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PARTS FORMULAS INVOLVING CONDITIONAL INTEGRAL TRANSFORMS ON FUNCTION SPACE

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Abstract. We obtain a formula for the conditional Wiener integral of the first variation of functionals and establish several integration by parts formulas of conditional Wiener integrals of functionals on a function space. We then apply these results to obtain various integration by parts formulas involving conditional integral transforms and conditional convolution products on the function space.

1. Definitions and preliminaries

In a unifying paper [11], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on an abstract Wiener space. For certain values of the parameters α and β and for certain classes of functionals, the Fourier-Wiener transform [2], the Fourier-Feynman transform [4] and the Gauss transform are special cases of the integral transform $\mathcal{F}_{\alpha,\beta}$. In [8], the authors established various integration by parts formulas involving integral transforms of functionals on a function space. Chang and Skoug [6], established the formula for the conditional analytic Feynman integral of the first variation of functionals on Wiener space and obtained several integration by parts formulas for conditional analytic Feynman integrals and conditional Fourier-Feynman transforms.

In this paper we obtain a formula for the conditional Wiener integral of the first variation of functionals of the form $F(x) = f(\langle\vec{\theta}, x\rangle)$,

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where $\langle \theta, x \rangle$ denotes the Paley-Wiener-Zygmund(PWZ) stochastic integral $\int_0^T \theta(t)\, dx(t)$ and establish several integration by parts formulas of conditional Wiener integrals of functionals on a function space. We then apply these results to obtain various integration by parts formulas involving conditional integral transforms and conditional convolution products on the function space.

Let $C_0[0,T]$ denote Wiener space; that is the space of all $\mathbb{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. Then $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a Wiener integrable functional F by

$$E_x[F(x)] = \int_{C_0[0,T]} F(x)\, m(dx). \tag{1.1}$$

Let $K = K[0,T]$ be the space of all $\mathbb{C}$-valued continuous functions defined on $[0,T]$ which vanish at $t = 0$ and let α and β be nonzero complex numbers.

Now we introduce the definitions of integral transform $\mathcal{F}_{\alpha,\beta}$, convolution product $(F * G)_\alpha$ and first variation δF for functionals defined on K. The main focus of [9] was to establish various relationships holding among $\mathcal{F}_{\alpha,\beta}F$, $\mathcal{F}_{\alpha,\beta}G$, $(F * G)_\alpha$, δF and δG.

Definition 1.1. Let F be a functional defined on K. Then the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F is defined by

$$\mathcal{F}_{\alpha,\beta}F(y) \equiv E_x[F(\alpha x + \beta y)], \quad y \in K \tag{1.2}$$

if it exists [9, 11].

Definition 1.2. Let F and G be functionals defined on K. Then the convolution product $(F * G)_\alpha$ of F and G is defined by

$$(F * G)_\alpha(y) \equiv E_x\Big[F\Big(\frac{y + \alpha x}{\sqrt{2}}\Big)G\Big(\frac{y - \alpha x}{\sqrt{2}}\Big)\Big], \quad y \in K \tag{1.3}$$

if it exists [7, 9, 13, 15].

Definition 1.3. Let F be a functional defined on K and let $w \in K$. Then the first variation δF of F is defined by

$$\delta F(y|w) \equiv \frac{\partial}{\partial t} F(y + tw)|_{t=0}, \quad y \in K \tag{1.4}$$

if it exists [1, 5, 9, 13].

Let $X : C_0[0,T] \to \mathbb{R}$ be a Wiener measurable functional and let $F : C_0[0,T] \to \mathbb{C}$ be a Wiener integrable functional. Then for $\eta \in \mathbb{R}$, $E_x[F\|X](\eta)$ denotes the conditional Wiener integral of F given X [6, 12, 16]. In [12], Park and Skoug gave a formula for expressing conditional Wiener integrals in terms of ordinary(i.e., non-conditional) Wiener integrals; namely that for $X(x) = x(T)$,

$$\begin{aligned} E_x[F(x)\|X(x)](\eta) &= E_x[F(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta)] \\ &= \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)m(dx). \end{aligned} \tag{1.5}$$

For a functional F defined on K we define the conditional integral transform $\mathcal{F}_{\alpha,\beta}(F\|X)(y,\eta)$ of F given X by the formula

$$\mathcal{F}_{\alpha,\beta}(F\|X)(y,\eta) = E_x[F(\alpha x + \beta y)\|X(x)](\eta) \tag{1.6}$$

for $y \in K$ and $\eta \in \mathbb{R}$ if it exists.

