The $*$-Nagata Ring of almost Pr"{u}fer $*$-multiplication Domains

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Abstract. Let $D$ be an integral domain with quotient field $K$, $\mathcal{D}$ denote the integral closure of $D$ in $K$ and $*$ be a star-operation on $D$. In this paper, we study the $*$-Nagata ring of AP$*$MDs. More precisely, we show that $D$ is an AP$*$MD and $D[X] \subseteq \mathcal{D}[X]$ is a root extension if and only if the $*$-Nagata ring $D[X]_{\times}$ is an AB-domain, if and only if $D[X]_{\times}$ is an AP-domain. We also prove that $D$ is a P$*$MD if and only if $D$ is an integrally closed AP$*$MD, if and only if $D$ is a root closed AP$*$MD.

1. Introduction

For the sake of clarity, we first review some definitions and notation. Let $D$ be an integral domain with quotient field $K$ and $\mathcal{F}(D)$ be the set of nonzero fractional ideals of $D$. A star-operation on $D$ is a mapping $I \mapsto I_{\ast}$ from $\mathcal{F}(D)$ into itself which satisfies the following three conditions for all $0 \neq a \in K$ and all $I, J \in \mathcal{F}(D)$:

1. $(aD)_{\ast} = aD$ and $(aI)_{\ast} = aI_{\ast}$;
2. $I \subseteq I_{\ast}$, and if $I \subseteq J$, then $I_{\ast} \subseteq J_{\ast}$; and
3. $(I_{\ast})_{\ast} = I_{\ast}$.

An $I \in \mathcal{F}(D)$ is said to be a $*$-ideal if $I = I_{\ast}$. A $*$-ideal of $D$ is called a maximal $*$-ideal of $D$ if it is maximal among proper integral $*$-ideals of $D$. Given any star-operation $*$ on $D$, we can construct a new star-operation $*_{f}$ as follows: For all $I \in \mathcal{F}(D)$, the $*_{f}$-operation is defined by $I_{\ast_{f}} = \bigcup\{J_{\ast} \mid J$ is a nonzero finitely generated fractional subideal of $I\}$. A star-operation $*$ on $D$ is said to be of finite character (or finite type) if $I_{\ast} = I_{\ast_{f}}$ for each $I \in \mathcal{F}(D)$. It is easy to see that the $*_{f}$-operation is of finite character. Let $*'$ be a finite character star-operation on $D$. It is well known that if $D$ is not a field, then each proper integral $*'$-ideal of $D$ is

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contained in a maximal $*f$-ideal of $D$, and hence a maximal $*f$-ideal of $D$ always exists. An $I \in \mathbf{F}(D)$ is said to be $*f$-invertible if $(IH^{-1})_{*f} = D$, or equivalently, $IH^{-1} \nsubseteq M$ for any maximal $*f$-ideal $M$ of $D$. If $*_1$ and $*_2$ are star-operations on $D$, then we mean by $*_1 \leq *_2$ that $I_{*_1} \subseteq I_{*_2}$ for all $I \in \mathbf{F}(D)$. Clearly, if $*_1$ and $*_2$ are star-operations of finite character with $*_1 \leq *_2$, then a $*_1$-invertible ideal is $*_2$-invertible.

The simplest example of a star-operation is the $d$-operation. Other well-known examples are the $v$- and $t$-operations. The $d$-operation is just the identity map on $\mathbf{F}(D)$, i.e., $I_d = I$ for all $I \in \mathbf{F}(D)$. The $v$-operation is defined by $I_v = (I^{-1})^{-1}$, where $I^{-1} := \{a \in K \mid aI \subseteq D\}$, and the $t$-operation is defined by $I_t = \bigcup \{J_v \mid J$ is a nonzero finitely generated fractional subideal of $I\}$, i.e., $t = v_f$. Clearly, if an $I \in \mathbf{F}(D)$ is finitely generated, then $I_v = I_t$. It is also well known that $d \leq v \leq t$ for all star-operations $\ast$. For more on star-operations, the readers can refer to [8, Section 32].

