

# Cost Analysis on Warranty Policies Using Freund's Bivariate Exponential Distribution

Minjae Park<sup>\*\*</sup> · Jae-Young Kim<sup>\*\*</sup>

<sup>\*</sup> College of Business Administration, Hongik University, Seoul, Korea

<sup>\*\*</sup> Department of Business Administration, Korea University, Seoul, Korea

## ABSTRACT

**Purpose:** In this paper, the minimal repair-replacement warranty policy is used to carry out a warranty cost analysis with warranty servicing times and failure times that are statistically correlated to bivariate distributions.

**Methods:** Based on the developed approach by Park and Pham (2012a), we investigate the property of the Freund's bivariate exponential distribution and obtain the number of warranty services using the field data to conduct the warranty cost analysis.

**Results:** Maximum likelihood estimates are presented to estimate the parameters and the warranty model is investigated using a Freund's bivariate exponential distribution. A numerical example is discussed to deal with the applicability of the developed approach in the paper.

**Conclusion:** A novel approach of analyzing the warranty cost is proposed for a product in which failure times and warranty servicing times are used simultaneously to investigate the eligibility of a warranty claim.

**Key words:** Bivariate Distributions, Field Data, Maximum Likelihood Estimation, Replacement Service, Warranty Service

---

• Received 24 November 2013, 1st revised 17 January 2014, 2nd revised 3 February 2014, accepted 4 February 2014

† Corresponding Author(mjpark@hongik.ac.kr)

© 2013, The Korean Society for Quality Management

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0>) which permits unrestricted non-Commercial use, distribution, and reproduction in any medium, provided the original work is properly cited.

※ This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-002380).

# 1. Introduction

Warranty policy is a promise from the seller to provide the buyer with a certain service such as repair or replacement in case of product failure. Different types of warranties can be provided based on the properties of warranty policies (Blischke 1994, Blischke and Murthy 1996). Warranty and maintenance have been studied by many researchers (Nam and Kim 2011, Cha et al. 2001).

Park and Pham (2012b) conduct the warranty cost analysis in terms of the long-run expected cost subject to warranty periods, warranty polices and maintenance services for  $k$ -out-of- $n$  systems. They optimize the length of warranty period, warranty service time limit and periodical maintenance cycles to minimize the long-run expected cost under the customers' perspective. They also focus on the cost analysis using the non-homogeneous Poisson process (NHPP) with minimal repair. Before the warranty expires, the repair service and replacement service are considered which are statistically independent to each other. After the warranty expires, maintenance services are considered such as corrective maintenance, preventive maintenance, and corrective and preventive maintenances. Recently, Park et al. (2013) develop a minimal repair-replacement warranty with a repair service and replacement service and propose an optimal maintenance model after the expiration of the warranty. They consider the warranty servicing time and failure time as two factors for their new approach instead of the traditional two dimensions of age and usage. Park et al. (2013) and Park and Pham (2012b) attempt to carry out the warranty cost analysis under the same assumption whereby if a failed product is delivered to the warranty service center, the warranty service other than the replacement service is then provided. If they do not complete the warranty service within a time limit, then they cease to provide any warranty service. Instead, the replacement service is provided. Park and Pham (2012b) use the two dimensional NHPP with repair times and failure times for the  $k$ -out-of- $n$  system considering both the warranty period and post warranty period, while Park et al. (2013) use the minimal repair-replacement warranty policy with warranty servicing times and failure times. In both models, it is assumed that the failure times and warranty servicing times are independent in order to implement the proposed approaches. On the contrary, in this study we consider the case in which the failure times and warranty servicing times are statistically dependent to each other. This is because when we use a product and it would naturally deteriorate with usage and age resulting in more severe failures. So it may take more warranty servicing time to fix a failed product. This indicates that the failure time and the warranty servicing time correlated to each other. Under the dependent relationship between failure times and warranty servicing times, Park and Pham (2012a) develop warranty cost model where the failure time and warranty servicing time are statistically correlated in bivariate distributions. In their cost model, Marshall and Olkins' bivariate exponential distribution is used to conduct the warranty cost analysis. If the failure time and warranty servicing time are independent, a minimal repair-replacement warranty (Park et al. 2013) is used to carry out the warranty cost analysis. If they are dependent, the bivariate distributions may be used to model the failure time and the warranty servicing time and estimate the number of warranty services (Park and Pham 2012a). Park (1985) investigate the estimation of the parameters of the Freund's bivariate exponential distribution using the moment method and maximum likelihood estimation method. In

this paper, when the failure time and the warranty servicing times are statistically correlated, we extend Park and Pham (2012a)'s model using Freund's bivariate exponential distribution (BED) with the Laplace transform, find the properties of Freund's BED, and conduct the warranty cost analysis using the field data.

