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Volumes of Solids in Joseon Mathematics

朝鮮 算學과 體積

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Joseon is mainly an agricultural country and its main source of national revenue is the farmland tax. Since the beginning of the Joseon dynasty, the assessment and taxation of agricultural land became one of the most important subjects in the national administration. Consequently, the measurement of fields, or the area of various plane figures and curved surfaces is a very much important topic for mathematical officials. Consequently Joseon mathematicians were concerned about the volumes of solids more for those of granaries than those of earthworks. The area and volume together with surveying have been main geometrical subjects in Joseon mathematics as well. In this paper we discuss the history of volumes of solids in Joseon mathematics and the influences of Chinese mathematics on the subject.

Keywords: Volumes of solids, Joseon mathematics, Gyeong SeonJing, Hong JeongHa, Jiuzhang Suanshu, Liu Hui, 慶善徵, 洪正夏, 九章算術, 劉徽.

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1 Introduction

Agriculture has been the backbone of Joseon economy and hence the farmland tax became the major portion of its national revenue. Since the beginning of the Joseon dynasty (1492–1910), the taxation of agricultural land was one of the most important subjects of the government. To assess the farmland tax, the government officials had to measure the farmlands. Agricultural productivity is strongly related to the weather, meteorology and astronomy were also very important affairs of Joseon government which is composed of the six departments (六曹). HoJo (戶曹) is the department which deals with census, taxation, compulsory services, national

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accounting and economy so that it needs mathematical officials (算員). Mathematical officials were chosen through the examination, called ChuiJae (取才). Mathematical officials from the end of the 15th century to 1888 were recorded in JuHak-IbGyeokAn (籌學入格案) and it contains 1,626 mathematicians who passed the examination [9]. For the astronomy and meteorology, the state observatory, Gwan-SangGam (觀象監) was established. Its officials were also selected by the national examination (雜科). In the era of the 4th King SeJong (世宗, r. 1418–1450), the law for HoJo including the farmland tax, was revised and consequently measured and recorded the farmlands of the whole country [13]. Also King SeJong guided the inventions of various instruments for meteorology and astronomy and developed the Korean system of calendar. During his era, mathematics, astronomy and study on agriculture experienced a great development [5]. Thus the area of plane figures together with curved surfaces was an important subject for the mathematical officials in HoJo. There was no difficulty in measuring basic figures like polygons and they used the approximations for the other figures and surfaces.

As plane figures were introduced as farmlands in the chapter Fangtian (方田) of Jiuzhang Suanshu (九章算術), their areas and the basic structure of fractions were dealt in the chapter. Further, volumes of earthworks and storehouses for grains were discussed in the chapter Shanggong(商功) [1, 2, 3]. In Joseon dynasty, they did not have much earthworks. Although volumes of earthworks were considered in Joseon mathematics, the volumes of solids were mainly introduced for the volumes of granaries.

The purpose of this paper is to discuss the history of volumes in Joseon mathematics. Unlike the other subjects, Joseon mathematicians mostly used the formulae of volumes in Chinese literatures with a few exceptions and their preferences were diverse depending on the period of their works. We also discuss the history of Chinese volumes to compare volumes between the two countries.

The reader may find all the Joseon and Chinese sources of this paper in HanGuk GwaHak GiSulSa JaRyo DaeGye (韓國科學技術史資料大系) [4] and ZhongGuo KeXue JiShu DianJi TongHui ShuXueJuan (中國科學技術典籍通彙 數學卷) [2] respectively and hence they will not be numbered as an individual reference.

2 Volumes of solids in Joseon mathematics

We first collect some unfamiliar terminologies of solids which were introduced in Jiuzhang Suanshu. For the polyhedra, it deals with prismatoids, i.e., polyhedra where all vertices lie in two parallel planes. They contain pyramids with various bases and their truncations, i.e., frusta, wedges, prisms and prismoids. Prismoid is a prismatoid whose cutting planes have the same number of vertices and the lateral faces are either parallelograms or trapezoids. We note that a prismoid is a frustum if and only if the corresponding edges in the two parallel planes are proportional. The prismoid with a rectangular base is called Chutong (芻童), Panchi (盤池) or Minggu (冥谷) and *none* of them in Jiuzhang Suanshu is a frustum. Clearly a prismoid with a square base, or Fangting (方亭) is a frustum. A rectangle based wedge is called Chumeng (芻甍). Prisms in Jiuzhang are right ones. Solids in Jiuzhang are not defined but indicated by their dimensions, namely sides and heights; circumferences and heights for circular solids.

