Abstract. Let $R$ be a commutative Noetherian (not necessarily local) ring, $I$ an ideal of $R$ and $M$ a finitely generated $R$-module. In this paper, by computing the local cohomology modules and Ext-modules via the injective resolution of $M$, we proved that, if for an integer $t > 0$, $$\dim_R H^i_I(M) \leq k$$ for $\forall i < t$, then $$\bigcup_{i=0}^{j} \text{Ass}_R H^i_I(M) \geq \bigcup_{i=0}^{j} \text{Ass}_R \text{Ext}_R^i(R/I^n, M) \geq k$$ for $\forall j \leq t$ and $\forall n > 0$. This shows that $\bigcup_{n>0} \text{Ass}_R \text{Ext}_R^i(R/I^n, M) \geq k$ is a finite set for $\forall i \leq t$. Also, we prove that $$\bigcup_{i=1}^{r} \text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M \geq k$$ if $x_1, x_2, \ldots, x_r$ is $M$-sequences in dimension $> k$ and $n_1, n_2, \ldots, n_r$ are some positive integers. Here, for a subset $T$ of $\text{Spec}(R)$, set $T_{\geq i} = \{ p \in T \mid \dim R/p \geq i \}$.

1. Introduction

Throughout this paper, let $R$ be a commutative Noetherian ring and $I$ an ideal of $R$. Let $M$ be a finitely generated $R$-module. For convenience, we use the following notations: for a subset $T$ of $\text{Spec}(R)$, we set $$T_{\geq i} := \{ p \in T \mid \dim R/p \geq i \},$$ $$T_{> i} := \{ p \in T \mid \dim R/p > i \},$$ $$T_i := \{ p \in T \mid \dim R/p = i \}.$$

It is well-known that the sequences of associated primes $\text{Ass}_R M/I^n M$ and $\text{Ass}_R I^n M/I^{n+1} M$, $n = 1, 2, \ldots$ eventually become constant for $n >> 0$ (see...
Dual to this results, Sharp has shown for the Artinian module $A$ in [9] that the sequences $\text{Att}_R(0 : A I^n)$ and $\text{Att}_R(0 : A I^n)/(0 : A I^{n+1})$ do not depend on $n$ for $n >> 0$. Next, Melkersson and Schenzel [8] showed that, for each integer $i$, the sets of prime ideals $\text{Ass}_R \text{Tor}_i^R(R/I^n, M)$ and $\text{Att}_R \text{Ext}_i^R(R/I^n, A)$ become independent of $n$ for $n >> 0$. They also asked whether the set of prime ideals $\bigcup_{n>0} \text{Ass}_R \text{Ext}_i^R(R/I^n, M)$ is finite.

In 2001, Khashyarmanesh and Salarian [7] proved that $\text{Ass}_R \text{Ext}_1^R(R/I^n, M)$ is independent of $n$ for $n >> 0$. Afterwards, in [5], it was proved, for an integer $t$, that if $\text{Supp}_R H_i^j(I)(M)$ is finite for all $i < t$, then $\bigcup_{n>0} \text{Ass}_R \text{Ext}_i^R(R/I^n, M)$ is independent of $n$ for $n >> 0$. Next, by using the notion of M-sequences in dimension $> k$, Brodmann and Nhan [3] proved that, for an integer $t >> 0$, if $\dim_R H_i^j(M) \leq k$ for all $i < t$, then $\bigcup_{i=0}^t \text{Ass}_R \text{Ext}_i^R(R/I, M)$ are two finite sets.

In this paper, by computing the local cohomology modules and Ext-modules via the injective resolution of $M$, we proved that, for an integer $t > 0$, if $\dim_R H_i^j(M) \leq k$ for all $i < t$, then

$$
\bigcup_{n>0} (\text{Ass}_R \text{Ext}_i^R(R/I^n, M))_{>1}
$$

is a finite set. Next, by using the notion of M-sequences in dimension $> k$, Brodmann and Nhan [3] proved that, for an integer $t > 0$, if $\dim_R H_i^j(M) \leq k$ for all $i < t$, then $\bigcup_{n>0} (\text{Ass}_R \text{Ext}_i^R(R/I^n, M))_{>1}$ is contained in the finite set $\bigcup_{i=0}^t \text{Ass}_R \text{Ext}_i^R(R/I, M)$. Moreover, in 2008, Khashyarmanesh and Khosh-Ahang [6] proved that, for an integer $t > 0$, if $\dim_R H_i^j(M) \leq k$ for all $i < t$, then $\bigcup_{n>0} (\text{Ass}_R \text{Ext}_i^R(R/I^n, M))_{>1}$ is a finite set.