Let $T(D)$ be the abelian group of $t$-invertible fractional $t$-ideals of an integral domain $D$ under the $t$-multiplication $I \ast J = (IJ)_t$, and let $\text{Prin}(D)$ be the subgroup of $T(D)$ of principal fractional ideals of $D$. Then the $t$-class group of $D$ is the quotient group $\text{Cl}_t(D) := T(D)/\text{Prin}(D)$. Let $\text{Inv}(D)$ be the group of invertible fractional ideals of $D$. Clearly, $\text{Inv}(D)$ is a subgroup of $T(D)$ containing $\text{Prin}(D)$. The Picard group is the group $\text{Pic}(D) := \text{Inv}(D)/\text{Prin}(D)$, and $\text{Pic}(D)$ is obviously a subgroup of $\text{Cl}_t(D)$.

Let $D$ be an integral domain with quotient field $K$, and $\overline{D}$ be the integral closure of $D$ in $K$. In [1, Definition 4.1], Anderson and Zafrullah first introduced the notions of almost Prüfer domains and almost Bézout domains. They defined $D$ to be an almost Prüfer domain (AP-domain) (respectively, almost Bézout domain (AB-domain)) if for any $0 \neq a, b \in D$, there exists a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is invertible (respectively, principal). It was shown that $D$ is an AP-domain with torsion $(t)$-class group if and only if $D$ is an AB-domain [1, Lemma 4.4]; and $D$ is an AP-domain (respectively, AB-domain) if and only if $D$ is a Prüfer domain (respectively, Prüfer domain with torsion Picard group) and $D \subseteq \overline{D}$ is a root extension [1, Corollary 4.8]. In [1, Definition 5.1], the authors also defined $D$ to be an almost valuation domain (AV-domain) if for any $0 \neq a, b \in D$, there exists an integer $n = n(a, b) \geq 1$ such that $a^n | b^n$ or $b^n | a^n$. Later, Li gave the notion of almost Prüfer $v$-multiplication domains which is the $t$-operation analogue of AP-domains. She defined $D$ to be an almost Prüfer $v$-multiplication domain (APeMD) if for any $0 \neq a, b \in D$, there exists a positive integer $n = n(a, b)$ such that $(a^n, b^n)$ is $t$-invertible. It was shown in [1, Theorem 5.8] (respectively, [14, Theorem 2.3]) that $D$ is an AP-domain (respectively, APeMD) if and only if $D_M$ is an AV-domain for all maximal ideals (respectively, maximal $t$-ideals) $M$ of $D$. Following [13, Definition 2.1], $D$ is an almost Prüfer $*f$-multiplication domain (AP+$*$MD) if for each $0 \neq a, b \in D$, there exists an integer $n = n(a, b) \geq 1$ such that $(a^n, b^n)$ is $*f$-invertible, where $*f$ is a star-operation on $D$. It was shown in [13, Theorem 2.4] that $D$ is an AP+$*$MD if and only if $D_M$ is an AV-domain for all maximal $*f$-ideals $M$ of $D$. Also, it is clear that if $*_1$ and $*_2$ are star-operations
with \( *_1 \leq *_2 \), then an AP\( *_1 \)MD is an AP\( *_2 \)MD; so for any star-operation \(*\), an AP-domain is an AP\(*\)MD, and an AP\(*\)MD is an AP\( \# \)MD.

In this paper, we study the \#-Nagata ring of AP\(*\)MDs, where \(*\) is a star-operation. More precisely, we show that \( D \) is an AP\(*\)MD and \( D[X] \subseteq \overline{D}[X] \) is a root extension if and only if the \#-Nagata ring \( D[X]_{N_v} \) is an AB-domain, if and only if \( D[X]_{N_v} \) is an AP-domain. We also prove that \( D \) is a P\(*\)MD if and only if \( D \) is an integrally closed AP\(*\)MD, if and only if \( D \) is a root closed AP\(*\)MD. (Preliminaries related to P\(*\)MDs will be reviewed before Lemma 5.) As a corollary, we recover a well-known fact that \( D \) is a P\(*\)MD if and only if \( D[X]_{N_v} \) is a Bézout domain, if and only if \( D[X]_{N_v} \) is a Prüfer domain.