Mathematical works of Joseon before the 16th century were all lost [8]. The first significant one in Joseon is MukSaJibSanBeob (默思集算法) written by a mathematical official, Gyeong SeonJing (慶善徵, 1616–?). In the first half of the 17th century, Joseon mathematicians had a completely ruined situation because of foreign invasions. Gyeong had a copy of Suanxue Qimeng (算學啓蒙, 1299) of Zhu Shijie (朱世傑) with some missing pages and Xiangming Suanfa (詳明算法, 1373) of An Zhizhai (安止齋). In the sections ChangDon JeokSokMun (倉囤積粟門) and SangGong SuChukMun (商功修築門) of the second book, Gyeong dealt with volumes of solids. He chose the former's title from Suanxue Qimeng but arranged his problems along the order of problems in the section Panliang Cangjiao (盤量倉窖) of Xiangming but changed the dimensions of storehouses in Xiangming, and he quoted two more problems from Qimeng. Further, he replaced the word Jiao (窖) in Xiangming by Don (囤) in Qimeng and followed Jumi (聚米) in Xiangming instead of Jusu (聚粟). We must point out that Jiuzhang Suanshu was first introduced to Joseon in the 19th century and hence Joseon mathematicians before the 19th century did not know the mathematical commentaries made by Liu Hui (劉徽, ca 3C.). It is interesting that Gyeong SeonJing included a formula for the volume of the frustum with a square base in Problem 5 about BangDon (方囤), i.e., Fangjiao (方窖) in Xiangming or Fangting. Indeed, let a, b be the lengths of the upper and lower squares of the frustum and h its height. He first find its volume by the formula

$$\frac{h(a^2+b^2+ab)}{3}$$

and then added the formula

$$a^2h + a(b-a)h + \frac{(b-a)^2h}{3}$$

which is precisely the formula made by subdivisions of the frustum as in Liu's commentary. We quote his explanation as follows for $a = 6, b = 10, h = 12.6$:

> 該曰 從下方直割下方 則其形如刀刃者爲四 方錐亦四也
> 盖刀刃者 下厚 各二尺 廣 六尺 刃高 一丈二尺六寸
> 顚倒交互合爲一箇 則上下各厚 四尺 廣 六尺 高如刃高同也
> 方錐者 下方 各二尺 高亦 一丈二尺六寸 四錐共併合爲一錐

則下方爲四尺 高亦與本高同也

By the subdivision of the BangDon, one has a square prism, 4 congruent right triangle prisms, called DoIn (刀刃), or Qiandu (塹堵) and 4 congruent pyramids (方錐). One turns over the two prisms and then fits them with the other two prisms, which results in a rectangular prism with dimensions $a, b-a, h$ so that its volume is $a(b-a)h$. Fitting the remaining 4 pyramids into one pyramid with the dimensions $b-a, h$, one has the volume $\frac{(b-a)^2h}{3}$.

We should remark that Liu Hui used two methods to obtain formulae of volumes in his commentary in Jiuzhang Suanshu. The first one is that Liu used the relationship between a given solid and those solids with known volumes and then obtained the volume of the solid. He used this method for trapezoidal prisms, the cylinders, truncations of circular cones, cones with square bases or circular bases, Qiandu, Yangma (陽馬), Bienao (鼈臑). Liu did not pay any attention to the order of problems in Jiuzhang and freely used the concepts introduced later. Indeed, when he calculated the approximation of π in the first chapter, he used the extraction of roots and Gougushu appearing in the later chapters. The second one is that first he verified the given formula and then using the subdivision, he added his own formula for the volume. This method is used for the volumes of Fangting, Chumeng and Chutong. He verified the given formulae in Jiuzhang by an example. We take the case of Fangting. Liu described the volumes of terms a^2h, b^2h, abh in the formula as sums of solids appearing in the subdivision of the Fangting into the basic solids, where he took the case of $a = 1, b = 3, h = 1$. Comparing the total basic solids in the terms and the original ones of the Fangting, Liu verified the formula. Then he added the formula

$$\frac{1}{3}(b-a)^2h + abh$$

directly from the subdivision. Indeed, using the process of turning over two prisms and fitting them (顚倒交互合) as Gyeong did, Liu had the rectangular prism with the volume abh. (also see [1, 3]).

In Problem 6, Gyeong dealt with the volume of a frustum of a right circular cone with the dimensions a, b, h, where a, b are circumferences of the upper and lower bases and h the height. It is called WonDon (圓囤). First he included the formula $\frac{h(a^2+b^2+ab)}{36}$ for the volume and then added another formula

$$\frac{a^2h}{12} + \frac{adh}{2} + \frac{1}{3}[\frac{(b-a)dh}{2}],$$

where d denotes the difference of radii of two circles, called SilGyeong (實經). Clearly the sum of the last two terms of the formula indicates the volume of the solid made by taking away the cylinder from the original frustum, which Gyeong called HwanDon (環囤). Thus the new formula should be obtained by subdivisions of the frustum. Moreover, the last two terms indicate the volumes of a rectangular prism and a cone respectively. Indeed, for the areas of circles, volumes of right circular cones and their frusta in Jiuzhang, π was assumed to be 3 and hence the volume of the frustum is precisely the volume of the BangDon with the dimensions $\alpha a, \alpha b, h$ where $\alpha = \frac{1}{\sqrt{12}}$. The terms in the added formula are precisely those appearing in the above BangDon. We do not know whether Gyeong had the new formula this way and could not so far find any literature in which the above two added formulae were included. The last three problems in the section are variations of the last problem of the corresponding section in Qimeng.