Also, by investigating the relationship among $M$-sequences in dimension $> k$, filter regular sequence and regular sequence, we prove that

$$
\bigcup_{i=1}^r \text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_i^{n_i})M \geq k = \bigcup_{i=1}^r \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M \geq k
$$
if $x_1, x_2, \ldots, x_r$ is $M$-sequences in dimension $> k$ and $n_1, n_2, \ldots, n_r$ are some positive integers. This shows that

$$(\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{\geq k} \setminus \bigcup_{i=1}^{r-1} (\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_i^{n_i})M)_{\geq k}$$

is independent of $n_1, n_2, \ldots, n_r$ for $n_1, n_2, \ldots, n_r$ large.

2. Auxiliary and preliminary results

Let $N$ be a non-zero $R$-module. The Krull dimension $\dim_R N$ of $N$ is the supremum of lengths of chains of prime ideals in $\text{Supp}_R N$ if this supremum exists, and $\infty$ otherwise. In the case when $N$ is finitely generated, this is equal to $\dim_R R/(0 : N)$. If $R$-module $N = 0$, we set $\dim_R N = -1$.

**Lemma 2.1.** Assume that $0 \to N_1 \to N_2 \to N_3 \to 0$ is an exact sequence of $R$-modules. Then $\dim_R N_2 = \max\{\dim_R N_1, \dim_R N_3\}$.

**Proof.** By virtue of the exactness of localization, it clear that

$$\text{Supp}_R N_2 = \text{Supp}_R N_1 \cup \text{Supp}_R N_3.$$ Then it follows that $\dim_R N_2 = \max\{\dim_R N_1, \dim_R N_3\}$. □

**Lemma 2.2.** Let $N$ be an $R$-module. Then

$$\text{Ass}_R \Gamma_I(N) = \text{Ass}_R \text{Hom}_R(R/I, N) = \text{Ass}_R N \cap V(I).$$

**Proof.** It follows from [1] that $\text{Ass}_R \text{Hom}_R(R/I, N) = \text{Ass}_R N \cap V(I)$. Now we will prove that

$$\text{Ass}_R \Gamma_I(N) = \text{Ass}_R N \cap V(I).$$

Let $p \in \text{Ass}_R \Gamma_I(N)$. Then $\Gamma_I R_R (N_p) \neq 0$, and then $p \in V(I)$. It is clear that $p \in \text{Ass}_R N$. So $\text{Ass}_R \Gamma_I(N) \subseteq \text{Ass}_R N \cap V(I)$. On the other hand, let $p \in \text{Ass}_R N \cap V(I)$. Then there exists $x \in N$ such that $p = (0 : R x)$. As $I \subseteq p$ we have $Ix = 0$, thus $x \in \Gamma_I(N)$. It follows that $p \in \text{Ass}_R \Gamma_I(N)$. Hence, $\text{Ass}_R \Gamma_I(N) = \text{Ass}_R N \cap V(I)$. This completes the proof. □

The following lemma is a well-known result. We can’t find a reference for it. For the convenience of the reader, we give a proof of it.

**Lemma 2.3.** Let $K, L$ be two $R$-modules. If $K \subseteq L$ is an essential extension, then $\text{Ass}_R K = \text{Ass}_R L$.

**Proof.** It is clear that $\text{Ass}_R K \subseteq \text{Ass}_R L$. On the other hand, let $p \in \text{Ass}_R L$. Then, there exists $x \in L$, $p = \text{Ann}_R x$. Since $K \subseteq L$ is an essential extension, there exists $r \in R$, $rx \in K$ and $rx \neq 0$. Thus, $r \notin p$. By virtue of this, it is easy to verify that $\text{Ann}_R(rx) \subseteq p$. This show that $\text{Ann}_R(rx) = \text{Ann}_R x = p$, and $p \in \text{Ass}_R K$. Hence, $\text{Ass}_R K = \text{Ass}_R L$. □
Let \( k \geq 0 \) be an integer. Let \( x_1, x_2, \ldots, x_r \) be a sequence of elements of \( R \). We say that \( x_1, x_2, \ldots, x_r \) is \( M \)-sequences in dimension \( > k \) if \( x_i \notin p \) for all \( p \in \text{Ass}_R M/(x_1, x_2, \ldots, x_{i-1})M \) and all \( i = 1, 2, \ldots, r \) (see [3, Definition 2.1]). It is easy to see that if \( x_1, x_2, \ldots, x_r \) is \( M \)-sequences in dimension \( > k \), then so is \( x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r} \) for all positive integers \( n_1, n_2, \ldots, n_r \). For the notion of \( M \)-sequences in dimension \( > k \), Brodmann and Nhan gave the following characterization:

**Remark 2.4** ([3, Lemma 2.4]). Let \( t > 0 \) be an integer. Then,

(i) \( \dim_R H^i_I(M) \leq k \) for all \( i < t \) if and only if there exists an \( M \)-sequence in dimension \( > k \) of length \( t \) in \( I \).

