A NOTE ON ∗-PARANORMAL OPERATORS AND RELATED CLASSES OF OPERATORS

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Abstract. We shall show that the Riesz idempotent $E_\lambda$ of every ∗-paranormal operator $T$ on a complex Hilbert space $\mathcal{H}$ with respect to each isolated point $\lambda$ of its spectrum $\sigma(T)$ is self-adjoint and satisfies $E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*$. Moreover, Weyl’s theorem holds for ∗-paranormal operators and more general for operators $T$ satisfying the norm condition $\|T^nx\| \leq \|T^n\|\|x\|^{n-1}$ for all $x \in \mathcal{H}$. Finally, for this more general class of operators we find a sufficient condition such that $E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*$ holds.

1. Introduction

Let $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $T \in B(\mathcal{H})$. An operator $T$ is said to be ∗-paranormal operator $T$ if

$$\|T^*x\|^2 \leq \|T^2x\| \|x\|$$

for all $x \in \mathcal{H}$. The class of ∗-paranormal operators is a generalization of the class of hyponormal operators (i.e., operators satisfying $T^*T \geq TT^*$), and several interesting properties have been proved by many authors. For example, if $T$ is a ∗-paranormal operator, then $T$ is normaloid, i.e., $\|T\| = r(T) = \sup\{|z| : z \in \sigma(T)\}$, and $(T - \lambda)x = 0$ implies $(T - \lambda)^*x = 0$ ([1], [8]). There is another natural generalization of hyponormal operators called paranormal operators, which satisfy

$$\|Tx\|^2 \leq \|T^2x\| \|x\|$$

for all $x \in \mathcal{H}$. It is known that a paranormal operator $T$ is normaloid and $T^{-1}$ is also paranormal if $T$ is invertible. Moreover $(T - \lambda)x = 0$ implies $(T - \lambda)^*x = 0$ if $\lambda \neq 0$ is an isolated point of spectrum of $T$. However it was not known whether $T^{-1}$ must also be ∗-paranormal if $T$ is an invertible ∗-paranormal operator. One of the main goals of this paper is to show that there

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exists an invertible \( * \)-paranormal operator \( T \) such that \( T^{-1} \) is not \( * \)-paranormal. We also show if \( T \) is an invertible \( * \)-paranormal operator, then
\[
\| T^{-1} \| \leq r(T^{-1})^3 r(T)^2.
\]
Using this and a more general inequality, we shall show several properties of \( * \)-paranormal operators and class \( \mathfrak{P}(n) \) operators, i.e., operators satisfying
\[
\| T^n x \| \leq \| T x \| \| x \|^{n-1}
\]
for all \( x \in \mathcal{H} \) for \( n \geq 2 \).

We remark that an operator in \( \mathfrak{P}(2) \) is called of class (N) by V. Istrătescu, T. Saitô and T. Yoshino in [6] and paranormal by T. Furuta in [4], and an operator in \( \mathfrak{P}(n) \) is called \( n \)-paranormal [2] and also called \((n-1)\)-paranormal, e.g., [3], [7]. In order to avoid confusion we use the notation \( \mathfrak{P}(n) \). S. M. Patel [8] proved that \( * \)-paranormal operators belong to the class \( \mathfrak{P}(3) \). It is known that paranormal operators are in \( \mathfrak{P}(n) \) for \( n \geq 3 \) (see the proof of Theorem 1 of [6]), but there is no inclusion relation between the class of paranormal operators and the class of \( * \)-paranormal operators.

The Riesz idempotent \( E_\lambda \) of an operator \( T \) with respect to an isolated point \( \lambda \) of \( \sigma(T) \) is defined as
\[
E_\lambda = \frac{1}{2\pi i} \int_{\partial D_\lambda} (z - T)^{-1} dz,
\]
where the integral is taken in the positive direction and \( D_\lambda \) is a closed disk with center \( \lambda \) and small enough radius \( r \) such as \( D_\lambda \cap \sigma(T) = \{ \lambda \} \). Then \( \sigma(T|_{E_\lambda \mathcal{H}}) = \{ \lambda \} \) and \( \sigma(T|_{(1-E_\lambda)\mathcal{H}}) = \sigma(T) \setminus \{ \lambda \} \). In [9], Uchiyama proved that for every paranormal operator \( T \) and each isolated point \( \lambda \) of \( \sigma(T) \) the Riesz idempotent \( E_\lambda \) satisfies that
\[
E_0 \mathcal{H} = \ker T,
E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^* \text{ and } E_\lambda \text{ is self-adjoint if } \lambda \neq 0.
\]
We shall show that for every \( * \)-paranormal operator \( T \) and each isolated point \( \lambda \in \sigma(T) \) the Riesz idempotent \( E_\lambda \) of \( T \) with respect to \( \lambda \) is self-adjoint with the property that \( E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^* \).

Let \( w(T) \) be the Weyl spectrum of \( T \), \( \pi_{00}(T) \) the set of all isolated points of \( \sigma(T) \) which are eigenvalues of \( T \) with finite multiplicities, i.e.,
\[
\begin{align*}
\sigma(T) &\setminus w(T) = \pi_{00}(T), \\
w(T) &\setminus \{ \lambda \in \sigma(T) \mid T - \lambda \text{ is not Fredholm with Fredholm index } 0 \}, \\
\pi_{00}(T) &\setminus \{ \lambda \in \text{iso}(\sigma(T)) \mid 0 < \dim \ker(T - \lambda) < \infty \}.
\end{align*}
\]
An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to satisfy Weyl’s theorem if
\[
\sigma(T) \setminus w(T) = \pi_{00}(T),
\]
also \( T \) is said to have the single valued extension property (SVEP) at \( \lambda \) if for any open neighborhood \( U \) of \( \lambda \) and analytic function \( f : U \to \mathcal{H} \) the zero function is only analytic solution of the equation
\[
(T - z)f(z) = 0,
\]
and $T$ is said to have the SVEP if $T$ has the SVEP at any $\lambda \in \mathbb{C}$ (or equivalently $\lambda \in \sigma(T)$).

It is well-known that every normal, hyponormal, $p$-hyponormal, $w$-hyponormal, class $A$, or paranormal operator satisfies Weyl's theorem and has the SVEP (see [9], [10] for definitions). We shall show that every $*$-paranormal operator satisfies Weyl's theorem. Y. M. Han and A. H. Kim [5] introduced totally $*$-paranormal operators $T$, i.e., operators for which $T - \lambda$ is $*$-paranormal for every $\lambda \in \mathbb{C}$, and they proved that every totally $*$-paranormal operator satisfies Weyl's theorem. Hence our result shows that the condition “totally” is not necessary. Also we shall show that every $*$-paranormal operator and every operator in the class $P(n)$ for $n \geq 2$ has the SVEP. The case of $P(n)$ for $n \geq 3$ was already proved by B. P. Duggal and C. S. Kubrusly [3] but we give another proof. We also show more general results for operators in the class $P(n)$ for $n \geq 2$.