Similarly for functionals F and G defined on K, we define the conditional convolution product $((F * G)_\alpha\|X)(y,\eta)$ of $(F * G)_\alpha$ given X by the formula

$$((F * G)_\alpha\|X)(y,\eta) = E_x[F\big(\frac{y+\alpha x}{\sqrt{2}}\big)G\big(\frac{y-\alpha x}{\sqrt{2}}\big)||X(x)](\eta) \tag{1.7}$$

for $y \in K$ and $\eta \in \mathbb{R}$ if it exists.

Next we describe the class of functionals that we work with in this paper. Let $\{\theta_n\}$ be a complete orthonormal set of $\mathbb{R}$-valued functions in $L^2[0,T]$. Furthermore assume that each θ_j is of bounded variation on $[0,T]$. Then for each $y \in K$ and $j \in \{1, 2, \ldots\}$, the PWZ integral $\langle\theta_j, y\rangle \equiv \int_0^T \theta_j(t)\,dy(t)$ exists.

For $0 \leq \sigma < 1$ let E_σ be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$F(y) = f(\langle\theta_1, y\rangle, \ldots, \langle\theta_n, y\rangle) = f(\langle\vec{\theta}, y\rangle) \tag{1.8}$$

for some positive integer n, where $f(\vec{\lambda})$ is an entire function of n complex variables $\lambda_1, \ldots, \lambda_n$ of exponential type; that is to say

$$|f(\vec{\lambda})| \leq A_F \exp\{B_F|\vec{\lambda}|^{1+\sigma}\} \tag{1.9}$$

for some positive constants A_F and B_F, where $|\vec{\lambda}|^{1+\sigma} = \sum\limits_{j=1}^{n} |\lambda_j|^{1+\sigma}$.

In addition we use the notation $F_j(y) = f_j(\langle\vec{\theta}, y\rangle)$ where $f_j(\vec{\lambda}) = \frac{\partial}{\partial\lambda_j} f_j(\lambda_1, \ldots, \lambda_n)$ for $j = 1, \ldots, n$.

In [10], the current authors and Skoug showed that for all F and G in E_σ, $\mathcal{F}_{\alpha,\beta}(F\|X)$ and $((F*G)_\alpha\|X)$ exist and belong to E_σ for all nonzero complex numbers α and β and the condition by $X(x) = x(T)$ while $\delta F(y|w)$ exists and belongs to E_σ for all y and w in K. For related work see [2, 7, 9, 11, 15] and for a detailed survey of previous work see [14].

2. Parts formulas involving conditional Wiener integral

We begin this section by stating the following well-known Cameron-Martin's translation theorem [3].

THEOREM 2.1. *Let $\phi \in C([0,T]) \cap BV([0,T])$, and $x_0(t) \equiv \int_0^t \phi(s)\,ds$. Then for any Wiener measurable function $F(x)$, $F(x+x_0)$ is Wiener measurable and we have*

(2.1)
$$\begin{aligned}&\int_{C_0[0,T]} F(x+x_0)\,m(dx)\\ &= \exp\Big\{-\frac{1}{2}\int_0^T [x_0'(t)]^2\,dt\Big\}\int_{C_0[0,T]} F(x)\exp\Big\{\int_0^T x_0'(t)\,dx(t)\Big\}\,m(dx).\end{aligned}$$

In [3], we see that the restriction on x_0' may be weakened to $x_0' \in L^2[0,T]$ provided that we interpret $\int_0^T x_0'(t)\,dx(t)$ in the sense of PWZ integral.