The next section, SangGong SuChukMun (商功修築門) with 19 problems deals with the volumes of earthworks. It quotes mainly problems from Suanxue Qimeng with altered dimensions and 4 problems from Xiangming Suanfa. The first 8 problems deals with the volumes of trapezoidal prisms and the next 5 problems with square and circular prisms, BangDon, WonDon and BangChu (方錐). Here Geyong used the same terminologies for prisms and frusta as BangDae (方臺) and WonDae (圓臺) instead of those for the granaries. Gyeong quoted Problem 4–7 of the section Xiuzhu (修築) in Xiangming as Problem 14–17. The final two problems are quotation of the final two problems in the corresponding section of Qimeng.

Kim SiJin (金始振, 1618–1667) republished Suanxue Qimeng in 1660 which gave a great stimulations for a revival of Joseon mathematics, in particular the theory of equations [7]. Park Yul (朴繘, 1621–1668) wrote SanHak WonBon (算學原本, 1700) mainly focused on Tianyuanshu (天元術) but he did not pay any attentions to the volumes except those of spheres [12].

We now discuss the volume in the most important mathematical work, GuIlJib (九一集, 1724) in the history of Joseon mathematics. It was written by Hong JeongHa (洪正夏, 1684–?), a mathematical official in HoJo. It consists of 3 books with 9 chapters. The first 8 chapters form a main body of the book and was written before 1713. Adding the last chapter as an appendix, Hong completed the book in 1724 [11]. Unlike Gyeong and Zhu, he included SangGong SuChukMun (商功修築門) in the first book before ChangDon JeokSokMun (倉囤積粟門) which he put in the second book. In SangGong SuChukMun, he dealt with volumes of trapezoidal prisms, BangDae (方臺 = 方亭), WonDae (圓臺 = 圓亭), BangChu (方錐), WonChu (圓錐) with $\pi = 3, 3.14, \frac{22}{7}$, called GoBeob (古法), HuiBeob (徽法), MilBeob (密法) and JikDae (直臺). As his book shows, he studied Suanxue Qimeng, YangHui Suanfa (楊輝算法, 1274–1275), Xiangming Suanfa and Suanfa Tongzong (算法統宗, 1592) written by Cheng Dawei (程大位, 1533–1606). Hong treated the volume very briefly. But Jik-

Dae (直臺) is the Chutong(芻童) and Hong might take the name from Suanfa Tongzong. Although Chutong was included in Xiangming as simply Tai (臺), it did not get any attention from Joseon mathematicians before Hong JeongHa. As mentioned already, Chutong need not be a frustum. In Cheng's book, Chumeng (芻甍) was mentioned. For the volume of Chumeng with the dimensions a, b, c, h of two sides a, b of the base rectangle, the upper edge c parallel to the side with the length b, and the height h, one has the formula $\frac{ah(2b+c)}{6}$ for its volume. As we discussed earlier, it was shown in Jiuzhang by the subdivision of the solid into the right prism with the height c and two side rectangular cones like the subdivision of Fangting. In this case, one assumes $c \leq b$. The formula is also true for the case $b \leq c$ by subtracting the volumes of the two side rectangular cones from the whole prism in the above subdivision. Now for the volume of Chutong with the dimensions a_1, b_1, a_2, b_2, h of two sides a_1, b_1 and a_2, b_2 of the upper and lower rectangles and the height h where sides with the lengths a_1, a_2 and those with b_1, b_2 are parallel respectively. Then the Chutong is divided into two Chumeng with the dimensions b_2, a_2, a_1, h and b_1, a_1, a_2, h in the order of the Chumeng as mentioned above and hence the volume of the Chutong is given by the sum

$$\frac{b_2h(2a_2+a_1)}{6} + \frac{b_1h(2a_1+a_2)}{6} = \frac{h[b_1(2a_1+a_2)+b_2(2a_2+a_1)]}{6}.$$

