasing for \( t > 1 \). From \((2.28)\) we get \( g'(t) > 0 \) for all \( t > 1 \). Therefore \( g(t) \) is strictly increasing for \( t > 1 \). \((2.27)\) implies that \( g(t) > 0 \) for all \( t > 1 \). Thus from \((2.19)\) we clearly see that \( f''(t) > 0 \) for \( t > 1 \), from which one has \( f'(t) \) is strictly increasing for \( t > 1 \). Notice that the second equality in \((2.18)\) becomes

\[ \lim_{t \to +\infty} f(t) = \frac{\pi}{4} - \frac{3}{5} > 0. \]

Hence \( f(t) > 0 \) and \((2.15)\) follows from \((2.18)\) and \((2.16)\).

**case 2** \( p = \frac{5}{4} \). From \((2.21), (2.23), (2.25)\) and \((2.26)\) we have

\((2.31)\) \[ \lim_{t \to 1^+} g(t) = \pi(3 - 5\pi/4) < 0, \quad \lim_{t \to +\infty} g(t) = +\infty, \]
(2.32) \[ \lim_{t \to 1^+} g'(t) = 2\pi(3 - 5\pi/4) < 0, \quad \lim_{t \to +\infty} g'(t) = +\infty, \]

(2.33) \[ \lim_{t \to 1^+} g''(t) = 4(-\pi^2 + \frac{5\pi}{2} + 1) < 0, \quad \lim_{t \to +\infty} g''(t) = +\infty, \]

(2.34) \[ \lim_{t \to 1^+} g'''(t) = \frac{9\pi^2}{8} + \frac{7\pi}{4} + 3 > 0, \quad \lim_{t \to +\infty} g'''(t) = +\infty, \]

Since

(2.35) \[ g^{(4)}(t) = 3\pi(4 - \pi) > 0, \]

then we clearly see that \( g''''(t) \) is strictly increasing for \( t > 1 \), which together with (2.34) implies that \( g''''(t) > 0 \) for \( t > 1 \). Thus \( g''''(t) \) is strictly increasing for \( t > 1 \). From (2.33), we derive that there exists \( \lambda_3 > 1 \) such that \( g''''(t) < 0 \) for \( t \in (1, \lambda_3) \) and \( g''''(t) > 0 \) for \( t \in (\lambda_3, +\infty) \). Hence \( g'(t) \) is strictly decreasing for \( t \in (1, \lambda_3) \) and strictly increasing for \( t \in (\lambda_3, +\infty) \). From (2.32), there exists \( \lambda_4 > 1 \) such that \( g'(t) < 0 \) for \( t \in (1, \lambda_4) \) and \( g'(t) > 0 \) for \( t \in (\lambda_4, +\infty) \). Thus \( g(t) \) is strictly decreasing for \( t \in (1, \lambda_4) \) and strictly increasing for \( t \in (\lambda_4, +\infty) \). (2.31) implies that there exists \( \lambda_5 > 1 \) such that \( g(t) < 0 \) for \( t \in (1, \lambda_5) \) and \( g(t) > 0 \) for \( t \in (\lambda_5, +\infty) \). (2.19) implies that \( f(t) \) is strictly decreasing for \( t \in (1, \lambda_5) \) and strictly increasing for \( t \in (\lambda_5, +\infty) \). Notice that in this case, the second equality in (2.18) becomes

\[ \lim_{t \to +\infty} f(t) = 0. \]

Thus \( f(t) < 0 \) for all \( t > 1 \), and (2.14) follows.

Secondly, we prove that \( \frac{4}{7}T(a, b) + \frac{2}{7}G(a, b) \) is the best possible lower convex combination bound of the second Seiffert and geometric means for arithmetic mean.

If \( \alpha_2 > \frac{3}{7} \), then (2.21) (with \( \alpha_2 \) in place of \( p \)) leads to

(2.36) \[ \lim_{t \to 1^+} g(t) = 4\alpha_2(3 - 5\alpha_2) < 0. \]

From (2.36) and the continuity of \( g(t) \) we see that there exists \( \delta = \delta(\alpha_2) > 0 \) such that

(2.37) \[ g(t) < 0 \]

for \( t \in (1, 1 + \delta) \). Then (2.19) and the first equality of (2.18) imply that

(2.38) \[ f(t) < 0 \]

for \( t \in (1, 1 + \delta) \). Therefore, by (2.16), \( A(t^2, 1) < \alpha_2 T(t^2, 1) + (1 - \alpha_2) G(t^2, 1) \) for \( t \in (1, 1 + \delta) \).
Finally, we prove that \( \frac{2}{4} T(a, b) + (1 - \frac{2}{4}) G(a, b) \) is the best possible upper convex combination bound of the second Seiffert and geometric means for arithmetic mean. If \( \beta_2 < \frac{2}{4} \), then from (1.1) one has

\[
\lim_{t \to +\infty} \frac{\beta_2 T(t, 1) + (1 - \beta_2) G(t, 1)}{A(t, 1)} = \lim_{t \to +\infty} \frac{\beta_2 (t - 1) + 2(1 - \beta_2) \sqrt{t \arctan \frac{t-1}{t+1}}}{(t + 1) \arctan \frac{t-1}{t+1}} = \frac{4\beta_2}{\pi} < 1.
\]

Inequality (2.39) implies that for any \( \beta_2 < \frac{2}{4} \) there exists \( X = X(\beta_2) > 1 \) such that

\[
\beta_2 T(t, 1) + (1 - \beta_2) G(t, 1) < A(t, 1)
\]

for \( t \in (X, +\infty) \).

References

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