Let
$$\begin{aligned}A = \{w \in C_0[0,T] : &\ w \text{ is absolutely continious on } [0,T]\\ &\text{with } w' \in L^2[0,T]\}.\end{aligned}$$

We note that if we choose $z \in L^2[0,T]$ and define $w(t) = \int_0^t z(s)\,ds$ for $t \in [0,T]$, then w is an element of A, $w' = z$ a.e. on $[0,T]$, and for all $v \in L^2[0,T]$, $\langle v, w\rangle = (v, w') = (v, z)$ where $(v,z) = \int_0^T v(s)z(s)\,ds$.

Our first result is a fundamental theorem which plays a key role throughout this paper. In this theorem the conditional Wiener integral of the first variation of a functional F is expressed in terms of the ordinary (that is, non conditional) Wiener integral of F multiplied by a linear factor. Furthermore it can be expressed as a sum of conditional Wiener integrals.

THEOREM 2.2. *Let $F \in E_\sigma$ be given by* (1.8) *and $w \in A$, then*

$$E_x\big[\delta F(x|w)\|X\big](\eta) = \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', x\rangle\, m(dx). \tag{2.2}$$

Also both sides of (2.2) *are given by the formula*

$$\begin{aligned} &E_x\big[F(x)\langle w', x\rangle\|X\big](\eta) + T(w', w_0')E_x\big[\delta F(x|w_0)\|X\big](\eta) \\ &- \eta(w', w_0')E_x\big[F(x)\|X\big](\eta) \end{aligned} \tag{2.3}$$

where $w_0(t) = \frac{t}{T}$.

Proof. In [10], we can see that for $F \in E_\sigma$ and $w \in A$,

$$\begin{aligned} &\int_{C_0[0,T]} \Big|\frac{\partial}{\partial k}F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta + kw(\cdot)\big)|_{k=0}\Big|\, m(dx) \\ &\le \sum_{j=1}^{n} \big|(\theta_j, w')\big| \int_{C_0[0,T]} \big|f_j(\langle\vec{\theta}, x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\rangle\big|\, m(dx) < \infty. \end{aligned} \tag{2.4}$$

Then using (1.4) and (1.5) we see that

$$\begin{aligned} &E_x\big[\delta F(x|w)\|X\big](\eta) \\ &= \int_{C_0[0,T]} \frac{\partial}{\partial k}F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta + kw(\cdot)\big)|_{k=0}\, m(dx) \\ &= \frac{\partial}{\partial k}\Big(\int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta + kw(\cdot)\big)\, m(dx)\Big)\Big|_{k=0}. \end{aligned}$$

Equation (2.4) justifies the second equality, interchange of differentiation and integration, in the above equation. Moreover by (2.1)

$$\begin{aligned} &E_x\big[\delta F(x|w)\|X\big](\eta) \\ &= \frac{\partial}{\partial k}\Big[\exp\Big\{-\frac{k^2}{2}\|w'\|^2\Big\} \\ &\int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\exp\{k\langle w', x\rangle\}\, m(dx)\Big]\Big|_{k=0} \end{aligned}$$

which is equal to the right hand side of (2.2). Furthermore we have

$$\begin{aligned}
&\int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', x\rangle\, m(dx)\\
&= \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\rangle\, m(dx)\\
&\quad + \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', \frac{\cdot}{T}x(T)\rangle\, m(dx)\\
&\quad - \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', \frac{\cdot}{T}\eta\rangle\, m(dx).
\end{aligned}$$

Now it is easy to see that, by (1.5), the last expression is equal to (2.3) and this completes the proof. □

Our first corollary of Theorem 2.2 follows from the equation (2.3) and yields a formula for the conditional integral of F multiplied by a linear factor $\langle w', x\rangle$.

Corollary 2.3. *Let w, w_0 and F be as in Theorem 2.2 above. Then we have*

$$\begin{aligned}
&E_x\big[F(x)\langle w', x\rangle \| X\big](\eta)\\
&= E_x\big[\delta F(x|w)\|X\big](\eta) - T(w', w_0')E_x\big[\delta F(x|w_0)\|X\big](\eta)\\
&\quad + \eta(w', w_0')E_x\big[F(x)\|X\big](\eta).
\end{aligned} \tag{2.5}$$

Our next corollary of Theorem 2.2 is a formula for the conditional integral of F multiplied by two linear factors $\langle w_1', x\rangle$ and $\langle w_2', x\rangle$.