(ii) If \( \dim M/IM > k \). Then each \( M \)-sequence in dimension \( > k \) in \( I \) may be extended to a maximal \( M \)-sequence in dimension \( > k \) in \( I \). Moreover, all maximal \( M \)-sequences in dimension \( > k \) in \( I \) have the same length, this common length is equal to the least integer \( i \) such that \( \dim_R H^i_I(M) > k \). We usually denote this length by \( \text{depth}_k(I, M) \).

(iii) If \( \dim M/IM \leq k \). Then there exists an \( M \)-sequence in dimension \( > k \) in \( I \) of length \( n \) for any integer \( n > 0 \).

The two lemmas below establish the relationships among a \( M \)-sequence in dimension \( > k \), a filter regular sequence on the localization of \( M \) and a regular sequence on the localization of \( M \).

**Lemma 2.5.** Let \( M \) be a finitely generated \( R \)-module. Let \( x_1, x_2, \ldots, x_r \) be \( M \)-sequences in dimension \( > k \). Then, for \( p \in \text{Spec} R \) satisfying \( x_1, x_2, \ldots, x_r \subseteq p \) and \( \dim_R/p \geq k \), \( x_1/1, x_2/1, \ldots, x_r/1 \in pR_p \) is a filter regular sequence on \( M_p \).

**Proof.** Let \( p \in \text{Spec} R \) satisfying \( x_1, x_2, \ldots, x_r \subseteq p \) and \( \dim_R/p \geq k \). Let \( qR_p \in \text{Spec} R_p \setminus \{ pR_p \} \). Then \( \dim_R/q > k \). Then \( x_1/1, x_2/1, \ldots, x_r/1 \) is a poor \( M_q \)-regular sequence. Note that \( M_q \cong (M_p)_{qR_p} \). So \( x_1/1, x_2/1, \ldots, x_r/1 \in pR_p \) is a filter regular sequence on \( M_p \). \( \square \)

For an \( R \)-module \( K \) and an ideal \( I \), we use \( 0 :_K \langle I \rangle \) to denote the submodule \( \{ x \in K \mid I^n x = 0 \text{ for some } n > 0 \} \).

**Lemma 2.6.** Let \( M \) be a finitely generated \( R \)-module. Let \( x_1, x_2, \ldots, x_r \) be \( M \)-sequences in dimension \( > k \). For \( p \in \text{Spec} R \) satisfying \( x_1, x_2, \ldots, x_r \subseteq p \) and \( \dim_R/p \geq k \), if \( p \notin \text{Ass}_R M/(x_1, x_2, \ldots, x_{i-1})M \) for every \( i, 1 \leq i \leq r \), then \( x_1/1, x_2/1, \ldots, x_r/1 \in pR_p \) is a poor \( M_p \)-regular sequence.

**Proof.** By Lemma 2.5, For \( p \in \text{Spec} R \) satisfying \( x_1, x_2, \ldots, x_r \subseteq p \) and \( \dim_R/p \geq k \), \( x_1/1, x_2/1, \ldots, x_r/1 \in pR_p \) is a filter regular sequence on \( M_p \). Then by the definition of the filter regular sequence, we have that

\[
0 :_{M_p/(x_1, x_2, \ldots, x_{i-1})M_p} x_1 \subseteq 0 :_{M_p/(x_1, x_2, \ldots, x_{i-1})M_p} (pR_p)
\]
for all $i = 1, \ldots, r$. Since $pR_p \notin \text{Ass}_{R_p} M_p/(x_1, x_2, \ldots, x_{i-1})M_p$ for every $i, 1 \leq i \leq r$, $0 : M_p/(x_1, x_2, \ldots, x_{i-1})M_p (pR_p) = 0$ for all $i = 1, \ldots, r$. So $x_1/1, x_2/1, \ldots, x_i/1 \in pR_p$ is a poor $M_p$-regular sequence. □

### 3. Main results

**Proposition 3.1.** Let $M$ be a finitely generated $R$-module and $E^•(M)$ a minimal injective resolution of $M$. Let $k, t$ be two integers. The following are equivalent:

(i) $\dim_R H^i_j(M) \leq k$ for all $i < t$;
(ii) $\dim_R \text{Ext}^i_p(R/I, M) \leq k$ for all $i < t$;
(iii) $\dim_R \Gamma_I(E^i(M)) \leq k$ for all $i < t$;
(iv) $\dim_R \text{Hom}_R(R/I, E^i(M)) \leq k$ for all $i < t$.