The remaining of this paper is organized as follows. In Section 2, we focus on the warranty cost analysis when failure times and warranty servicing times are dependent. The renewal function is obtained using Freund's bivariate exponential distribution for several types of warranty policies. An illustrative example is given in Section 3 to show the warranty models with bivariate distributions using field data and finally, we give concluding remarks in Section 4.

## 1.1 Nomenclature

r.v. : random variable  
pdf, cdf : probability density function, cumulative distribution function, respectively  
MLE : Maximum Likelihood Estimates  
BED : Bivariate Exponential Distribution  
 $W_1, W_2$ : r.v. warranty period and time limit of the warranty service, respectively  
 $f(x), F(x), \bar{F}(x)$ : pdf, cdf and reliability function, respectively  
 $L(\cdot)$ : likelihood function  
 $g(w_1, w_2), G(w_1, w_2)$ : bivariate pdf and cdf, respectively for r.v.  $W_1$  and  $W_2$   
 $f(x, y), F(x, y)$ : bivariate pdf and cdf, respectively for failure times X and warranty servicing times Y  
 $M(W_1, W_2)$ : expected number of warranty services per warranty period  $W_1$  and per warranty servicing time limit  $W_2$ .

## 2. Model Formulation

### 2.1 Renewal function using bivariate distribution

Let  $N(W_1, W_2)$  be the number of warranty services within the warranty period. Let  $W_1$  be the warranty period and  $W_2$  be the warranty servicing time limit.  $W_1$  and  $W_2$  are exponential random variables. Let  $(x, y)$  be the bivariate exponential random variables of the failure times and the warranty servicing times, respectively. When the failure time and warranty servicing time are dependent, the expected number of warranty services and the variance of the number of warranty services within the warranty servicing time limit  $w_2$  and the warranty period  $w_1$  are given (Park and Pham 2012a) as

$$E[N(w_1, w_2)] = \frac{\iint P(w_1 \geq t, w_2 \leq s) f(t, s) dt ds + \iint P(w_1 \geq t, w_2 > s) f(t, s) dt ds}{1 - \iint P(w_1 \geq t, w_2 > s) f(t, s) dt ds} \quad (1)$$

and

$$\begin{aligned} & \text{Var}(N(w_1, w_2)) \\ &= \frac{\left( \int_0^\infty \int_0^\infty P\{w_1 \geq t, w_2 \leq s\} f(t, s) dt ds + \int_0^\infty \int_0^\infty P\{w_1 \geq t, w_2 > s\} f(t, s) dt ds \right) \left( 1 - \int_0^\infty \int_0^\infty P\{w_1 \geq t, w_2 > s\} f(t, s) dt ds \right)}{\left( 1 - \int_0^\infty \int_0^\infty P\{w_1 \geq t, w_2 \leq s\} f(t, s) dt ds \right)^2}. \end{aligned} \tag{2}$$

## 2.2 Freund's bivariate exponential distribution

While the renewal function plays an important role in investigating the warranty policies, it is not easy to obtain analytic expressions for  $M(W_1, W_2)$  and computational procedures are generally required. Hunter (1974) obtains the analytical expression for  $M(W_1, W_2)$  using Downton's BED. However, cases where the transform is invertible in closed form are rare. For most of the bivariate models, closed form transform inverses are not available (Nachlas 2005). So, let  $f^*(s_1, s_2)$  be the bivariate Laplace transform of  $f(x, y)$  and let  $F^*(s_1, s_2)$  be the bivariate Laplace transform of  $F(x, y)$ , the cdf of  $f(x, y)$ . We then determine that

$$F^*(s_1, s_2) = \frac{f^*(s_1, s_2)}{s_1 s_2} \tag{3}$$

Let  $F^n(x, y)$  be the n-fold convolution function of  $F(x, y)$ . From the properties of Laplace transforms, we have

$$f^{*(n)}(s_1, s_2) = f^*(s_1, s_2)^n, F^{*(n)}(s_1, s_2) = \frac{f^*(s_1, s_2)^n}{s_1 s_2} \tag{4}$$