2. $*$-paranormal operators

Let $T$ be a $*$-paranormal operator, i.e., $\|T^*x\|^2 \leq \|T^2x\|\|x\|$ for all $x \in \mathcal{H}$. It is well-known that $T \in P(3)$. Indeed,

$$\|Tx\|^2 \leq \|T^*Tx\|\|x\| \leq \sqrt{\|T^3x\|\|Tx\|\|x\|}$$

for all $x \in \mathcal{H}$.

(1) $\|Tx\|^3 \leq \|T^3x\|\|x\|^2$ for all $x \in \mathcal{H}$.

Therefore every $*$-paranormal operator belongs to the class $P(3)$ (see [1], [8]).

Proposition 1 and Lemmas 2, 3 and 4 are also well-known (see [1], [3], [7], [8]). For the convenience we give proofs of them.

**Proposition 1.** Every $*$-paranormal operator $T$ and every operator in the class $P(n)$ for $n \geq 2$ is normaloid, i.e., the operator norm $\|T\|$ is equal to the spectral radius $r(T)$.

Proposition 1 follows from Lemma 1.

**Lemma 1.** If $T$ is $*$-paranormal or belongs to class $P(n)$ for $n \geq 2$ and $\{x_m\}$ is a sequence of unit vectors in $\mathcal{H}$ which satisfies $\lim_{m \to \infty} \|Tx_m\| = \|T\|$, then

$$\lim_{m \to \infty} \|T^kx_m\| = \|T\|^k$$

for all $k \in \mathbb{N}$. Hence $\|T^k\| = \|T\|^k$ for all $k \in \mathbb{N}$.

**Proof.** Let $T$ be $*$-paranormal. By the inequality (1), for every unit vector $x \in \mathcal{H}$ we have

$$\|Tx\|^3 \leq \|T^2x\| \leq \|T\|\|T^2x\| \leq \|T\|^3,$$

therefore if $\|Tx_m\| \to \|T\|$ as $m \to \infty$, then $\|T^2x_m\| \to \|T\|^2$ and $\|T^3x_m\| \to \|T\|^3$ as $m \to \infty$. Let $k \in \mathbb{N}$ satisfy $\lim_{m \to \infty} \|T^lx_m\| = \|T\|^l$ for all $l = 1, \ldots, k$. Since $T \in P(3)$, it follows that

$$\|T^kx_m\|^3 = \|T \cdot T^{k-1}x_m\|^3 \leq \|T^3 \cdot T^{k-1}x_m\|\|T^{k-1}x_m\|^2.$$
\[
\leq \|T^{k+2}x_m\| \|T^{k-1}x_m\|^2 \leq \|T\|^2 (k-1) \|T^{k+2}x_m\|
\]
\[
\leq \|T\|^{2k-1} \|T^{k+1}x_m\| \leq \|T\|^{3k}.
\]
This implies that \(\lim_{m \to \infty} \|T^{k+1}x_m\| = \|T\|^{k+1}\). By the induction, the assertion follows.

Next, let \(T \in \mathfrak{P}(n)\) for \(n \geq 2\). The inequality
\[
\|Tx_m\|^n \leq \|T^n x_m\| \leq \|T\|^{n-1} \|T^l x_m\| \leq \|T\|^n
\]
implies that \(\lim_{m \to \infty} \|T^l x_m\| = \|T\|^l\) for all \(l = 1, \ldots, n\). By using same argument as above we also have \(\lim_{m \to \infty} \|T^k x_m\| = \|T\|^k\) for all \(k \in \mathbb{N}\).

**Theorem 1.** Let \(T\) be an invertible \(*\)-paranormal operator. Then
\[
(2) \quad \|T^{-1}\| \leq r(T^{-1})^3 r(T)^2.
\]
More generally, if \(T \in \mathfrak{P}(n)\) for \(n \geq 3\) is invertible, then
\[
(3) \quad \|T^{-1}\| \leq r(T^{-1})^{\frac{n(n-1)}{2}} r(T)^{-\frac{3(n+1)(n+2)}{2}}.
\]
In particular, if \(T\) is \(*\)-paranormal or in the class \(\mathfrak{P}(n)\) for \(n \geq 3\) and \(\sigma(T) \subset S^1 := \{z \in \mathbb{C} \mid \|z\| = 1\}\), then \(T\) is unitary.