**Proof.** (i)$\iff$(iii): We denote $H^i_j(\bullet)$ by $T^i(\bullet)$, $i \geq 0$, and denote $\Gamma_I(\bullet)$ by $T(\bullet)|= T^0(\bullet)$.

We have the following commutative graph:

\[ \cdots \xrightarrow{T(d^r-1)} T(E^r(M)) \xrightarrow{T(d^r)} T(E^{r+1}(M)) \xrightarrow{T(d^{r+1})} \cdots \]
\[ \cdots \xrightarrow{d^r} E^r(M) \xrightarrow{d^r} E^{r+1}(M) \xrightarrow{d^{r+1}} \cdots \]

Since $\text{Ker} d^r \subseteq E^r(M)$ is an essential extension, then $\text{Ker} T(d^r) = \text{Ker} d^r \cap T(E^r(M)) \subseteq T(E^r(M))$ is an essential extension. By Lemma 2.3,

\[ \text{Ass}_R \text{Ker} T(d^r) = \text{Ass}_R T(E^r(M)). \]

Then
\[ \text{Supp}_R \text{Ker} T(d^r) = \text{Supp}_R T(E^r(M)). \]

and so that

(a) $\dim_R \text{Ker} T(d^r) \leq k$ if and only if $\dim_R T(E^r(M)) \leq k$ for some integer $k$. On the other hand, it follows by Lemma 2.1 that

(b) $\dim_R \text{Im} T(d^r) \leq k$ if $\dim_R T(E^r(M)) \leq k$.

By using the following exact sequence

\[ 0 \longrightarrow \text{Im} T(d^{r-1}) \longrightarrow \text{Ker} T(d^r) \longrightarrow T^r(M) \longrightarrow 0 \]

for $r = 1, 2, \ldots, t - 1$ and $\text{Ker} T(d^r) \cong T^0(M)$, it follows from the results (a) and (b) that

\[ \dim_R T^r(M) \leq k \text{ for all } i < t \text{ if and only if } \dim_R T(E^r(M)) \leq k \text{ for all } i < t. \]

This completes the proof of (i)$\iff$(iii).

(ii)$\iff$(iv): By the same argument as above (we only replace $H^i_j(\bullet), \Gamma_I(\bullet)$ by $\text{Ext}^i_R(R/I, \bullet), \text{Hom}_R(R/I, \bullet)$ respectively), it follows that (ii)$\iff$(iv).
(iii) ⇐⇒ (iv): By Lemma 2.2, \( \text{Ass}_R \Gamma_i(E^i(M)) = \text{Ass}_R \text{Hom}_R(R/I, E^i(M)) \), then
\[
\text{Supp}_R \Gamma_i(E^i(M)) = \text{Supp}_R \text{Hom}_R(R/I, E^i(M))
\]
and so we have that (iii) and (iv) are equivalent.

This completes the proof. □

**Lemma 3.2.** Let \( M \) be a finitely generated \( R \)-module and \( E^\bullet(M) \) a minimal injective resolution of \( M \). For an integer \( t > 0 \), if \( \dim_R H^i_I(M) \leq k \) for \( \forall i < t \).

Then there are some equalities:

(i) \[
\bigcup_{i=0}^j (\text{Ass}_R H^i_I(M))_{>k} = \bigcup_{i=0}^j (\text{Ass}_R \Gamma_i(E^i(M)))_{>k} = \bigcup_{i=0}^j (\text{Ass}_R \text{Hom}_R(R/I, E^i(M)))_{>k} = \bigcup_{i=0}^j (\text{Ass}_R \text{Ext}^i_R(R/I, M))_{>k}
\]
for \( \forall j < t \).

(ii) \[
\bigcup_{i=0}^j (\text{Supp}_R H^i_I(M))_{>k} = \bigcup_{i=0}^j (\text{Supp}_R \Gamma_i(E^i(M)))_{>k} = \bigcup_{i=0}^j (\text{Supp}_R \text{Hom}_R(R/I, E^i(M)))_{>k} = \bigcup_{i=0}^j (\text{Supp}_R \text{Ext}^i_R(R/I, M))_{>k}
\]
for \( \forall j < t \).