A requisite property of  $f(x, y)$  is that the conditional expectation of X and Y must be an increasing function. Clearly, the expected warranty cost and the variance increase linearly with y. Let  $N(x, y)$  denote the number of renewals over the rectangle  $[(0, x) \times (0, y)]$  with the origin being a renewal point. Let  $M_\rho(x, y)$  be the bivariate renewal function with correlation coefficient  $\rho$ .

$$M_\rho(x, y) = \sum_{n=1}^\infty F^{(n)}(x, y) \tag{5}$$

We obtain the bivariate Laplace transform of  $M_\rho(x, y)$  as:

$$M_\rho^*(s_1, s_2) = \frac{f^*(s_1, s_2)}{s_1 s_2 (1 - f^*(s_1, s_2))} \tag{6}$$

A bivariate function of the exponential distribution is proposed as a model for issues in reliability. Its popularity is witnessed by the very broad literature available in this area (Nadarajah and Kotz 2006, Downton 1970, Hawkes 1972). However, unlike the normal distribution, the exponential distribution unfortunately does not have a natural bivariate function or the multivariate case. Therefore, since 1960 many classes of bivariate distributions with exponential marginals have been developed. Freund (1961) suggested a bivariate distribution of the exponential distribution as a model for problems in life testing for a two-component system. Freund's BED has the memoryless property. Further, estimating their parameters is relatively straightforward. Therefore, among bivariate distributions, Freund's BED is selected for the warranty cost analysis. Usually this model represents one type of BED that is applicable to the lifetime distributions of a two components system. However, we consider the BED for two factors, including the warranty servicing time and failure time. From Freund (1961), the joint probability density function of Freund's BED is given by

$$f(y_1, y_2) = \alpha\beta' \exp(-\beta'y_2 - (\alpha + \beta - \beta')y_1) \tag{7}$$

for  $0 < y_1 < y_2$  and

$$f(y_1, y_2) = \beta\alpha' \exp(-\alpha'y_1 - (\alpha + \beta - \alpha')y_2) \tag{8}$$

for  $0 < y_2 < y_1$ .

Then,  $f^*(s_1, s_2)$  and  $F^*(s_1, s_2)$  are obtained respectively as follows.

$$f^*(s_1, s_2) = L(f(x, y)) = \frac{\alpha\beta'}{(\alpha + \beta - \beta' + s_1)(\beta' + s_1)} \tag{9}$$

$$F^*(s_1, s_2) = L(F(x, y)) = \frac{f^*(s_1, s_2)}{s_1 s_2} = \frac{\alpha\beta'}{s_1 s_2 (\alpha + \beta - \beta' + s_1)(\beta' + s_1)} \tag{10}$$

We obtain the bivariate Laplace transform of  $M_\rho(x, y)$  as:

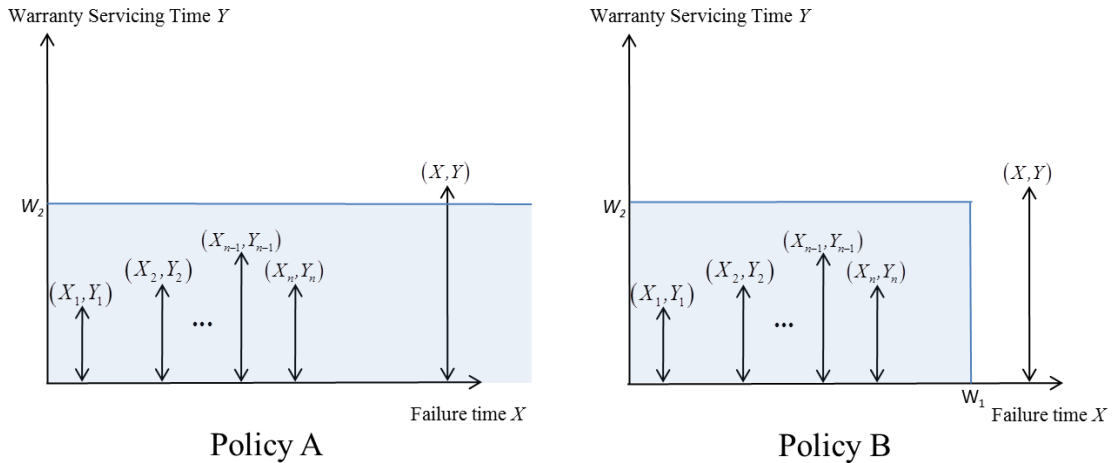
$$M_\rho^*(s_1, s_2) = \frac{f^*(s_1, s_2)}{s_1 s_2 (1 - f^*(s_1, s_2))} = \frac{\alpha\beta'}{s_1 s_2 ((\beta' + s_1)(\beta - \beta' + s_1) + \alpha s_1)} \tag{11}$$