Corollary 2.4. *Let w_1, w_2 be elements of A, and $F \in E_\sigma$ be given by* (1.8). *Then we have*

$$\begin{aligned}
&E_x\big[F(x)\langle w_2', x\rangle\langle w_1', x\rangle \| X\big](\eta)\\
&= E_x\big[(w_1', w_2')F(x) + \langle w_2', x\rangle \delta F(x|w)\|X\big](\eta)\\
&\quad - T(w_1', w_0')E_x\big[(w_0', w_2')F(x) + \delta F(x|w_0)\langle w_2', x\rangle\|X\big](\eta)\\
&\quad + \eta(w_1', w_0')E_x\big[F(x)\langle w_2', x\rangle\|X\big](\eta).
\end{aligned} \tag{2.6}$$

Proof. Let $G(x) = F(x)\langle w_2', x\rangle$, then we have

$$\delta G(x|w_1) = F(x)(w_2', w_1') + \delta F(x|w_1)\langle w_2', x\rangle.$$

Now equation (2.6) follows directly from equation (2.5). □

In our next theorem we express the conditional Wiener integral of the product of functionals in E_σ in terms of the Wiener integral and obtain an integration by parts formula for the functionals.

THEOREM 2.5. *Let F and $G \in E_\sigma$ be given by* (1.8) *with corresponding entire functions f and g respectively. Then for $w \in A$, we have the following*

$$\begin{aligned} &E_x\big[F(x)\delta G(x|w) + \delta F(x|w)G(x)\|X\big](\eta) \\ &= \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big) \\ &G\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', x\rangle\, m(dx). \end{aligned} \tag{2.7}$$

Also both sides of (2.7) *are given by the formula*

$$\begin{aligned} &E_x\big[F(x)G(x)\langle w', x\rangle\|X\big](\eta) \\ &+ T(w', w_0')E_x\big[F(x)\delta G(x|w_0) + \delta F(x|w_0)G(x)\|X\big](\eta) \\ &- \eta(w', w_0')E_x\big[F(x)G(x)\|X\big](\eta) \end{aligned} \tag{2.8}$$

where $w_0(t) = \frac{t}{T}$.

Proof. For $H(x) = F(x)G(x)$, we have

$$\delta H(x|w) = F(x)\delta G(x|w) + \delta F(x|w)G(x).$$

Then equations (2.7) and (2.8) follow from equations (2.2) and (2.3), respectively. □

By choosing $F = G$ in Theorem 2.5, we have corollary.

COROLLARY 2.6. *Let $F \in E_\sigma$ be given by* (1.8) *and $w \in A$, then we have*

$$\begin{aligned} &E_x\big[F(x)\delta F(x|w)\|X\big](\eta) \\ &= \frac{1}{2}\int_{C_0[0,T]} \big[F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\big]^2\langle w', x\rangle\, m(dx). \end{aligned} \tag{2.9}$$

Also both sides of (2.9) *are given by the formula*

$$\begin{aligned} &\frac{1}{2}E_x\big[[F(x)]^2\langle w', x\rangle\|X\big](\eta) + T(w', w_0')E_x\big[F(x)\delta F(x|w_0)\|X\big](\eta) \\ &- \frac{\eta}{2}(w', w_0')E_x\big[[F(x)]^2\|X\big](\eta) \end{aligned} \tag{2.10}$$

where $w_0(t) = \frac{t}{T}$.

If $F \in E_\sigma$, then $\delta F(x|w_1)$ belongs to E_σ [10]. Thus if we replace $G(x)$ with $\delta F(x|w_1)$ in Theorem 2.5 we have the following corollary.