(iii) \[
(\text{Ass}_R H^i_I(M))_{>k} = (\text{Ass}_R \Gamma_i(E^i(M)))_{>k} = (\text{Ass}_R \text{Hom}_R(R/I, E^i(M)))_{>k} = (\text{Ass}_R \text{Ext}^i_R(R/I, M))_{>k}
\]

**Proof.** (i) Since \( \dim_R H^i_I(M) \leq k \) for \( \forall i < t \), it follows by Proposition 3.1 that
\[
\bigcup_{i=0}^j (\text{Supp}_R H^i_I(M))_{>k} = \bigcup_{i=0}^j (\text{Supp}_R \text{Ext}^i_R(R/I, M))_{>k} = \emptyset
\]
for \( \forall j < t \).

(ii) We denote \( H^i_I(\bullet) \) by \( T^i(\bullet) \), \( i \geq 0 \). And we denote \( \Gamma_I(\bullet) \) by \( T(\bullet) (= T^0(\bullet)) \).

As the proof of Proposition 3.1, we have the following commutative graph:
\[
\begin{array}{cccccc}
\cdots & \xrightarrow{T(d_{i-1})} & T(E^i(M)) & \xrightarrow{T(d_i)} & T(E^{i+1}(M)) & \xrightarrow{T(d_{i+1})} & \cdots \\
\downarrow & & & & & \\
\cdots & \xrightarrow{d_{i-1}} & E^i(M) & \xrightarrow{d_i} & E^{i+1}(M) & \xrightarrow{d_{i+1}} & \cdots
\end{array}
\]
Since \( \text{Ker} d^i \subseteq E^i(M) \) is an essential extension, then \( \text{Ker} T(d^i) = \text{Ker} d^i \cap T(E^i(M)) \subseteq T(E^i(M)) \) is an essential extension. Then, by Lemma 2.3, for \( \forall i \geq 0 \),

(1) \[
\text{Ass}_R \text{Ker} T(d^i) = \text{Ass}_R T(E^i(M)).
\]
Let \( j \leq t \) be an integer. In the following, we use induction on \( j \) to prove that
\[
\bigcup_{i=0}^{j}(\text{Ass}_R T(E^i(M)))_k = \bigcup_{i=0}^{j}(\text{Ass}_R T^i(M))_k.
\]

Let \( j = 0 \). Since \( \text{Ker} T(d^0) \cong T^0(M) \), it follows from the equality (1) that
\[
(\text{Ass}_R T(E^0(M)))_k = (\text{Ass}_R T^0(M))_k.
\]

Then we suppose that \( j > 1 \) and that the result have been proved for \( j - 1 \):
\[
\bigcup_{i=0}^{j-1}(\text{Ass}_R T(E^i(M)))_k = \bigcup_{i=0}^{j-1}(\text{Ass}_R T^i(M))_k.
\]

For every \( i \in \{1, 2, \ldots, j\} \), from the exact sequence
\[
0 \rightarrow \text{Im} T(d^{i-1}) \rightarrow \text{Ker} T(d^i) \rightarrow T^i(M) \rightarrow 0,
\]
it follows that
\[
(\text{Ass}_R T(E^i(M)))_k = (\text{Ass}_R \text{Ker} T(d^i))_k \subseteq (\text{Supp}_R \text{Im} T(d^{i-1}))_k \bigcup (\text{Ass}_R T^i(M))_k
\]
\[
= (\text{Supp}_R T(E^{i-1}(M)))_k \bigcup (\text{Ass}_R T^i(M))_k.
\]

Then we have that
\[
\bigcup_{i=0}^{j}(\text{Ass}_R T(E^i(M)))_k \subseteq \bigcup_{i=0}^{j}(\text{Ass}_R T^i(M))_k.
\]

On the other hand, let \( p \in \bigcup_{i=0}^{j-1}(\text{Ass}_R T^i(M))_k \). If \( p \in \bigcup_{i=0}^{j-1}(\text{Ass}_R T^i(M))_k \), by the inductive assumption, it is clear that \( p \in \bigcup_{i=0}^{j-1}(\text{Ass}_R T^i(M))_k \). So we assume that \( p \not\in \bigcup_{i=0}^{j-1}(\text{Ass}_R T^i(M))_k \). Then, by the inductive assumption again,
\[
p \not\in \bigcup_{i=0}^{j-1}(\text{Ass}_R T(E^i(M)))_k = \bigcup_{i=0}^{j-1}(\text{Supp}_R T(E^i(M)))_k
\]
and \( p \not\in \text{Supp}_R \text{Im} T(d^{i-1}) \). The exact sequence
\[
0 \rightarrow \text{Im} T(d^{j-1}) \rightarrow \text{Ker} T(d^j) \rightarrow T^j(M) \rightarrow 0
\]
implies that
\[
(\text{Ker} T(d^j))_p \cong (T^j(M))_p.
\]