**Proof.** It is sufficient to consider the case where \(T \in \mathfrak{P}(n)\) for \(n \geq 3\). Since \(S = T^{-1}\) satisfies
\[
\|S^{n-1}x\|^n \leq \|S^n x\|^n \|x\|
\]
for \(x \in H\), we have \(\|S^{n-1+k}x\|^n \leq \|S^n x\|^n \|S^k x\|\) for every non-negative integer \(k\). Then for any \(x \neq 0\) we have
\[
\prod_{k=0}^{l} \left( \frac{\|S^{n-1+k}x\|}{\|S^n x\|} \right)^{n-1} \leq \prod_{k=0}^{l} \frac{\|S^k x\|}{\|S^{n-1+k}x\|}
\]
and hence
\[
\left( \frac{\|S^{n-1}x\|^{n-1}}{\|S^n x\|^{n-1}} \right)^{n-1} \leq \frac{1}{\|S^{l+1}x\| \|S^{l+2}x\| \cdots \|S^{n-1+l}x\|}.
\]
Then
\[
\left( \frac{\|S^{n-1}x\|^{(L+1)(n-1)}}{\|S^n x\|^{(L+1)n-2} \|S^{l+1}x\| \|S^{l+2}x\| \cdots \|S^{n-1+l}x\|} \right)^{n-1}
\]
and
\[
\|S^{n-1}x\|^{(L+1)(n-1)} \|S^n x\|^{n-2} \|S^{n-2}x\| \|S^{n-2}x\| \cdots \|S^{n-2}x\| \|S^{n-2}x\|\]
\[
\leq \left( \left( \|S^n x\| \|S^{n+1}x\| \cdots \|S^{n+L}x\| \right)^{L+1} \right)^{n-1}.
\]
So we have
\[ \|S^{n-1}x\|^{(L+2)(n-1)} \]
\[ \leq \|x\|^{L+1}\|Sx\|^L \cdots \|S^{n-2}x\|^L \cdots \|S^{L+2}x\|^L \cdot \|S^{L+3}x\|^L \cdots \|S^{L+n}x\|^{n-1}, \]
and
(4)
\[ \|S^{n-1}x\|^{\frac{(L+2)(n-1)}{L+1}} \]
\[ \leq \|x\|\|Sx\|^\frac{L}{L+1} \cdots \|S^{n-2}x\|^\frac{L}{L+1} \cdots \|S^{L+2}x\|^\frac{L}{L+1} \cdots \|S^{L+n}x\|^{\frac{n-1}{n}}, \]
By letting \( L \to \infty \) in (4) we have
\[ \|S^{n-1}x\|^{n-1} \leq \|x\|\|Sx\| \cdots \|S^{n-2}x\| r(S)^2 \cdots r(S)^{n-1}. \]
Therefore
\[ \left\| S S^{n-2} x \right\| \leq \left( \prod_{k=2}^{n-1} T^k \frac{S^{n-1} x}{\|S^{n-1} x\|} \right) r(S)^{\frac{n(n-1)}{2}} \]
\[ \leq \left( \prod_{k=2}^{n-1} \|T\|^k \right) r(S)^{\frac{n(n-1)}{2}} = r(T)^{\frac{n(n-1)}{2} + \frac{2(n-2)}{n}} r(S)^{\frac{n(n-1)}{2}} , \]
and
\[ \|T^{-1}\| \leq r(T)^{\frac{n+1(n-2)}{2}} r(T^{-1})^{\frac{n(n-1)}{2}}. \]
Since every \(*\)-paranormal operator \( T \) belongs to the class \( \mathfrak{P}(3) \), so if \( T \) is invertible, then
\[ \|T^{-1}\| \leq r(T)^2 r(T^{-1})^3 . \]
Finally, if \( T \) is \(*\)-paranormal or belongs to the class \( \mathfrak{P}(n) \) such that \( \sigma(T) \subset S^1 \), then \( r(T) = r(T^{-1}) = 1 \). Hence, \( \|T\| = r(T) = 1 \) and \( 1 = r(T^{-1}) \leq \|T^{-1}\| \leq r(T)^{\frac{n+1(n-2)}{2}} r(T^{-1})^{\frac{n(n-1)}{2}} = 1 \) implies \( \|T^{-1}\| = 1 \). It follows that \( T \) is invertible and an isometry because
\[ \|x\| = \|T^{-1}Tx\| \leq \|Tx\| \leq \|x\| \]
for all \( x \in \mathcal{H} \), so \( T \) is unitary. \( \square \)

Remark 1. Theorem 1 also holds for \( n = 2 \). If \( T \) is in the class \( \mathfrak{P}(2) \), then \( T \) is paranormal and normaloid. Hence if \( T \) is invertible, then \( T^{-1} \) is also paranormal and normaloid, i.e., \( r(T) = \|T\| \) and \( r(T^{-1}) = \|T^{-1}\| \). Hence if \( \sigma(T) \subset S^1 \), then \( T \) is unitary.

Corollary 1. Let \( T \) be \(*\)-paranormal or belong to the class \( \mathfrak{P}(n) \) for \( n \geq 2 \). If \( \sigma(T) = \{ \lambda \} \), then \( T = \lambda \).

Proof. If \( \lambda = 0 \), then \( \|T\| = r(T) = 0 \) by Theorem 1. Hence \( T = 0 \).
If \( \lambda \neq 0 \), then \( \frac{1}{\lambda} T \) is unitary with \( \sigma(\frac{1}{\lambda} T) = \{1\} \). Hence \( T = \lambda \). \( \square \)

Lemma 2. If \( T \) is \(*\)-paranormal and \( M \) is a \( T \)-invariant closed subspace, then the restriction \( T_{\mid M} \) of \( T \) to \( M \) is also \(*\)-paranormal.
Proof. Let $P$ be the orthogonal projection onto $\mathcal{M}$. Since $TP = PTP$, we have
\[ \| (T|_{\mathcal{M}})^* x \|^2 = \| PT^* P x \|^2 = \| PT^* x \|^2 \leq \| T^* x \|^2 \leq \| T^2 x \| \| x \| = \| (T|_{\mathcal{M}})^2 \| \| x \| \]
for all $x \in \mathcal{M}$. Thus $T|_{\mathcal{M}}$ is $*$-paranormal. \qed

Similarly, the following is proved by C. S. Kubrusly and B. P. Duggal [7].

Lemma 3 ([7]). If $T \in \Psi(n)$ for $n \geq 2$ and $\mathcal{M}$ is a $T$-invariant closed subspace, then the restriction $T|_{\mathcal{M}}$ of $T$ to $\mathcal{M}$ also belongs to the class $\Psi(n)$.

Lemma 4 ([1]). If $T$ is $*$-paranormal, $\lambda \in \sigma_p(T)$ and a vector $x \in \mathcal{H}$ satisfies $(T - \lambda)x = 0$, then $(T - \lambda)^* x = 0$.

Proof. Without loss of generality we may assume $\|x\| = 1$.
\[ \| T^* x \|^2 \leq \| T^2 x \| \| x \| = |\lambda|^2 \| x \|^2 = |\lambda|^2 \]
implies that $\| T^* x \| \leq |\lambda|$. Hence
\[ 0 \leq \| (T - \lambda)^* x \|^2 = \| T^* x \|^2 - 2 \Re(T^* x, \overline{\lambda} x) + |\lambda|^2 \leq |\lambda|^2 - 2 \Re(x, T \overline{\lambda} x) + |\lambda|^2 = 2|\lambda|^2 - 2|\lambda|^2 = 0. \]
\qed

Lemma 5. Let $T$ be $*$-paranormal or belong to the class $\Psi(n)$ for $n \geq 2$, $\lambda \in \mathbb{C}$ an isolated point of $\sigma(T)$ and $E_\lambda$ the Riesz idempotent with respect to $\lambda$. Then
\[ (T - \lambda)E_\lambda = 0. \]
The $\lambda$ is an eigenvalue of $T$. Therefore $T$ is isoloid, i.e., every isolated point of $\sigma(T)$ is an eigenvalue of $T$.

Proof. The Riesz idempotent $E_\lambda$ satisfies $\sigma(T|_{E_\lambda \mathcal{H}}) = \{ \lambda \}$ and $\sigma(T|_{(1 - E_\lambda)\mathcal{H}}) = \sigma(T) \setminus \{ \lambda \}$.