2. Main Results

Throughout this section, \( D \) always denotes an integral domain with quotient field \( K \), \( \overline{D} \) is the integral closure of \( D \) in \( K \) and \( D[X] \) means the polynomial ring over \( D \). For a polynomial \( g \in D[X] \), \( c(g) \) stands for the content ideal of \( D \), i.e., the ideal of \( D \) generated by the coefficients of \( g \). Let \(*\) be a star-operation on \( D \) and set \( N_v := \{ g \in D[X] \mid c(g)_* = D \} \). If we need to make the integral domain \( D \) explicit, then we use \( N_v(D) \) instead of \( N_v \). Clearly, \( N_v = N_v \). Also, note that \( N_v = D[X] \setminus \bigcup M \), where \( M \) runs over all maximal \(*_f\)-ideals of \( D \) [11, Proposition 2.1(1)]: so \( N_v \) is a saturated multiplicative subset of \( D[X] \). We call the quotient ring \( D[X]_{N_v} \) the \#-Nagata ring of \( D \). Recently, the authors in [3] studied the \( t \)-Nagata ring of AP\( \# \)MDs. In fact, they showed that \( D \) is an AP\( \# \)MD and \( D[X] \subseteq \overline{D}[X] \) is a root extension if and only if \( D[X]_{N_v} \) is an AP-domain, if and only if \( D[X]_{N_v} \) is an AB-domain [3, Theorem 2.5]. (Recall that an extension \( R \subseteq T \) of integral domains is a root extension if for each \( z \in T \), \( z^n \in R \) for some integer \( n = n(z) \geq 1 \)).

In order to study the \#-Nagata ring of AP\(*\)MDs, we need the following lemma.

Lemma 1. The following assertions hold.

(1) If \( D \) is an AV-domain and \( F \) is a subfield of \( K \), then \( D \cap F \) is an AV-domain.

(2) Let \(*\) be a star-operation on \( D \). Then \( D \) is an AP\(*\)MD if and only if \( D_M \) is an AV-domain for all maximal \(*_f\)-ideals \( M \) of \( D \).

Proof. (1) Let \( 0 \neq x \in F \). Then \( x = \frac{x}{a} \) for some \( 0 \neq a, b \in D \). Since \( D \) is an AV-domain, we can find a suitable integer \( n = n(a, b) \geq 1 \) such that \( a^n \mid b^n \) or \( b^n \mid a^n \); so \( x^n \in D \) or \( x^{-n} \in D \). Hence \( x^n \in D \cap F \) or \( x^{-n} \in D \cap F \). Thus \( D \cap F \) is an AV-domain.

(2) This appears in [13, Theorem 2.4].

Recall that \( D \) is root closed if for \( a \in K \), \( a^n \in D \) for some positive integer \( n \) implies that \( a \in D \).
Lemma 2. Let $S$ be a (not necessarily saturated) multiplicative subset of $D$. Then the following assertions hold.

1. If $D \subseteq \mathcal{D}$ is a root extension, then $D_S \subseteq \mathcal{D}_S$ is a root extension.

2. If $D$ is root closed, then $D_S$ is root closed.

Proof. (1) Let $\xi \in \mathcal{D}_S$, where $e \in \mathcal{D}$ and $s \in S$. Since $D \subseteq \mathcal{D}$ is a root extension, $e^n \in D$ for some integer $n = n(e) \geq 1$; so $(\xi^n) \in D_S$. Thus $D_S \subseteq \mathcal{D}_S$ is a root extension.

(2) Let $a \in K$ such that $a^n \in D_S$ for some integer $n \geq 1$. Then $sa^n \in D$ for some $s \in S$; so $(sa)^n \in D$. Since $D$ is root closed, $sa \in D$, and hence $a \in D_S$. Thus $D_S$ is root closed. \qed

Now, we give the main result in this article.

Theorem 3. Let $*$ be a star-operation on $D$ and let $N_* := \{g \in D[X] \mid c(g)_* = D\}$. Then the following statements are equivalent.

1. $D$ is an AP+MD and $D[X] \subseteq \mathcal{D}[X]$ is a root extension.

2. $D[X]_{N_*}$ is an AB-domain.

3. $D[X]_{N_*}$ is an AP-domain.