COROLLARY 2.7. *Let $F \in E_\sigma$ be given by* (1.8). *Then for each $w_1, w_2 \in A$, we have*

$$\begin{aligned} &E\big[F(x)\delta^2 F(\cdot|w_1)(x|w_2) + \delta F(x|w_2)\delta F(x|w_1)\|X\big](\eta) \\ &= \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big) \\ &\quad \delta F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta|w_1\big)\langle w_2', x\rangle\, m(dx). \end{aligned} \tag{2.11}$$

Also both sides of (2.11) *are given by the formula*

$$\begin{aligned} &E_x\big[F(x)\delta F(x|w_1)\langle w_2', x\rangle\|X\big](\eta) \\ &+ T(w_2', w_0')E_x\big[F(x)\delta^2 F(\cdot|w_1)(x|w_0) + \delta F(x|w_0)\delta F(x|w_1)\|X\big](\eta) \\ &- \eta(w_2', w_0')E_x\big[F(x)\delta F(x|w_1)\|X\big](\eta) \end{aligned} \tag{2.12}$$

where $w_0(t) = \frac{t}{T}$.

3. Additional results

In this section we obtain the various integration by parts formulas involving conditional integral transforms and conditional convolution products.

Since G belongs to E_σ, $\mathcal{F}_{\alpha,\beta}(G\|X)$ also belongs to E_σ [10] and so we have the following various formulas for conditional Wiener integrals and the integration by parts formulas involving integral transforms and conditional integral transforms. Furthermore we can obtain the formulas involving conditional integral transform of a conditional convolution products.

FORMULA 3.1. Replacing $G(x)$ with $\mathcal{F}_{\alpha,\beta}G(x)$ in Theorem 2.5 yields

$$\begin{aligned} &E_x\big[F(x)\delta\mathcal{F}_{\alpha,\beta}G(x|w) + \delta F(x|w)\mathcal{F}_{\alpha,\beta}G(x)\|X\big](\eta) \\ &= \int_{C_0[0,T]} F\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big) \\ &\quad \mathcal{F}_{\alpha,\beta}G\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta\big)\langle w', x\rangle\, m(dx). \end{aligned} \tag{3.1}$$

Also both sides of (3.1) are given by the formula

$$\begin{aligned}
&E_x\big[F(x)\mathcal{F}_{\alpha,\beta}G(x)\langle w',x\rangle\|X\big](\eta)\\
&+T(w',w_0')E_x\big[F(x)\delta\mathcal{F}_{\alpha,\beta}G(x|w_0)+\delta F(x|w_0)\mathcal{F}_{\alpha,\beta}G(x)\|X\big](\eta)\\
&-\eta(w',w_0')E_x\big[F(x)G(x)\|X\big](\eta)
\end{aligned}\tag{3.2}$$

where $w_0(t)=\frac{t}{T}$.

FORMULA 3.2. Replacing $F(x)$ and $G(x)$ by $\mathcal{F}_{\alpha,\beta}F(x)$ and $\mathcal{F}_{\alpha,\beta}G(x)$, respectively in Theorem 2.5 yields

$$\begin{aligned}
&E_x\big[\mathcal{F}_{\alpha,\beta}F(x)\delta\mathcal{F}_{\alpha,\beta}G(x|w)+\delta\mathcal{F}_{\alpha,\beta}F(x|w)\mathcal{F}_{\alpha,\beta}G(x)\|X\big](\eta)\\
&=\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}F\big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\big)\\
&\mathcal{F}_{\alpha,\beta}G\big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta\big)\langle w',x\rangle\,m(dx).
\end{aligned}\tag{3.3}$$

Also both sides of (3.3) are given by the formula

$$\begin{aligned}
&E_x\big[\mathcal{F}_{\alpha,\beta}F(x)\mathcal{F}_{\alpha,\beta}G(x)\langle w',x\rangle\|X\big](\eta)\\
&+T(w',{w_0}')E_x\big[\mathcal{F}_{\alpha,\beta}F(x)\delta\mathcal{F}_{\alpha,\beta}G(x|w_0)\\
&+\delta\mathcal{F}_{\alpha,\beta}F(x|w_0)\mathcal{F}_{\alpha,\beta}G(x)\|X\big](\eta)\\
&-\eta(w',w_0')E_x\big[\mathcal{F}_{\alpha,\beta}F(x)\mathcal{F}_{\alpha,\beta}G(x)\|X\big](\eta)
\end{aligned}\tag{3.4}$$

where $w_0(t)=\frac{t}{T}$.