Since $T|_{E_\lambda \mathcal{H}}$ is also $*$-paranormal or belongs to the class $\Psi(n)$ it follows that $(T - \lambda)E_\lambda = (T|_{E_\lambda \mathcal{H}} - \lambda)E_\lambda = 0$ by Corollary 1. Hence $\lambda \in \sigma_p(T)$ \qed

Theorem 2. Let $T \in \Psi(n)$ for $n \geq 2$, $\lambda$ an isolated point of $\sigma(T)$ and $E_\lambda$ the Riesz idempotent with respect to $\lambda$. Then
\[ E_\lambda \mathcal{H} = \ker(T - \lambda). \]

Proof. In Lemma 5, we have already shown $E_\lambda \mathcal{H} \subset \ker(T - \lambda)$. Let $x \in \ker(T - \lambda)$. Then
\[ E_\lambda x = \frac{1}{2\pi i} \int_{\partial D_\lambda} (z - T)^{-1} x \, dz = \left( \frac{1}{2\pi i} \int_{\partial D_\lambda} \frac{1}{z - \lambda} \, dz \right) x = x, \]
so $x \in E_\lambda \mathcal{H}$. This completes the proof of $E_\lambda \mathcal{H} = \ker(T - \lambda)$. \qed
**Theorem 3.** Let $T$ be a $*$-paranormal operator, $\lambda \in \sigma(T)$ an isolated point and $E_\lambda$ the Riesz idempotent with respect to $\lambda$. Then

$$E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^*.$$  

In particular, $E_\lambda$ is self-adjoint, i.e., it is an orthogonal projection.

**Proof.** It suffices to show that $\ker(T - \lambda) = \ker(T - \lambda)^*$. The inclusion $\ker(T - \lambda) \subset \ker(T - \lambda)^*$ holds by Lemma 4 and hence $E_\lambda H = \ker(T - \lambda)$ reduces $T$. 

Put $T = \lambda \oplus T_2$ on $H = E_\lambda H \oplus (E_\lambda H)^\perp$. If $\lambda \in \sigma(T_2)$, then $\lambda$ is an isolated point of $\sigma(T_2)$. Since $T_2$ is also $*$-paranormal by Lemma 2, $\lambda \in \sigma_p(T_2)$ by Lemma 5. Since $\ker(T_2 - \lambda) \subset \ker(T - \lambda)$, we have

$$\{0\} \neq \ker(T_2 - \lambda) \subset \ker(T - \lambda) \cap (\ker(T - \lambda))^\perp = \{0\},$$

and it is a contradiction. Hence $\lambda \notin \sigma(T_2)$ and $T_2 - \lambda$ is invertible. This implies $\ker(T - \lambda)^* \subset \ker(T - \lambda)$ and $\ker(T - \lambda)^* = \ker(T - \lambda)$. Finally, we show $E_\lambda = E^*_\lambda$. Consider the $E_\lambda$ on $H = E_\lambda H \oplus (E_\lambda H)^\perp$ in its block operator form $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$. Observe that $E_\lambda$ and $T = \lambda \oplus T_2$ commute and that $(T_2 - \lambda)$ is invertible. This implies $B = 0$. Hence $E_\lambda$ is self-adjoint. \qed

**Theorem 4.** Weyl’s theorem holds for $*$-paranormal operator, i.e.,

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

**Proof.** Let $\lambda \in \sigma(T) \setminus w(T)$. Then $T - \lambda$ is Fredholm with $\text{ind}(T - \lambda) = 0$ and is not invertible. Hence $\lambda \notin \sigma_p(T)$ and $0 < \dim \ker(T - \lambda) < \infty$. By Theorem 3, $\ker(T - \lambda)$ reduces $T$, so $T = \lambda \oplus T_2$ on $H = \ker(T - \lambda) \oplus (\ker(T - \lambda))^\perp$. If $\lambda \notin \text{iso}(\sigma(T))$, then $\lambda \in \sigma(T_2)$. Since $T - \lambda$ is a Fredholm operator with $\text{ind}(T - \lambda) = 0$ and $\ker(T - \lambda)$ is finite dimensional subspace the operator $T_2 - \lambda$ is also Fredholm with $\text{ind}(T_2 - \lambda) = 0$. Hence, $\ker(T_2 - \lambda) \neq \{0\}$. However, this is a contradiction since

$$\{0\} \neq \ker(T_2 - \lambda) \subset (\ker(T - \lambda))^\perp \cap \ker(T - \lambda) = \{0\}.$$

Therefore $\lambda \in \text{iso}(\sigma(T))$ and $\lambda \in \pi_{00}(T)$.

Conversely, let $\lambda \in \pi_{00}(T)$. Then $0 < \dim \ker(T - \lambda) < \infty$. Since $\ker(T - \lambda)$ reduces $T$, the operator $T$ is of the form $T = \lambda \oplus T_2$ on $H = \ker(T - \lambda) \oplus (\ker(T - \lambda))^\perp$. If $\lambda \in \sigma(T_2)$, then $\lambda$ is an isolated point of $\sigma(T_2)$ and hence $\lambda \in \sigma_p(T_2)$ by Lemma 5. However $\ker(T_2 - \lambda) \subset \ker(T - \lambda) \cap (\ker(T - \lambda))^\perp = \{0\}$ implies $\ker(T_2 - \lambda) = \{0\}$, contradiction. So, $T_2 - \lambda$ is invertible and $\text{ind}(T - \lambda) = \text{ind}(T_2 - \lambda) = 0$. Hence $\lambda \in \sigma(T) \setminus w(T)$. \qed

For an operator $T$, we denote the approximate point spectrum of $T$ by $\sigma_a(T)$, i.e., $\sigma_a(T)$ is the set of all $\lambda \in \mathbb{C}$ such that there exists a sequence $\{x_n\}$ of unit vectors in $H$ which satisfies

$$\|T - \lambda\| x_n \| \to 0 \quad (as \ n \to \infty).$$

In [10], the authors defined spectral properties (I) and (II) as follows and proved that each property implies SVEP.
(I) if \( \lambda \in \sigma_a(T) \) and \( \{x_n\} \) is a sequence of bounded vectors of \( \mathcal{H} \) satisfying \( \| (T - \lambda)x_n \| \to 0 \) (as \( n \to \infty \)), then \( \| (T - \lambda)^*x_n \| \to 0 \) (as \( n \to \infty \)).

(II) if \( \lambda, \mu \in \sigma_a(T) \) (\( \lambda \neq \mu \)) and sequences of bounded vectors \( \{x_n\} \) and \( \{y_n\} \) of \( \mathcal{H} \) satisfy \( \| (T - \lambda)x_n \| \to 0 \) and \( \| (T - \mu)y_n \| \to 0 \) (as \( n \to \infty \)), then \( \langle x_n, y_n \rangle \to 0 \) (as \( n \to \infty \)), where \( \langle \cdot, \cdot \rangle \) is the inner product on \( \mathcal{H} \).