Proof. (1) $\Rightarrow$ (2) Assume that $D$ is an AP+MD, and let $Q$ be a maximal ideal of $D[X]_{N_*}$. Then $Q = MD[X]_{N_*}$ for some maximal $*_f$-ideal $M$ of $D$ [11, Proposition 2.1(2)]. Note that $D_M$ is an AV-domain by Lemma 1(2); so $D_M$ is an APvMD and $MD_M$ is a maximal $t$-ideal of $D_M$ [1, Proof of Theorem 5.6]. Also, note that $D_M[X] = \mathcal{D}_M[X]$ (cf. [7, Theorem 12.10(2)]); so by Lemma 2(1), $D_M[X] \subseteq \mathcal{D}_M[X]$ is a root extension, because $D[X] \subseteq \mathcal{D}[X]$ is a root extension. Therefore $D_M[X]$ is an APvMD [14, Theorem 3.13]. Since $MD_M[X]$ is a maximal $t$-ideal of $D_M[X]$ [5, Lemma 2.1(4)], $D_M[X]_{MD_M[X]}$ is an AV-domain by Lemma 1(2). Note that $(D[X]_{N_*})_Q = (D[X]_{N_*})_{MD[X]_{N_*}} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$ [2, Lemma 2]; so $(D[X]_{N_*})_Q$ is an AV-domain. Hence $D[X]_{N_*}$ is an AP-domain by Lemma 1(2). Note that $\text{Pic}(D[X]_{N_*}) = 0$ [11, Theorem 2.14]. Thus $D[X]_{N_*}$ is an AB-domain [1, Lemma 4.4].

(2) $\Rightarrow$ (3) This implication is obvious.

(3) $\Rightarrow$ (1) Let $M$ be a maximal $*_f$-ideal of $D$. Then $MD[X]_{N_*}$ is a maximal ideal of $D[X]_{N_*}$ [11, Proposition 2.1(2)]. Since $D[X]_{N_*}$ is an AP-domain, $(D[X]_{N_*})_{MD[X]_{N_*}}$ is an AV-domain by Lemma 1(2). Note that $(D[X]_{N_*})_{MD[X]_{N_*}} = D[X]_{MD[X]} = D_M[X]_{MD_M[X]}$ [2, Lemma 2]; so $D_M[X]_{MD_M[X]}$ is an AV-domain. Since $D_M = D_M[X]_{MD_M[X]} \cap K$ [11, Proposition 2.8(1)], $D_M$ is an AV-domain by Lemma 1(1). Thus Lemma 1(2), $D$ is an AP+MD.

Let $N_v := \{f \in D[X] \mid c(f)_v = D\}$. Then $N_* \subseteq N_v$; so $D[X]_{N_v} = (D[X]_{N_*})_{N_v}$. Since $D[X]_{N_*}$ is an AP-domain, $D[X]_{N_*}$ is an APvMD; so $D[X]_{N_v}$ is also an APvMD [3, Lemma 2.4]. Thus $D[X] \subseteq \mathcal{D}[X]$ is a root extension [3, Theorem 2.5]. \qed
By applying \( * = d \) to Theorem 3, we obtain

**Corollary 4.** The following assertions are equivalent.

(1) \( D \) is an AP-domain and \( D[X] \subseteq \overline{D}[X] \) is a root extension.

(2) \( D[X]_{N_d} \) is an AB-domain.

(3) \( D[X]_{N_d} \) is an AP-domain.

Let \( * \) be a star-operation on \( D \). Recall that \( D \) is a Prüfer \(*\)-multiplication domain (P\(*\)MD) if every nonzero finitely generated ideal of \( D \) is \( *_f \)-invertible, or equivalently, \( D_M \) is a valuation domain for all maximal \( *_f \)-ideals \( M \) of \( D \) [10, Theorem 1.1]. When \( * = d \) or \( t \), it was shown in [1, Theorem 4.7] (respectively, [14, Theorem 2.4]) that \( D \) is a Prüfer domain (respectively, PrMD) if and only if \( D \) is an integrally closed AP-domain (respectively, APMD), if and only if \( D \) is a root closed AP-domain (respectively, AP\(d\)MD). We next extend these results to P\(*\)MDs for any star-operation \( * \).

**Lemma 5.** Let \( * \) be a star-operation on \( D \). Then the following assertions are equivalent.

(1) \( D \) is a P\(*\)MD.

(2) \( D \) is an integrally closed AP\(*\)MD.

(3) \( D \) is a root closed AP\(*\)MD.

**Proof.** (1) \( \Rightarrow \) (2) Clearly, a P\(*\)MD is an AP\(*\)MD. Thus this implication follows directly from a well-known fact that a P\(*\)MD is integrally closed [10, Theorem 1.1].

(2) \( \Rightarrow \) (3) It suffices to note that an integrally closed domain is always root closed.