Then, since \( p \in (\text{Ass}_R T^j(M))_k \), it follows that
\[
p R_p \in \text{Ass}_R T^j(M)_p = \text{Ass}_R T^j(M)_p,
\]
and so by the equality (1), \( p \in (\text{Ass}_R \text{Ker} T(d^j))_k = (\text{Ass}_R T(E^j(M)))_k \). Hence, 
\[ \bigcup_{i=0}^j (\text{Ass}_R T^i(M))_k \subseteq \bigcup_{i=0}^j (\text{Ass}_R T(E^i(M)))_k. \]
This shows the equality in the previous formula.

Thus, we have that 
\[ \bigcup_{i=0}^j (\text{Ass}_R T^i(M))_k \subseteq \bigcup_{i=0}^j (\text{Ass}_R T(E^i(M)))_k, \]
and 
\[ \bigcup_{i=0}^j (\text{Ass}_R T^i(M))_k \supseteq \bigcup_{i=0}^j (\text{Ass}_R T(E^i(M)))_k. \]
This shows the equality in the previous formula.

By the same argument as above (we only replace \( H_i^n(M) \), \( \Gamma_I(M) \) by \( \text{Ext}^i_R (R/I, \bullet) \), \( \text{Hom}_R (R/I, \bullet) \), respectively), we have that 
\[ \bigcup_{i=0}^j (\text{Ass}_R \text{Ext}^i_R (R/I))_k = \bigcup_{i=0}^j (\text{Ass}_R \text{Hom}(R/I, E^i(M)))_k \]
for \( \forall j \leq t \).

Finally, the result (ii) follows from Lemma 2.2.

(iii) We continue to use the notations as (ii). Since \( \dim_R H_i^1(M) \leq k \) for \( \forall i < t \), it follows that \( \dim_R \text{Im} T(d^i-1) \leq k \) by (i) \( \iff \) (iii) of Proposition 3.1. From the exact sequences 
\[ 0 \rightarrow \text{Im} T(d^i-1) \rightarrow \text{Ker} T(d^i) \rightarrow T^i(M) \rightarrow 0, \]
it follows that for \( \forall p \in \text{Spec} R \) satisfying \( \dim R/p > k \), we have that 
\[ (\text{Ker} T(d^i))_p \cong (T^i(M))_p. \]
Then, \( p \in \text{Ass}_R \text{Ker} T(d^i)_{>k} \) if and only if \( p \in (\text{Ass}_R T^i(M))_{>k} \). Hence, by the equality (1) 
\[ (\text{Ass}_R T^i(M))_{>k} = \text{Ass}_R \text{Ker} T(d^i)_{>k} = (\text{Ass}_R T(E^i(M)))_{>k}. \]
This shows that 
\[ (\text{Ass}_R H^1_I(M))_{>k} = (\text{Ass}_R \Gamma_I(E^i(M)))_{>k}. \]
By the same argument as above, it follows that 
\[ (\text{Ass}_R \text{Hom}(R/I, E^i(M)))_{>k} = (\text{Ass}_R \text{Ext}^i_R (R/I, M))_{>k}. \]
Then by Lemma 2.2, the result (iii) follows. \( \Box \)

Note that \( H^1_I(M) \cong H^1_{I^n}(M) \) for any positive integer \( n \). The following corollaries are two immediate consequences of Theorem 3.2.
Corollary 3.3. Let $M$ be a finitely generated $R$-module. For an integer $t > 0$, $\dim_R H^j_i(M) \leq k$ for $\forall i < t$. Then

$$\bigcup_{i=0}^{j} (\text{Ass}_R H^j_i(M))_{\geq k} = \bigcup_{i=0}^{j} (\text{Ass}_R \text{Ext}^i_R(R/I^n, M))_{\geq k}$$

for $\forall j \leq t$ and $\forall n > 0$. In particular, $(\text{Ass}_R H^j_i(M))_{\geq k}$ is a finite set for $\forall i < t$.