**Theorem 5.** Every \( \ast \)-paranormal operator \( T \) has the spectral property (I), so \( T \) has SVEP.

**Proof.** Let \( \lambda \in \sigma_a(T) \) and \( \{x_n\} \) be a sequence of bounded vectors of \( \mathcal{H} \) satisfying \( \| (T - \lambda)x_n \| \to 0 \) (as \( n \to \infty \)). Then

\[
\| (T - \lambda)^*x_n \|^2 = \| T^*x_n \|^2 - 2\text{Re}(\lambda x_n, T^*x_n) + |\lambda|^2 \| x_n \|^2
= \| T^2x_n \| \| x_n \| - 2\text{Re}(\lambda T^2x_n, x_n) + |\lambda|^2 \| x_n \|^2
= |\lambda|^2 \| x_n \|^2 - 2|\lambda|^2 \| x_n \|^2 + \lambda^2 \| x_n \|^2 + O(\| (T - \lambda)x_n \|)
\]

\[
\to 0 \quad (n \to \infty).
\]

\( \square \)

**Remark 2.** According to [7], it is still unknown whether the inverse of an invertible operator in \( \mathfrak{P}(n) \) is normaloid and so whether an operator in \( \mathfrak{P}(n) \) is totally hereditarily normaloid. Example 1 shows that there exists a \( \ast \)-paranormal operator which is not paranormal. Example 2 shows there are invertible \( \ast \)-paranormal operators \( T \) such that \( T^{-1} \) are not normaloid, so not \( \ast \)-paranormal. Hence a \( \mathfrak{P}(n) \) operator is not totally hereditarily normaloid in general. Moreover, these examples show that the inequality (2) is sharp.

**Example 1.** Let \( \{e_n\}_{n=1}^\infty \) be an orthonormal base of \( \mathcal{H} \) and \( T \) be a weighted shift operator defined by

\[
Te_n = \begin{cases} \sqrt{2}e_2 & (n = 1), \\
e_3 & (n = 2), \\
2e_{n+1} & (n \geq 3). 
\end{cases}
\]

Then \( T^2T^2 = 2 \oplus 1 \oplus \left( \bigoplus_{n=3}^\infty 16 \right), \ T^2T = 2 \oplus 1 \oplus \left( \bigoplus_{n=3}^\infty 4 \right) \) and \( TT^* = 0 \oplus 2 \oplus 1 \oplus \left( \bigoplus_{n=4}^\infty 1 \right) \). It is well-known that an operator \( S \) is \( \ast \)-paranormal if and only if \( S^2 S^2 - 2kSS^* + k^2 \geq 0 \) for all \( k > 0 \) and also well-known that \( S \) is paranormal if and only if \( S^2 S^2 - 2kSS^* + k^2 \geq 0 \) for all \( k > 0 \). We shall show that \( T \) is \( \ast \)-paranormal but not paranormal. Since

\[
T^2T^2 - 2kTT^* + k^2
= (2 + k^2) \oplus (4 - 4k + k^2) \oplus (16 - 2k + k^2) \oplus \left( \bigoplus_{n=4}^\infty (16 - 8k + k^2) \right)
\]

\[
= (2 + k^2) \oplus (k - 2)^2 \oplus \left( \bigoplus_{n=4}^\infty (k - 4)^2 \right) \geq 0
\]

for all \( k > 0 \), \( T \) is \( \ast \)-paranormal. However, since

\[
T^2T^2 - 2kTT^* + k^2
\]
Thus, the inequality \(|c|\|\| \leq \left( \sum_{n=3}^{\infty} (16 - 8k + k^2) \right)
= (2 - 4k + k^2) \oplus (4 - 2k + k^2) \oplus \left( \sum_{n=3}^{\infty} (k^2 - 2) \oplus \left( \left(16 - 8k + k^2\right) \right) \right)
= \left\{(k - 2)^2 - 2\right\} \oplus \left\{(k - 1)^2 + 3\right\} \oplus \left( \sum_{n=3}^{\infty} (k - 4)^2 \right) \geq 0
\]
for \(k = 2\), \(T\) is not paranormal.

**Example 2.** Let \(a > 1\), \(\{e_n\}_{n=1}^{\infty}\) be an orthonormal base of \(\mathcal{H}\) and \(T_a\) a weighted shift defined by
\[
T_a e_n = \begin{cases}
\sqrt{a}e_{n+1} & (n \leq -2), \\
a_0 & (n = -1), \\
e_1 & (n = 0), \\
a^2e_{n+1} & (n \geq 1).
\end{cases}
\]

Then
\[
(T_a^*)^2(T_a)^2 = \left( \sum_{n=3}^{\infty} a^2 \right) \oplus a^3 \oplus a^2 \oplus a^1 \oplus (\sum_{n=1}^{\infty} a^8),
T_a(T_a)^* = \left( \sum_{n=-1}^{0} a \right) \oplus a^2 \oplus 1 \oplus (\sum_{n=2}^{\infty} a^4).
\]

Thus,
\[
(T_a^*)^2(T_a)^2 - 2kT_a(T_a)^* + k^2
= \left( \sum_{n=-3}^{0} (a - k)^2 \right) \oplus \left\{(a - k)^2 + a^2 - a^2\right\} \oplus (a - k)^2
\oplus (a^2 - k)^2 \oplus \left\{(1 - k)^2 + a^8 - 1\right\} \oplus (\sum_{n=2}^{\infty} (a^4 - k^2)) \geq 0
\]
for all \(k > 0\). Therefore \(T_a\) is \(*\)-paranormal. Since \(\|T_a^{-1}\| = 1\), \(r(T_a) = a^2\) and \(r(T_a^{-1}) = \frac{1}{\sqrt{a}}\), we have that \(T_a^{-1}\) is not normaloid and not paranormal. Since
\[
\frac{r(T_a^{-1})^3 \cdot r(T_a)^2}{a^2} \rightarrow 1 \quad (a \downarrow 1),
\]
the inequality \(\|T^{-1}\| \leq \frac{r(T^{-1})^3 \cdot r(T)^2}{a^2}\) is sharp in the sense that the least constant \(c\) which satisfies
\[
\|T^{-1}\| \leq c \cdot \frac{r(T^{-1})^3 \cdot r(T)^2}{a^2}
\]
for every \(*\)-paranormal operator \(T\) which is not paranormal is \(c = 1\).