FORMULA 3.3. Applying Theorem 2.5 to the product of conditional integral transforms of F and G yields

$$\begin{aligned}
&E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\delta\mathcal{F}_{\alpha,\beta}(G\|X)(x|w_1,\eta_2)\\
&+\delta\mathcal{F}_{\alpha,\beta}(F\|X)(x|w_1,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(x,\eta_2)\|X\big](\eta_3)\\
&=\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}(F\|X)\big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_3,\eta_1\big)\\
&\mathcal{F}_{\alpha,\beta}(G\|X)\big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_3,\eta_2\big)\langle w_1',x\rangle\,m(dx).
\end{aligned}\tag{3.5}$$

Also both sides of (3.5) are given by the formula

$$
\begin{aligned}
&E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(x,\eta_2)\langle w_1', x\rangle\|X\big](\eta_3)\\
&+T(w_1', w_0')E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\delta\mathcal{F}_{\alpha,\beta}(G\|X)(x|w_0,\eta_2)\\
&+\delta\mathcal{F}_{\alpha,\beta}(F\|X)(x|w_0,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(x,\eta_2)\|X\big](\eta_3)\\
&-\eta_3(w', w_0')E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(x,\eta_2)\|X\big](\eta_3)
\end{aligned}
\tag{3.6}
$$

where $w_0(t) = \frac{t}{T}$.

Proof. Since the first variation satisfies the Leibnitz rule, that is, $\delta(PQ)(x|w_1) = P(x)\delta Q(x|w_1) + \delta P(x|w_1)Q(x)$, we have

$$
\begin{aligned}
&E_x\big[\delta\big(\mathcal{F}_{\alpha,\beta}(F\|X)(\cdot,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(\cdot,\eta_2)\big)(x|w_0)\|X\big](\eta_3)\\
&= E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\delta\mathcal{F}_{\alpha,\beta}(G\|X)(x|w_0,\eta_2)\\
&\quad+\delta\mathcal{F}_{\alpha,\beta}(F\|X)(x|w_0,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(x,\eta_2)\|X\big](\eta_3).
\end{aligned}
$$

Applying Theorem 2.5 with

$$
h(\cdot;\eta_1,\eta_2) \equiv \mathcal{F}_{\alpha,\beta}(F\|X)(\cdot,\eta_1)\mathcal{F}_{\alpha,\beta}(G\|X)(\cdot,\eta_2),
$$

we have the formulas (3.5) and (3.6). □

Formula 3.4. Choosing $G(x)$ to be identically equal to one on $C_0[0,T]$ in (3.5) and (3.6) yields

$$
\begin{aligned}
&E_x\big[\delta\mathcal{F}_{\alpha,\beta}(F\|X)(x|w_1,\eta_1)\|X\big](\eta_3)\\
&= \int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}(F\|X)\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta_3, \eta_1\big)\langle w_1', x\rangle\, m(dx).
\end{aligned}
\tag{3.7}
$$

Also both sides of (3.7) are given by the formula

$$
\begin{aligned}
&E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\langle w_1', x\rangle\|X\big](\eta_3)\\
&+T(w_1', {w_0}')E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\delta\mathcal{F}_{\alpha,\beta}(G\|X)(x|w_0,\eta_2)\\
&-\eta_3(w_1', {w_0}')E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\|X\big](\eta_3)
\end{aligned}
\tag{3.8}
$$

where $w_0(t) = \frac{t}{T}$.