Corollary 3.4 ([6, Theorem 1.1]). Let $M$ be a finitely generated $R$-module. For an integer $t > 0$, $\dim_R H^j_i(M) \leq k$ for $\forall i < t$. Then

1. $\bigcup_{n>0} (\text{Ass}_R \text{Ext}^i_R(R/I^n, M))_{\geq k} \subseteq \bigcup_{i=0}^{j} (\text{Ass}_R \text{Ext}^i_R(R/I, M))$ for $\forall i \leq t$.
2. $\bigcup_{n>0} (\text{Supp} \text{Ext}^i_R(R/I^n, M))_{\geq k} = \bigcup_{i=0}^{j} (\text{Ass}_R \text{Ext}^i_R(R/I^n, M))_{\geq k}$ for $\forall i < t$.

In particular, $\bigcup_{n>0} (\text{Supp} \text{Ext}^i_R(R/I^n, M))_{\geq k}$ is a finite set for $\forall i < t$.

Theorem 3.5. Let $M$ be a finitely generated $R$-module. Let $x_1, x_2, \ldots, x_r$ be $M$-sequences in dimension $> k$. Then, for any positive integers $n_1, n_2, \ldots, n_r$,

1. $(\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{\geq k} \cap V(x_{i+1}) = \emptyset$, $\forall i < r$;
2. $(\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{\geq k} = (\text{Ass}_R M/(x_1, x_2, \ldots, x_r)M)_{\geq k}$$;
3. $\bigcup_{n=0}^t (\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_k = \bigcup_{n=0}^t (\text{Ass}_R M/(x_1, x_2, \ldots, x_r)M)_k$.

In particular,

$$\bigcup_{n_1, n_2, \ldots, n_r} (\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{\geq k}$$

is a finite set.

Proof. (i) Let $\forall i < r$. For $p \in \text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M \cap V(x_{i+1})$, then, by the definition of $M$-sequences in dimension $> k$, $\dim_R p \leq k$. So

$$(\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{\geq k} \cap V(x_{i+1}) = \emptyset.$$

(ii) Let $n_1, n_2, \ldots, n_r$ be any positive integers and

$$p \in (\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{> k}.$$

Then $x_1^{n_1}/1, x_2^{n_2}/1, \ldots, x_r^{n_r}/1 \in pR_p$ is a poor $M_p$-regular sequence. By [4, Lemma 1.2.4],

$$(\text{Hom}(R_p/(x_1, x_2, \ldots, x_r)R_p, M_p/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M_p) \cong \text{Ext}^r_{R_p}(R_p/(x_1, x_2, \ldots, x_r)R_p, M_p).$$

So we have that

$$pR_p \in (\text{Ass}_{R_p} M_p/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M_p)_{> k}$$

if and only if

$$pR_p \in (\text{Ass}_{R_p} \text{Ext}^r_{R_p}(R_p/(x_1, x_2, \ldots, x_r)R_p, M_p))_{> k}.$$
if and only if
\[ pR_p \in (\text{Ass}_{R_p} M_p/(x_1, x_2, \ldots, x_r)M_p)_{> k}. \]
Hence,
\[ p \in (\text{Ass}_R M/(x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r})M)_{> k} \]
if and only if
\[ p \in (\text{Ass}_R M/(x_1, x_2, \ldots, x_r)M)_{> k}. \]

(iii) Since \( x_1, x_2, \ldots, x_r \) is \( M \)-sequences in dimension \( > k \), then, for all positive integers \( n_1, n_2, \ldots, n_r \), so is \( x_1^{n_1}, x_2^{n_2}, \ldots, x_r^{n_r} \). The result follows by the more general statement of Theorem 3.6 (proved in the following) for \( I = (x_1, x_2, \ldots, x_r) \). □

The following theorem is a generalization of [3, Theorem 1.2].

**Theorem 3.6.** Let \( M \) be a finitely generated \( R \)-module. Let \( x_1, x_2, \ldots, x_r \in I \) be \( M \)-sequences in dimension \( > k \). Then
\[ (\bigcup_{i=0}^r \text{Ass}_R \text{Ext}_R^i(R/I, M))_{> k} = (\bigcup_{i=0}^r (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{> k} \cap V(I). \]

In particular,
\[ (\bigcup_{n} \text{Ass}_R \text{Ext}_R^n(R/I^n, M))_{> k} \]
is contained in the finite set
\[ (\text{Ass}_R M/(x_1, x_2, \ldots, x_r)M)_{> k} \cup (\bigcup_{i=0}^r \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{> k}. \]