If the Laplace transform is used, we obtain the reverse renewal function. However it is difficult to develop the renewal function using the reverse Laplace transform. Therefore in this study, instead of Laplace transform, the maximum likelihood estimation (MLE) is used to estimate its parameters. We maximize the likelihood and use logarization. Eventually, the estimated parameters for  $\lambda_1$  and  $\lambda_2$  can be obtained by

$$\hat{\lambda}_1 = \frac{\sum_{i=1}^n x_i}{n}, \quad \hat{\lambda}_2 = \frac{\left( \sum_{i=1}^n y_i - \sum_{i=1}^n x_i \right)}{n} \tag{12}$$

respectively.

### 2.3 Warranty cost analysis



**Figure 1.** Warranty policies with time limits of the warranty servicing time.

We develop number of warranty services based on the proposed warranty policies in Fig. 1 and obtain the expected warranty cost. For Policy A, we do not adopt the warranty period. We consider the warranty servicing time limit only and it indicates that a one-factor warranty policy is investigated. Park and Pham (2010) obtain the renewal function for the one-factor warranty model. If  $N(W_2)$  denotes the number of warranty services under warranty for the warranty policy, then the expected number of warranty services is given by

$$E[N(W_2)] = \frac{\int P(W_2 \leq y) f_Y(y) dy}{1 - \int P(W_2 \leq y) f_Y(y) dy} \tag{13}$$

and its variance is given by

$$Var[N(W_2)] = \frac{(1 + 2E[N(W_2)]) \int P(W_2 \geq y) f_Y(y) dy}{1 - \int P(W_2 \geq y) f_Y(y) dy} - (E[N(W_2)])^2 \tag{14}$$

where  $E[N(W_2)]$  is obtained from Eq. (13).

Let  $c$  be a warranty cost per failure and  $N(W_2)$  be the number of renewals over the warranty servicing time limit  $W_2$ . The expected cost is given by

$$E_1(C) = c \cdot E[N(W_2)] \quad (15)$$

where  $E[N(W_2)]$  is obtained from Eq. (13).

$$Var_1(C) = c^2 \cdot Var[N(W_2)] \quad (16)$$

where  $Var[N(W_2)]$  is obtained from Eq. (14).

If  $N(W_1, W_2)$  denotes the number of repair services in the warranty period for Policy B, then the expected warranty cost is given by

$$E_2(C) = c \cdot E[N(W_1, W_2)] = c \cdot M(W_1, W_2) \quad (17)$$

where  $M(W_1, W_2)$  is obtained from Eq. (1),

$$Var_2(C) = c^2 \cdot Var[N(W_1, W_2)] \quad (18)$$

where  $Var[N(W_1, W_2)]$  is obtained from Eq. (2).

For the numerical examples, as the computations have an infinite sum to have the expected number of failures and their variances, a numerical method is adopted in the calculation. We pick a certain number of terms in the series, and then run an extrapolation to investigate the contributions of other terms.

### 3. Illustrative Example Using the Field Data

Four nuclear sites are located in South Korea and, in 2009, 20 nuclear power plants were in operation with a total licensed output amount of 17,716 MWe (MegaWatt electrical). Additional 8 nuclear power plants are under construction, to achieve a total of 28 plants in operation by the end of 2016 (Park and Pham 2012a). The field data are investigated to check the dependency using the nonparametric method. We implement the proposed approaches to conduct warranty cost analysis using the field data.