Formula 3.5. Choosing $G(x) = F(x)$ in Formula 3.3 yields

$$
\begin{aligned}
&E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\delta\mathcal{F}_{\alpha,\beta}(F\|X)(x|w_1,\eta_1)\|X\big](\eta_3)\\
&= \frac{1}{2}\int_{C_0[0,T]} \Big[\mathcal{F}_{\alpha,\beta}(F\|X)\big(x(\cdot) - \frac{\cdot}{T}x(T) + \frac{\cdot}{T}\eta_3, \eta_1\big)\Big]^2\langle w_1', x\rangle\, m(dx).
\end{aligned}
\tag{3.9}
$$

Also both sides of (3.9) are given by the formula

$$
\begin{aligned}
&\frac{1}{2}E_x\big[\big(\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\big)^2\langle w_1',x\rangle\|X\big](\eta_3)\\
(3.10)\quad &+T(w_1',w_0')E_x\big[\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\delta\mathcal{F}_{\alpha,\beta}(F\|X)(x|w_0,\eta_1)\|X\big](\eta_3)\\
&-\frac{1}{2}\eta_3(w',w_0')E_x\big[\big(\mathcal{F}_{\alpha,\beta}(F\|X)(x,\eta_1)\big)^2\|X\big](\eta_3)
\end{aligned}
$$

where $w_0(t)=\frac{t}{T}$.

In our final formula we express the conditional Wiener integral of the product of the first variation of conditional integral transform and conditional integral transform as the sum of Wiener integrals involving conditional integral transform of a conditional convolution products of the functionals.

In [10], we see that

$$
\begin{aligned}
&\mathcal{F}_{\alpha,\beta}\big(((F*G)_\alpha\|X)(\cdot,\eta_1)\|X\big)(x,\eta_2)\\
&=\mathcal{F}_{\alpha,\beta}(F\|X)\big(\frac{x}{\sqrt{2}},\frac{\eta_1+\eta_2}{\sqrt{2}}\big)\mathcal{F}_{\alpha,\beta}(G\|X)\big(\frac{x}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}\big).
\end{aligned}
$$

Taking the first variation of the last expression and applying Theorem 2.2, we have the following formula.

Formula 3.6. Applying Theorem 2.5 to $\mathcal{F}_{\alpha,\beta}(F*G)_\alpha$ yields the formula

(3.11)

$$
\begin{aligned}
&E_x\Big[\delta\mathcal{F}_{\alpha,\beta}(F\|X)\big(\cdot,\frac{\eta_1+\eta_2}{\sqrt{2}}\big)\big(\frac{x}{\sqrt{2}}|\frac{w_1}{\sqrt{2}}\big)\mathcal{F}_{\alpha,\beta}(G\|X)\big(\frac{x}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}\big)\\
&+\mathcal{F}_{\alpha,\beta}(F\|X)\big(\frac{x}{\sqrt{2}},\frac{\eta_1+\eta_1}{\sqrt{2}}\big)\\
&\delta\mathcal{F}_{\alpha,\beta}(G\|X)\big(\cdot,\frac{\eta_2-\eta_1}{\sqrt{2}}\big)\big(\frac{x}{\sqrt{2}}|\frac{w_1}{\sqrt{2}}\big)\|X\Big](\eta_3)\\
=&\int_{C_0[0,T]}\mathcal{F}_{\alpha,\beta}((F*G)_\alpha\|X)\big(x(\cdot)-\frac{\cdot}{T}x(T)+\frac{\cdot}{T}\eta_2,\eta_1\big)\langle w_1',x\rangle\, m(dx).
\end{aligned}
$$

Also both sides of (3.11) are given by the formula

$$
\begin{aligned}
&E_x\Big[\mathcal{F}_{\alpha,\beta}\big(((F*G)_\alpha\|X)(\cdot,\eta_1)\|X\big)(x,\eta_2)\langle w_1',x\rangle\|X\Big](\eta_3)\\
(3.12)\quad &+T(w_1',w_0')E_x\Big[\delta\big(\mathcal{F}_{\alpha,\beta}\big(((F*G)_\alpha\|X)\big)(\cdot,\eta_1)\|X\big)(x|w_0)\|X\Big](\eta_3)\\
&-\eta_3(w_1',w_0')E_x\Big[\mathcal{F}_{\alpha,\beta}\big(((F*G)_\alpha\|X)\big)(\cdot,\eta_1)(x,eta_2)\|X\Big](\eta_3)
\end{aligned}
$$

where $w_0(t)=\frac{t}{T}$.

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