**Proof.** We use induction on \( t \) to prove that
\[ (\bigcup_{i=0}^t \text{Ass}_R \text{Ext}_R^i(R/I, M))_{> k} = (\bigcup_{i=0}^t (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{> k} \cap V(I) \]
for every \( t, 0 \leq t \leq r \). When \( t = 0 \), then it is nothing to prove since it is well known that \( \text{Ass}_R \text{Hom}(R/I, M) = \text{Ass}_R M \cap V(I) \). Then we suppose that \( t > 1 \) and that the result have been proved for \( t - 1 \):
\[ (\bigcup_{i=0}^{t-1} \text{Ass}_R \text{Ext}_R^i(R/I, M))_{> k} = (\bigcup_{i=0}^{t-1} (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{> k} \cap V(I). \]

Let \( p \in (\bigcup_{i=0}^t (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{> k} \cap V(I). \)
If \( p \in (\bigcup_{i=0}^t \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M) \), then by the inductive assumption, we have that
\[ p \in (\bigcup_{i=0}^t \text{Ass}_R \text{Ext}_R^i(R/I, M))_{> k}. \]
If \( p \not\in \bigcup_{i=0}^{t-1} \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M \). Then \( p \in \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M \), moreover by Lemma 2.6, \( x_1/1, x_2/1, \ldots, x_i/1 \in IR_p \) is a poor \( M_p \)-regular sequence. Then there is an isomorphism:

\[
\text{Hom}(R_p/IR_p, M_p/(x_1, x_2, \ldots, x_t)M_p) \cong \text{Ext}^t_{R_p}(R_p/IR_p, M_p).
\]

Since \( p \in \text{Ass}_R M/(x_1, x_2, \ldots, x_t)M \), it follows that

\[
pR_p \in \text{Ass}_R M_p/(x_1, x_2, \ldots, x_t)M_p \cap V(IR_p) = \text{Ass}_R \text{Ext}^t_{R_p}(R_p/IR_p, M_p).
\]

This shows that \( p \in \text{Ass}_R \text{Ext}^t_{R}(R/I, M) \). Therefore, we have that

\[
(\bigcup_{i=0}^{t}(\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{\geq k} \cap V(I) \subseteq (\bigcup_{i=0}^{t} \text{Ass}_R \text{Ext}^i_{R}(R/I, M))_{\geq k}.
\]

On the other hand, let \( p \in (\bigcup_{i=0}^{t} \text{Ass}_R \text{Ext}^i_{R}(R/I, M))_{\geq k}. \)

If \( p \in \bigcup_{i=0}^{t} \text{Ass}_R \text{Ext}^i_{R}(R/I, M) \), it is clear that

\[
p \in \left( \bigcup_{i=0}^{t} (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{\geq k} \cap V(I) \right)
\]

by the inductive assumption.

If \( p \not\in \bigcup_{i=0}^{t} \text{Ass}_R \text{Ext}^i_{R}(R/I, M) \), by the inductive assumption, we have that

\[
p \not\in \bigcup_{i=0}^{t-1} \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M. \]

This shows that \( x_1/1, x_2/1, \ldots, x_i/1 \in IR_p \) is a poor \( M_p \)-regular sequence by Lemma 2.6. Then, similar to the proof above, we can also prove that

\[
p \in \left( \bigcup_{i=0}^{t} (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{\geq k} \cap V(I) \right).
\]

Therefore,

\[
(\bigcup_{i=0}^{t} \text{Ass}_R \text{Ext}^i_{R}(R/I, M))_{\geq k} \subseteq (\bigcup_{i=0}^{t} (\text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{\geq k} \cap V(I).
\]

This proves the equality in the previous formula.

By Theorem 3.5(i), \( (\bigcup_{i=0}^{t} \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{\geq k} \cap V(I) = \emptyset \). So we have that

\[
(\bigcup_{n} \text{Ass}_R \text{Ext}^n_{R}(R/I^n, M))_{\geq k}
\]

is contained in the finite set

\[
(\text{Ass}_R M/(x_1, x_2, \ldots, x_r)M)_{\geq k} \cup (\bigcup_{i=0}^{t} \text{Ass}_R M/(x_1, x_2, \ldots, x_i)M)_{k}.
\]

\[\square\]

**Acknowledgment.** The authors thank the referee for his or her carefully reading of this manuscript.
References


LIZHONG CHU  
DEPARTMENT OF MATHEMATICS  
SUOCHOW UNIVERSITY  
SUZHOU 215006, P. R. CHINA  
E-mail address: chulizhong@suda.edu.cn

XIAN WANG  
DEPARTMENT OF MATHEMATICS  
CHINA UNIVERSITY OF MINING AND TECHNOLOGY  
XUZHOU, 221116, P. R. CHINA  
E-mail address: wx2008117@cumt.edu.cn