3. The class \(\Psi(n)\)

**Lemma 6.** Let \(T \in \Psi(n)\) for \(n \geq 2\), \(\lambda \in \sigma_p(T)\). Put \(T = \left( \begin{array}{c} a & b \\ c & d \end{array} \right)\) on \(\mathcal{H} = \ker(T - \lambda) \oplus (\ker(T - \lambda))^\perp\). Then \(\lambda \not\in \sigma_p(T_2)\). In particular, if \(\lambda\) is isolated in \(\sigma(T)\), then \(T_2 - \lambda\) is invertible.

**Proof.** If \(\lambda \in \sigma_p(T_2)\), then \(\mathcal{M} := \ker(T - \lambda) \oplus \ker(T_2 - \lambda)\) is an invariant subspace of \(T\) and \((T - \lambda)^2\mathcal{M} = \{0\}\). The operator \(T|_{\mathcal{M}}\) belongs to the class
\(\Psi(n)\) by Lemma 3 and \(\sigma(T|_{\mathcal{M}}) = \{\lambda\}\), so \(T|_{\mathcal{M}} = \lambda\) by Corollary 1. This means that
\[
\{0\} \neq \ker(T_2 - \lambda) \subset \ker(T - \lambda) \cap (\ker(T - \lambda)^\perp) = \{0\},
\]
which is a contradiction. Hence \(\lambda \notin \sigma_p(T_2)\).

Next, we shall show the remaining assertion. Assume \(\lambda\) is isolated in \(\sigma(T)\). Suppose \(\lambda \in \sigma(T_2)\). Then \(\lambda\) is an isolated point of \(\sigma(T_2)\). Let \(F\) be the Riesz idempotent of \(T_2\) with respect to \(\lambda\). Then \(\mathcal{M}' := \ker(T - \lambda) \oplus F(\ker(T - \lambda))\perp\) is an invariant subspace of \(T\) and \(T|_{\mathcal{M}'}\) is of the form \(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\) with \(\sigma(T_3) = \{\lambda\}\). Thus \(T|_{\mathcal{M}'} \in \Psi(n)\) and \(\sigma(T|_{\mathcal{M}'}) = \{\lambda\}\), so \(T|_{\mathcal{M}'} = \lambda\) and hence \(T_3 = \lambda\) by Corollary 1 and hence \(S_1 = 0\). This implies \(\lambda \in \sigma_p(T_2)\), a contradiction. Therefore \(T_2 - \lambda\) is invertible.

**Theorem 6.** *Weyl’s theorem holds for operators in \(\Psi(n)\) for \(n \geq 2\).*

**Proof.** Let \(\lambda \in \sigma(T) \setminus w(T)\). Then \(0 < \dim \ker(T - \lambda) < \infty\) and \(T - \lambda\) is Fredholm with \(\text{ind}(T - \lambda) = 0\). Put \(T = \begin{pmatrix} \lambda & S \\ 0 & T_2 \end{pmatrix}\) on \(\mathcal{H} = \ker(T - \lambda) \oplus (\ker(T - \lambda))\perp\). Since \(\ker(T - \lambda)\) is finite dimensional, the operator \(S\) is a finite rank operator and the operator \(T_2 - \lambda\) is Fredholm with \(\text{ind}(T_2 - \lambda) = 0\). By Lemma 6, \(\ker(T_2 - \lambda) = \{0\}\) so \(T_2 - \lambda\) is invertible and hence \(\lambda\) is isolated in \(\sigma(T)\). Thus \(\lambda \in \pi_00(T)\).

Conversely, if \(\lambda \in \pi_00(T)\), then \(\lambda\) is isolated in \(\sigma(T)\) and \(0 < \dim \ker(T - \lambda) < \infty\). Put \(T = \begin{pmatrix} \lambda & S \\ 0 & T_2 \end{pmatrix}\) on \(\mathcal{H} = \ker(T - \lambda) \oplus (\ker(T - \lambda))\perp\). Then \(T_2 - \lambda\) is invertible by Lemma 6, so it is Fredholm with index 0. Since \(\ker(T - \lambda)\) is finite dimensional, \(T - \lambda\) is also Fredholm with index 0. Hence \(\lambda \in \sigma(T) \setminus w(T)\). \(\square\)

In [9], Uchiyama showed that if \(T\) is paranormal, i.e., \(T \in \Psi(2)\), then Weyl’s theorem holds for \(T\), and if \(\lambda\) is a non-zero isolated point of \(\sigma(T)\), then the Riesz idempotent \(E_\lambda\) of \(T\) with respect to \(\lambda\) is self-adjoint and
\[
E_\lambda \mathcal{H} = \ker(T - \lambda) = \ker(T - \lambda)^*.
\]

In the case of \(\lambda = 0\), it is well-known that the Riesz idempotent \(E_0\) is not necessarily self-adjoint.

The following example is a paranormal operator having zero as an isolated point of the spectrum, but the Riesz idempotent is not self-adjoint.

**Example 3.** Let \(\{e_n\}_{n=1}^\infty\) be an orthonormal base of \(\mathcal{H}\) and \(\{a_n\}_{n=-\infty}^\infty \subset [1, 2]\) satisfy \(a_n < a_{n+1}\) for all \(n \in \mathbb{Z}\). Let \(A\) be the weighted bilateral shift defined by
\[
 Ae_n = a_n e_{n+1} \quad (n \in \mathbb{Z}),
\]
\(S = (A^* A - AA^*)^{1/2}\). Then the operator \(T = \begin{pmatrix} \lambda & S \\ 0 & T_2 \end{pmatrix}\) on \(\mathcal{H} \oplus \mathcal{H}\) satisfies \(T^2 - T^2 = (T^* T)^2\) which means that \(T\) is paranormal. Observe that \(\sigma(T) = \sigma(A) \cup \{0\}\) and that \(A\) is invertible. This implies that 0 is an isolated point of \(\sigma(T)\). Let \(E_0\) the Riesz idempotent with respect to 0. Then
\[
 E_0 = \frac{1}{2\pi i} \int_{D_0} (z - T)^{-1} dz = \frac{1}{2\pi i} \int_{D_0} \left( z - A \right)^{-1} \begin{pmatrix} \frac{1}{2} & \frac{1}{z} \\ 0 & 1 \end{pmatrix} S \right) dz
\]
This $E_0$ satisfies $E_0 \mathcal{H} = \ker T$, but $E_0$ is not self-adjoint since $A^{-1}S \neq 0$.