(3) \( \Rightarrow \) (1) Assume that \( D \) is a root closed AP\(*\)MD, and let \( M \) be a maximal \( *_f \)-ideal of \( D \). Then \( D_M \) is an AV-domain by Lemma 1(2). Let \( a \) and \( b \) be nonzero elements of \( D_M \). Then there exists a positive integer \( n = n(a,b) \) such that \( a^n \mid b^n \) or \( b^n \mid a^n \). Hence \( (\frac{a}{b})^n \in D_M \) or \( (\frac{b}{a})^n \in D_M \). Note that \( D_M \) is root closed by Lemma 2(2); so \( \frac{a}{b} \in D_M \) or \( \frac{b}{a} \in D_M \), which indicates that \( D_M \) is a valuation domain. Thus \( D \) is a P\(*\)MD [10, Theorem 1.1].

Recall that \( D \) is a Bézout domain if every finitely generated ideal of \( D \) is principal. It is well known that \( D \) is a Bézout domain if and only if \( D \) is a Prüfer domain with trivial Picard group.

**Corollary 6.** ([6, Theorem 3.1]) Let \( * \) be a star-operation on \( D \). Then the following statements are equivalent.

(1) \( D \) is a P\(*\)MD.

(2) \( D[X]_{N_*} \) is a Bézout domain.
(3) $D[X]_{N_*}$ is a Prüfer domain.

Proof. Note that by suitable combinations of [12, Theorems 51 and 52], [7, Corollary 12.11(2)] and [11, Proposition 2.8(1)], it is easy to see that $D$ is integrally closed if and only if $D[X]_{N_*}$ is integrally closed.

$(1) \Rightarrow (2)$ If $D$ is a P*MD, then by Lemma 5, $D$ is an integrally closed AP*MD; so $D[X]_{N_*}$ is an integrally closed AP-domain by Theorem 3. Hence $D[X]_{N_*}$ is a Prüfer domain [1, Theorem 4.7] (or Lemma 5). Note that Pic($D[X]_{N_*}$) = 0 [11, Theorem 2.14]. Thus $D[X]_{N_*}$ is an integrally closed AP-domain by Theorem 3. Hence $D[X]_{N_*}$ is a Prüfer domain [1, Theorem 4.7] (or Lemma 5). Note that Pic($D[X]_{N_*}$) = 0 [11, Theorem 2.14]. Thus $D[X]_{N_*}$ is an integrally closed AP-domain by Theorem 3. Hence $D[X]_{N_*}$ is an integrally closed AP*MD by Theorem 3. Thus the result follows from Lemma 5. □

A particular case of Corollary 6 is when $*=d$ or $t$.

Corollary 7. ([2, Theorem 4] (respectively, [11, Theorem 3.7])) The following assertions are equivalent.

(1) $D$ is a Prüfer domain (respectively, Pr*MD).

(2) $D[X]_{N_d}$ (respectively, $D[X]_{N_v}$) is a Bézout domain.

(3) $D[X]_{N_d}$ (respectively, $D[X]_{N_v}$) is a Prüfer domain.

Let $*$ be a star-operation on $D$. Note that the $*$-Nagata ring $D[X]_{N_*}$ is a quotient ring of the polynomial ring $D[X]$. We end this article by mentioning a remark for the polynomial extensions of AP*MDs.

Remark 8. (1) Let $*$ be a star-operation on $D[X]$. Then the mapping $\tau : F(D) \to F(D)$ defined by $I* = (I D[X])_* \cap D$ for all $I \in F(D)$ is a star-operation on $D$ [15, Proposition 2.1]. It is well known that if $*$ denotes the $d$-operation (respectively, $t$-operation, $v$-operation) on $D[X]$, then $\tau$ is the $d$-operation (respectively, $t$-operation, $v$-operation) on $D$ [15, Remark 2.2] (or [9, Proposition 4.3]).

(2) If $D$ is an AP$v$MD and $D[X] \subseteq \overline{D}[X]$ is a root extension, then $D[X]$ is also an AP$v$MD [14, Theorem 3.13]. (Note that the condition “$D[X] \subseteq \overline{D}[X]$ is a root extension” is essential [14, Remark 3.12(3)].)

(3) Let $*$ and $\tau$ be star-operations as in (1). By (2), it might be natural to ask whether AP$v$MD properties of the base ring can be ascended to AP*MD properties of the polynomial extension (under some assumptions if needed), i.e., if $D$ is an AP$v$MD with some additional conditions, then $D[X]$ is an AP*MD. However the answer is not generally affirmative. For example, the polynomial ring over an AP-domain is not generally an AP-domain. In fact, $D[X]$ is an AP-domain if and only if $D$ is a field (cf. [4, Theorem 2.15]).

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