#### 3.1 Exploratory data analysis

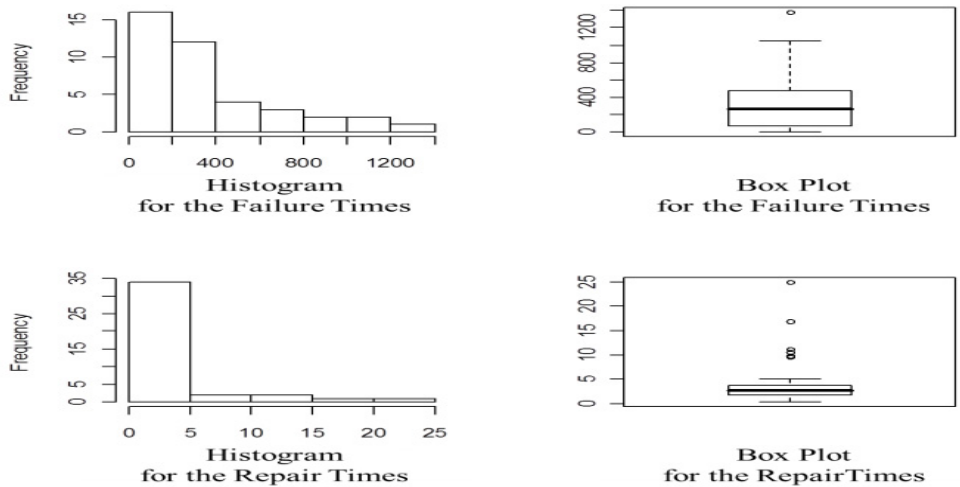
Among the 20 nuclear power plants of the 4 nuclear sites in South Korea, 30 failure data are summarized for relatively recent events or failures as shown in Table 1. The data includes the failure time and the warranty servicing time. This data can be obtained from the Operational Performance Information System for Nuclear Power Plants (Park and Pham 2012a).

**Table 1.** Failure times and warranty servicing times for Nuclear Power Plants (Unit: days)

No.	Failure times	Warranty servicing times	No.	Failure times	Warranty servicing times
1	353.04	4.37	16	30.27	3.63
2	334.72	1.91	17	117.37	2.73
3	80.04	2.04	18	126.27	2.55
4	6.49	1.72	19	56.45	0.72
5	1.34	0.29	20	45.28	3.69
6	467.19	1.93	21	267.31	0.36
7	0.35	1.82	22	615.64	10.63
8	398.86	1.77	23	115.37	11.24
9	1048.23	9.61	24	359.76	9.70
10	829.39	3.80	25	412.30	3.31
11	227.20	2.86	26	276.69	4.96
12	260.14	0.31	27	601.04	2.99
13	14.00	0.85	28	1021.01	2.36
14	14.15	2.04	29	192.17	1.63
15	38.96	2.73	30	0.36	0.26

From Operational Performance Information System for nuclear Power Plant (Park and Pham 2012a)

The exploratory data analysis on the failure time and the warranty servicing time is conducted to determine their attribution. From the shape of each box plot and histogram shown in Figure 2, we can determine that the failure times and warranty servicing times are not normally distributed.



**Figure 2.** Exploratory data analysis for the failure time and the warranty servicing time



Using a nonparametric method, Kendall's  $\tau$  we investigate the dependence between the failure times and the warranty servicing times. As a result, if two times are independent, then we may use the minimal repair-replacement warranty in Park et al. (2013), otherwise the bivariate function with the failure time and the warranty servicing time is used to calculate the warranty cost. Kendall's rank correlation measures the strength of the monotonic association. An advantage of the Kendall rank correlation over other non-parametric methods is that the score function  $S$  is nearly normally distributed for small  $n$  and the distribution of  $S$  is easier to use. It may also be noted that the Pearson correlation is fairly robust and it usually agrees well in terms of statistical significance with results obtained using Kendall's rank correlation (Park 2010). The null hypotheses and the alternative hypotheses are as follows:

$$\begin{cases} H_0 : \text{The failure time and the warranty servicing time are independent.} \\ H_a : \text{The failure time and the warranty servicing time are dependent.} \end{cases}$$