For operators in $\Psi(n)$ ($n \geq 3$), it is still not known whether the Riesz idempotent $E_\lambda$ with respect to a non-zero isolated point $\lambda$ of the spectrum is self-adjoint or not.

Next, we shall show that the self-adjointness of the Riesz idempotent $E_\lambda$ for a $\Psi(n)$ operator ($n \geq 3$) with respect to a non-zero isolated point $\lambda$ of its spectrum under some additional assumptions.

Let $n \in \mathbb{N}, \lambda \in \mathbb{C}$. The polynomial

$$F_{n, \lambda}(z) := -(n - 1)\lambda^{n-1} + \lambda^{n-2}z + \lambda^{n-3}z^2 + \cdots + \lambda z^{n-2} + z^{n-1}$$

is important to study the class $\Psi(n)$.

**Theorem 7.** Let $T \in \Psi(n)$ for an $n \geq 3$ and $\lambda$ be a non-zero isolated point $\sigma(T)$. Put $T = \frac{1}{\lambda} S_\lambda$ on $\mathcal{H} = \ker(T - \lambda) \oplus (\ker(T - \lambda))^\perp$. Then

$$S(\lambda^{n-1} + \lambda^{n-2}A + \cdots + \lambda A^{n-1} + A^{n-1}) = n\lambda^{n-1}S.$$ 

In particular, if $\sigma(T) \cap \{z \in \mathbb{C} | F_{n, \lambda}(z) = 0\} = \{\lambda\}$, then the Riesz idempotent $E_\lambda$ with respect to $\lambda$ is self-adjoint and

$$E_\lambda \mathcal{H} = \ker(T - \lambda) = (T - \lambda)^*.$$

**Proof.** We remark that $\lambda \in \sigma_p(T)$ by Lemma 5. Without loss of generality, we may assume $\lambda = 1$. Let $x \in \ker(T - 1)$ and $y \in (\ker(T - 1))^\perp$ be arbitrary unit vectors and let $0 < \epsilon < 1$ be arbitrary. It follows that $T^n = (\frac{1}{\lambda} S_n)$ where $S_n = S(1 + A + \cdots + A^{n-1})$. Since $T \in \Psi(n)$, the inequality

$$||T((\sqrt{1 - \epsilon}x) \oplus (\sqrt{\epsilon}y))||^n \leq ||T((\sqrt{1 - \epsilon}x) \oplus (\sqrt{\epsilon}y))||^n$$

implies that

$$\left(||\sqrt{1 - \epsilon}x + \sqrt{\epsilon}Sy||^2 + ||\sqrt{\epsilon}Ay||^2\right)^n \leq ||\sqrt{1 - \epsilon}x + \sqrt{\epsilon}Sy||^2 + ||\sqrt{\epsilon}Ay||^2.$$

Hence

$$\left\{(1 - \epsilon) + 2\sqrt{\epsilon(1 - \epsilon)}\Re\langle x, S y \rangle + \epsilon(||Sy||^2 + ||Ay||^2)\right\}^n \leq (1 - \epsilon) + 2\sqrt{\epsilon(1 - \epsilon)}\Re\langle x, S y \rangle + \epsilon(||Sy||^2 + ||Ay||^2),$$

and

$$(1 - \epsilon)^n + 2n(1 - \epsilon)^{n-1}\sqrt{\epsilon(1 - \epsilon)}\Re\langle x, S y \rangle + O(\epsilon) \leq (1 - \epsilon) + 2\sqrt{\epsilon(1 - \epsilon)}\Re\langle x, S y \rangle + O(\epsilon).$$

Since $(1 - \epsilon) - (1 - \epsilon)^n = (1 - \epsilon)\epsilon(1 + (1 - \epsilon) + \cdots + (1 - \epsilon)^{n-2}) = O(\epsilon)$, we have

$$n(1 - \epsilon)^{n-1}\Re\langle x, S y \rangle - \Re\langle x, S y \rangle \leq \frac{1}{2}\sqrt{\frac{\epsilon}{1 - \epsilon}} \frac{1}{\epsilon} O(\epsilon).$$
Letting $\epsilon \to 0$, we have
\[
\text{Re}\langle x, (nS - S_n)y \rangle \leq 0,
\]
and hence $S_n = nS$.

Next, let $\sigma(T) \cap \{z \in \mathbb{C} \mid F_{n,1}(z) = 0\} = \{1\}$. Since $1 \not\in \sigma(A)$ by Lemma 6 and
\[
\sigma(A) \cap \{z \in \mathbb{C} \mid F_{n,1}(z) = 0\} \subset \sigma(T) \cap \{z \in \mathbb{C} \mid F_{n,1}(z) = 0\} = \{1\},
\]
it follows that $\sigma(A) \cap \{z \in \mathbb{C} \mid F_{n,1}(z) = 0\} = \emptyset$ and $F_{n,1}(A) = 1 - n + A + \cdots + A^{n-1}$ is invertible. Since $S_n = nS$, we have $S(1 - n + A + \cdots + A^{n-1}) = 0$ and hence $S = 0$. Then $T$ is of the form $1 \oplus A$ with $1 \not\in \sigma(A)$, this implies that
\[
E_1 = \frac{1}{2\pi i} \int_{\partial D_1} ((z - 1)^{-1} \oplus (z - A)^{-1}) \, dz = 1 \oplus 0,
\]
so $E_1$ is self-adjoint and $E_1 \mathcal{H} = \ker(T - 1) = \ker(T - 1)^*$. \hfill \Box

If we assume that $-n + 1 + A + \cdots + A^{n-1}$ has a dense range instead of the assumption $\sigma(T) \cap \{z \in \mathbb{C} \mid F_{n}(z) = n\lambda^{n-1}\} = \{\lambda\}$ in Theorem 7, we also have the same conclusions.

**Lemma 7.** Let $T \in \Psi(n)$ for $n \geq 3$ and $\lambda, \mu \in \sigma_p(T)$ such as $\lambda \neq \mu$. Then $\ker(T - \lambda) \perp \ker(T - \mu)$.

**Proof.** Without loss of generality, we may assume $\lambda = 1$ and $|\mu| \leq 1$. Consider the subspace $\mathcal{M} = \ker(T - 1) \vee \ker(T - \mu)$, the closed subspace generated by $\ker(T - 1)$ and $\ker(T - \mu)$, then $\mathcal{M}$ is invariant under $T$ and $\sigma(T|_{\mathcal{M}}) = \{1, \mu\}$. Since $T|_{\mathcal{M}}$ belongs to the class $\Psi(n)$ by Lemma 3 we have $\|T|_{\mathcal{M}}\| \leq r(T|_{\mathcal{M}}) = 1$. For any $u \in \mathcal{M}$
\[
\|T|_{\mathcal{M}}u\|^n \leq \|(T|_{\mathcal{M}})u\||u||^{n-1} \leq \|T|_{\mathcal{M}}||u||^n \leq \|u||^n.
\]
Therefore, for any $x \in \ker(T - 1)$ and any $y \in \ker(T - \mu)$,
\[
\|T|_{\mathcal{M}}(x + y)\|^n = \|x + \mu y\|^n = \left(\|x\|^2 + |\mu|^2\|y\|^2 + 2\text{Re}\langle x, \mu y \rangle\right)^n
\leq \|x + y\|^n = \left(\|x\|^2 + \|y\|^2 + 2\text{Re}\langle x, y \rangle\right)^n,
\]
so we have
\[
2\text{Re}\left(\overline{\mu}\langle x, y \rangle\right) \leq (1 - |\mu|^2)\|y\|^2.
\]
If necessary, replace $x$ by $ne^{i\theta}x$ for a $\theta \in \mathbb{R}$ and any $n \in \mathbb{N}$ so that $(1 - \overline{\mu})(ne^{i\theta}x, y) = n(1 - \overline{\mu})\langle x, y \rangle$ it follows that
\[
|\langle x, y \rangle| \leq \frac{(1 - |\mu|^2)\|y\|^2}{2n|1 - \overline{\mu}|}
\]
for any $n \in \mathbb{N}$ and hence $\langle x, y \rangle = 0$. \hfill \Box

**Theorem 8.** If $T \in \Psi(n)$ for $n \geq 2$, then $T$ has the SVEP.
Proof. If \( T \in \mathcal{B}(2) \), \( T \) has SVEP by [9]. Let \( T \in \mathcal{B}(n) \) for \( n \geq 3 \). Let \( \lambda \in \mathbb{C} \) be arbitrary, \( \mathcal{U} \) any neighborhood of \( \lambda \) and \( f : \mathcal{U} \to \mathcal{H} \) an analytic function which is a solution of the equation
\[
(T - z)f(z) = 0 \quad \text{for all} \quad z \in \mathcal{U}.
\]
Since \( f(z) \in \ker(T - z) \) for all \( z \in \mathcal{U} \) and \( \ker(T - z) \perp \ker(T - w) \) for all \( z, w \in \mathcal{U} \) such as \( z \neq w \), we have
\[
\|f(z)\|^2 = \lim_{w \to z} \langle f(z), f(w) \rangle = 0,
\]
and \( f = 0 \). \( \square \)

In [10], the authors show that every paranormal operator, i.e., an \( \mathcal{P} \)-operator (\( \mathcal{P} \)), has the spectral property (II). We extend this result as follows.

**Theorem 9.** If \( T \in \mathcal{P}(n) \) for \( n \geq 3 \), then \( T \) satisfies the spectral property (II).

**Proof.** Let \( \lambda, \mu \in \sigma_a(T) \) such as \( \lambda \neq \mu \) with \(|\mu| \geq |\lambda|\), \( \{x_m\} \) and \( \{y_m\} \) be arbitrary sequences of unit vectors such that
\[
\|(T - \lambda)x_m\| \to 0, \quad \|(T - \mu)y_m\| \to 0 \quad (m \to \infty).
\]
We shall show that \( \langle x_m, y_m \rangle \to 0 \) as \( m \to \infty \). Suppose \( \langle x_m, y_m \rangle \not\to 0 \). By considering subsequence we may assume that \( \langle x_m, y_m \rangle \) converges to some number \( a \). Also, we may assume \( a > 0 \), if necessary replace \( x_m \) by \( e^{it_m}x_m \) for some \( t_m \in \mathbb{R} \) such as \( \langle e^{it_m}x_m, y_m \rangle = |\langle x_m, y_m \rangle| \). Let \( 0 < \epsilon < 1 \) and \( c \in S^1 \) be arbitrary. Then
\[
\|T(\sqrt{\epsilon}cx_m + \sqrt{1 - \epsilon}y_m)\|^2 \leq \|T^n(\sqrt{\epsilon}cx_m + \sqrt{1 - \epsilon}y_m)\|^2 \|(\sqrt{\epsilon}cx_m + \sqrt{1 - \epsilon}y_m)\|^{2(n-1)}.
\]
Letting \( m \to \infty \), we have
\[
\left(\epsilon|\lambda|^2 + (1 - \epsilon)|\mu|^2 + 2a\sqrt{\epsilon(1 - \epsilon)}\Re(c\lambda\mu)\right)^n
\leq \left(\epsilon|\lambda|^{2n} + (1 - \epsilon)|\mu|^{2n} + 2a\sqrt{\epsilon(1 - \epsilon)}\Re(c\lambda\mu)\right) + (1 + 2a\sqrt{\epsilon(1 - \epsilon)}\Re(c))^{n-1}.
\]
Hence
\[
|\mu|^{2n} + n|\mu|^{2n-2}2a\sqrt{\epsilon(1 - \epsilon)}\Re(c\lambda\mu) + O(\epsilon)
\leq |\mu|^{2n} + |\mu|^{2n}(n - 1)2a\sqrt{\epsilon(1 - \epsilon)}\Re(c) + 2a\sqrt{\epsilon(1 - \epsilon)}\Re(c\lambda\mu) + O(\epsilon),
\]
and
\[
(n - 1)\Re(c|\mu|^{2n}) + \Re(c\lambda\mu) - n\Re(c\lambda\mu) + O(\epsilon) \geq 0.
\]
and
\[
\text{Re}\left\{ c \left( (n-1) - \frac{n\lambda}{\mu} + \left( \frac{\lambda}{\mu} \right)^n \right) \right\} = 0
\]
for all \( c \in S^1 \). Hence
\[
(n-1) - \frac{n\lambda}{\mu} + \left( \frac{\lambda}{\mu} \right)^n = 0. \tag{6}
\]
If \( \lambda = 0 \), then \( n-1 = 0 \) by (6). This is a contradiction. Hence \( \lambda \neq 0 \). Let \( z = \frac{\lambda}{\mu} \). Then \( 0 < |z| \leq 1, z \neq 1 \) and
\[
(n-1) - nz + z^n = (z-1)F_{n,1}(z) = 0.
\]
Hence
\[
F_{n,1}(z) = 1 + z + z^2 + \cdots + z^{n-1} - n = 0.
\]
Then
\[
n = 1 + z + \cdots + z^{n-1} \leq 1 + |z| + \cdots + |z^{n-1}| \leq n.
\]
This implies \( z = 1 \). This is a contradiction. \( \square \)

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