Based on the result of Kendall's  $\tau$  method using R software (McLeod 2005),  $\tau$  is 0.277 and the p value is 0.012246. Therefore, at  $\alpha = 0.05$ , we do not have sufficient evidence to support the null hypotheses that the failure times and warranty servicing times are independent. Using the non-parametric methods, we can conclude that the failed times and warranty servicing times are dependent for the nuclear power plant data in Table 1. With the failure time and the warranty servicing time, we attempt to determine the best fit distributions for the data. Calculations are based on the distribution specified. The computer software indicates that for the failure times, the gamma, exponential, and Weibull distributions are the best three well fit distributions. For the warranty servicing times, the exponential with 2 parameters and with 1 parameter, the lognormal, and gamma distributions are well fit distributions. Based on the output, we choose the BED, and then conduct a warranty cost analysis because the warranty servicing time and the failure time are exponentially distributed with one parameter. We need to choose the commonly fit distribution for both data because we need to obtain the bivariate distribution for dependable two-dimensional data. The BED is implemented for the failure time and warranty servicing time because it is the most commonly used model for the joint distribution of failure and warranty servicing times (Park and Pham 2012a). In the following section, we conduct the warranty cost analysis using Freund's BED and the field data.

### 3.2 Warranty cost analysis

To illustrate the proposed method, we assume that a minimal repair-replacement warranty (Park et al. 2013) has been provided by the manufacturer to determine the warranty period and the time limit of the warranty service. Using the warranty servicing time and the failure time, we attempt to conduct a warranty cost analysis using Freund's BED. Using the MLE method, we estimate the four parameters of Freund's BED using R software (McLeod 2005) as shown in Table 2.

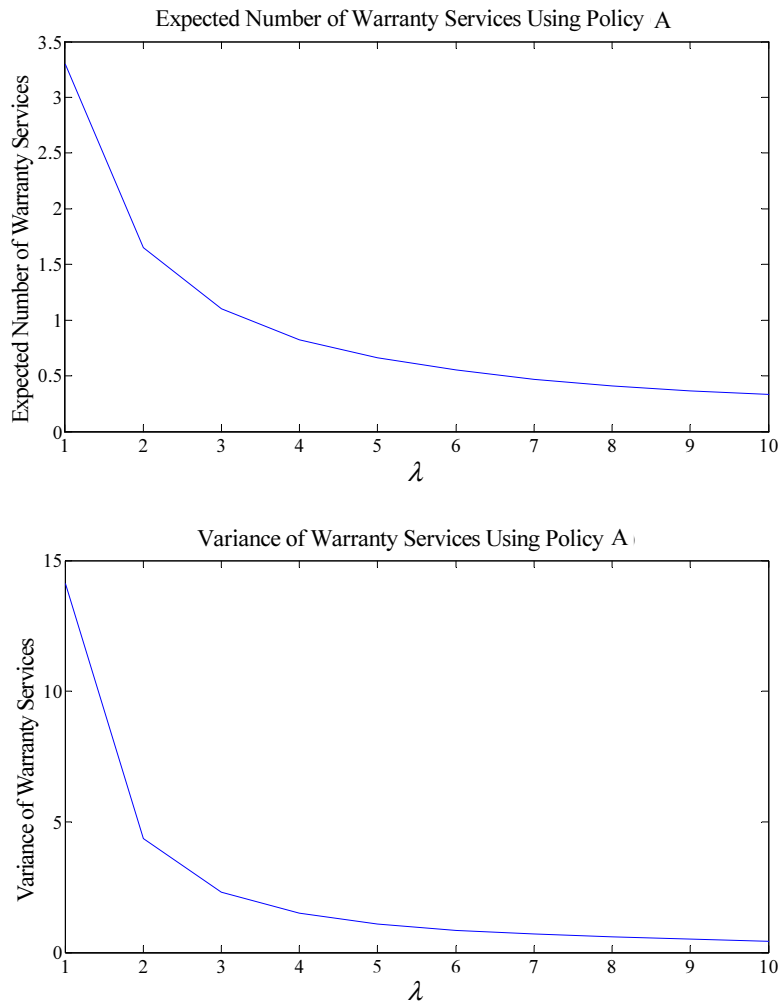
**Table 2.** Estimated parameters using Freund's bivariate exponential distribution

$\alpha$	$\alpha'$	$\beta$	$\beta'$
0.0103	0.0035	0.2979	0.6825

Based on the parameters in Table 2, we show the numerical example and conduct the sensitivity analysis. From the result of the warranty cost analysis using BED, we obtain the expected number of warranty servicing in Table 3. Table 3 and Fig. 3 show the expected number of failures under warranty for the limit parameters for Policy A. Using Eqs. (15) and (16), we investigate the warranty servicing cost. Using the warranty servicing times, we calculate the exponential parameter,  $\lambda$  value, because the best fit distribution for the warranty servicing times is exponential distribution.  $W_2$  is assumed to follow another exponential distribution and by sensitivity analysis, we input different parameters for exponential distribution for  $W_2$ . As a result of the sensitivity analysis, Table 3 and Fig. 3 are obtained.

**Table 3.** Expected number of warranty services and their variance for Policy A

$\lambda$	Expected number of warranty services	Variance of warranty service number
1	3.2937	14.1419
2	1.6468	4.3589
3	1.0979	2.3033
4	0.8234	1.5014
5	0.6587	1.0927
6	0.5489	0.8503
7	0.4705	0.6919
8	0.4117	0.5812
9	0.3660	0.4999
10	0.3294	0.4378

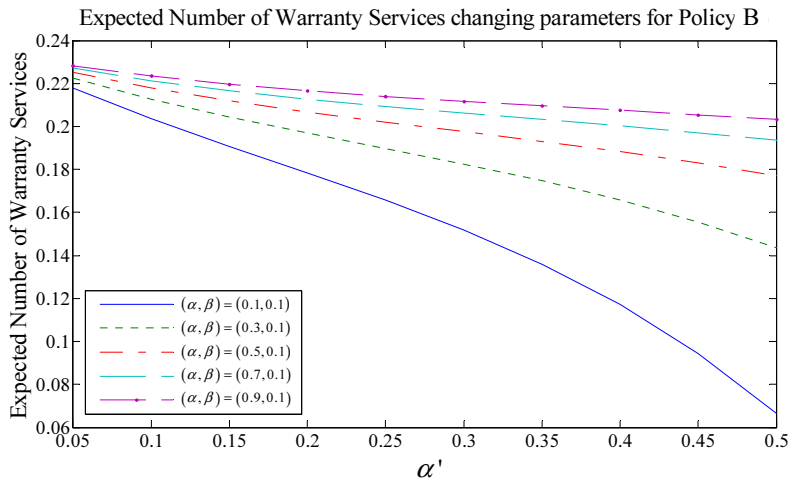


**Figure 3.** Expected number of warranty services and their variance for Policy A

In Fig. 3, we determine that as  $\lambda$  increases, the number of warranty services decreases. The manufacturer or customer service center can handle  $W_2$  and adjust the  $\lambda$  value and expect the number of warranty services. Using Eqs. (17) and (18), we investigate the warranty servicing cost and as a result of the analysis Table 4 and Fig. 4 are obtained.

**Table 4.** Expected number of warranty services for Policy B

$\alpha'$	$(\alpha, \beta)$				
	(0.1,0.1)	(0.3,0.1)	(0.5,0.1)	(0.7,0.1)	(0.9,0.1)
0.05	0.2178	0.2226	0.2253	0.2271	0.2283
0.1	0.2035	0.2127	0.2179	0.2212	0.2234
0.15	0.1907	0.2044	0.2119	0.2165	0.2196
0.2	0.1785	0.1969	0.2067	0.2126	0.2165
0.25	0.1658	0.1898	0.2021	0.2093	0.2140
0.3	0.1519	0.1824	0.1976	0.2062	0.2117
0.35	0.1360	0.1746	0.1930	0.2032	0.2095
0.4	0.1172	0.1658	0.1882	0.2002	0.2075
0.45	0.0945	0.1556	0.1830	0.1971	0.2054
0.5	0.0666	0.1436	0.1770	0.1937	0.2032



**Figure 4.** Expected number of warranty services under warranty for Policy B.

When we have fixed  $\beta$  and changed  $\alpha$  and  $\alpha'$ , we conduct a sensitivity analysis. When  $\alpha'$  increases gradually, the number of warranty services decreases. On the other hand, when  $\alpha$  increases gradually, the number of warranty services increases. The manufacturer manages the warranty period and the warranty servicing time limit; they can then change the BED's parameters. Based on the arrangement of the warranty period and the warranty servicing time limit, the warranty cost can be estimated and can be controlled by the manufacturer.

## 4. Concluding Remarks

The warranty model has been developed and investigated for warranty cost analysis from the perspective of the manufacturer. The contribution of this paper is summarized as follows. First, we have obtained both the expected value and variance of the warranty cost. While the warranty cost is a good measure on the overall cost of warranty, it provides little information of the risk which have contained in a warranty policy, Therefore, it is dissatisfactory to express the data based on the expected values alone. Thus, by obtaining the variance of warranty cost with the expected value, the result provides more accurate cost analysis. Second, we have used field data from nuclear power plants instead of simulated data which better shows the model's application. Because we analyze the warranty cost based on the field data in the numerical example, the results of the numerical example are more convincing and practical. Third, we suggest the novel approach, which indicates that we consider the warranty servicing times and the failure times, not age and usage.

This study provides new research opportunities for the warranty policy. First, we can consider the warranty period and warranty servicing time limit as random variables. If various lengths of warranty period can exist based on the location and point of purchase of policy. For example, in the U.S., Hyundai car makers sell vehicles with a 10 year warranty, while in South Korea, same models are sold with a 5 year warranty policy. Also, if company permits, the warranty period can be adjusted for customers who want to extend the warranty period by purchasing additional policy. This indicates that the warranty periods are not fixed but changeable. Second, instead of Freund's BED, other BEDs may be used. We may also develop the warranty cost model based on different bivariate distributions.

## ACKNOWLEDGEMENTS

The author would like to thank the associate editor and anonymous referees for their valuable and constructive comments and suggestions which greatly improved the final version of this paper.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-002380).

## REFERENCES

- Blischke, Wallace R. 1994. *Warranty Cost Analysis*: CRC Press.
- Blischke, Wallace R., and Murthy, D. N. Prabhakar. 1996. *Product Warranty Handbook*. CRC Press.
- Cha, Ji Hwan, Jung, Jong Tae, and Kim, Jae Joo. 2001. "Optimal Preventive Maintenance Policy for a Repairable System." *Journal of the Korean Society for Quality Management* 29:46-53.
- Downton, F. 1970. "Bivariate exponential distributions in reliability theory." *Journal of the Royal Statistical Society, Series B (Methodological)*:408-417.
- Freund, John E. 1961. "A bivariate extension of the exponential distribution." *Journal of the American Statistical Association*:971-977.

- Hawkes, Alan G. 1972. "A bivariate exponential distribution with applications to reliability." *Journal of the Royal Statistical Society, Series B (Methodological)*:129–131.
- Hunter, Jeffrey J. 1974. "Renewal Theory in Two Dimensions: Basic Results." *Advances in Applied Probability* 6:376–391.
- McLeod, A. Ian. 2005. "Kendall rank correlation and Mann-Kendall trend test." R Package (Kendall). <http://cran.r-project.org/web/packages/Kendall>.
- Nachlas, Joel A. 2005. *Reliability engineering: probabilistic models and maintenance methods*. CRC Press.
- Nadarajah, Saralees., and Kotz, Samuel. 2006. "Reliability models based on bivariate exponential distributions." *Probabilistic Engineering Mechanics* 21:338–51.
- Nam, Kyung H., and Kim, Do Hoon. 2011. "A Study on Optimal Software Maintenance Policies with Warranty Period." *Journal of the Korean Society for Quality Management* 39:170–178.
- Park, Byung-gu. 1985. "Statistical inference for bivariate exponential distribution in reliability and life testing problems." *Journal of the Korean Society for Quality Management* 13:31–40.
- Park, Minjae. 2010. "Cost Analyses on Warranty Policies for Systems Subject to Two Types of Warranty Periods." PhD diss., Rutgers, the State University of New Jersey.
- Park, Minjae, Jung, Ki Mun, and Park, Dong Ho. 2013. "Optimal post-warranty policy with repair time threshold for minimal repair." *Reliability Engineering and System Safety* 111:147–153.
- Park, Minjae, and Pham, Hoang. 2012. "A new warranty policy with failure times and warranty servicing times." *IEEE Transactions on Reliability* 61:822–831.
- Park, Minjae, and Pham, Hoang. 2012. "Warranty Cost Analysis for k-out-of-n Systems With 2-D Warranty." *IEEE Transactions on Systems Man and Cybernetics-Part A-Systems and Humans* 42:947–957.
- Park, Minjae, and Pham, Hoang. 2010. "Altered quasi-renewal concepts for modeling renewable warranty costs with imperfect repairs." *Mathematical and Computer Modelling* 52:1435–1450.
- Ross, Sheldon M. 2000. *Introduction to Probability Models*. San Diego: Academic Press.