As the case of Fangting in the previous discussion, Liu Hui added another formula for the volume of the Chutong as follows:

$$\frac{h}{3}[a_1b_1 + a_2b_2 + \frac{a_1b_2+b_1a_2}{2}].$$

Liu Hui's formula contains that for Fangting as a special case of $a_i = b_i$. As we mentioned earlier, solids in Jiuzhang are not exactly defined. Fangting is assumed to be a truncation of a right cone with a square base so that the square prism in the division of Fangting sits exactly in the middle of the solid. As is well known, the formula for the volume of Fangting works for a truncation of an oblique cone with a square base. To have the volume of Chumeng in the above discussion, the position of the upper edge is irrelevant because the given dimensions of a Chumeng determine uniquely two sides of any cross sectional rectangle for the same height. Thus using the Cavalieri's principle, its volume is uniquely determined by its dimensions. The same remark holds for Chutong. Liu Hui dealt with special cases of Chumeng and Chutong where the upper base sit exactly in the middle of the lower base in his commentary. We also point out that Chutong in Shuli Jingyun (数理精蘊, 1723) is a truncation of a pyramid with a rectangular base in Problem 12 of Chapter 25. Basic blocks of polyhedra, Yangma (陽馬), Bienao (鼈臑) in Jiuzhang and Chumeng as Churao (芻蕘) were included in Shuli Jingyun. We do not know the reason why its authors dealt with a special case of Chutong with same formulae in Jiuzhang.

We now return to GuIlJib. In its second book, Hong JeongHa dealt with volumes of granaries in the section ChangDon JeokSokMun (倉囤積粟門) as we mentioned above. It contains 26 problems. Hong dealt with the usual granaries in the first six problems as Gyeong SeonJing did, but he used the term Gyo, or Jiao (窖) instead of Don (囤). Problem 7 deals with the volume of a granary in a ship and is a variation of the second problem in the section Suanliang Chuanzaimifa (算量船載米法) of Zhiming Suanfa (指明算法). We have a version corrected by Zheng YuanMei (鄭元美, ca. 17th C.) [6]. We first discuss the volume of a trapezoidal right prismoid whose lateral faces are all trapezoids, which we call Wang's dyke, di (隄), because Wang Xiaotong (王孝通) first had the formula of its volume in his work Jigu Suanjing (緝古算經, ca. 7th C.). Its volume is given by

$$\frac{l}{6}[\frac{(a_1+b_1)(2h_1+h_2)}{2}+\frac{(a_2+b_2)(2h_2+h_1)}{2}],$$

where a_1, b_1, h_1 are the dimensions of two parallel sides and height of a lower base trapezoid, a_2, b_2, h_2 those of the upper base and l is the height of the prismoid.

A granary in a ship, called SeonChang or Chinese Chuancang (船倉) is given by combining two Wang's dykes in which the upper face of the lower one is the lower face of the upper one so that each base is a hexagon made of two trapezoids, called Sanguangtian (三廣田). Its dimensions are given by $a_1, b_1, c_1, a_2, b_2, c_2, h, l$, where c_i is the length of the common side of two trapezoids and h the sum of heights of two trapezoids. It is well known that the sum of two heights is not enough to evaluate the area. Thus in this case, it is assumed that the two trapezoids have the same height $\frac{h}{2}$. Using these and the volume of Wang's dyke, Hong has the volume of SeonChang by a slightly different but equivalent formula in Zhiming as follows:

$$\frac{hl}{2}[\frac{m_1+m_2}{2}+m_3],$$

where $m_1 = \frac{a_1+b_1}{2}, m_2 = \frac{a_2+b_2}{2}$ and $m_3 = \frac{c_1+c_2}{2}$.

In Suanxue Baojian (算學寶鑑, 1524), Wang Wensu (王文素, ca. 15th C.) claimed that Xiangming Suanfa and Zhiming Suanfa had the wrong formula for Wang's dyke as divided by 5 instead of 6. The wrong one is also quoted in Suanfa Tongzong. We can not find the wrong formula in Zhiming so that it might be deleted by Zheng YuanMei. Hong JeongHa used the remaining 17 problems in ChangDon JeokSokMun to construct equations mostly cubic ones. It is well known that Hong's main purpose to write GuIlJib is the theory of equations [7, 11].

Jo TaeGu (趙泰耈, 1660–1725) was the minister of HoJo (戶曹判書) when Hong JeongHa served as a mathematical offical in HoJo, and wrote JuSeo GwanGyeon (籌書管見, 1718), one of the great mathematical works in Joseon dynasty. Jo dealt with volumes of solids which appeared in the previous two authors and used Liu's formula for Chutong first in Joseon mathematical works. We can not find its source so far.

Hwang YunSeok (黃胤錫, 1729–1791) is a unique encyclopedist in Joseon dynasty who wrote mathematical works, indeed two books and a commentary on Park Yul's SanHak WonBon. He included the volumes of solids in the sections Panliang Cangjiao as in Xiangming Suanfa and SangGong SuChuk (商功修築). He invented terminologies YangBangChang (兩方倉), YangJuChang (兩周倉) for Fangting and Yuanting (圓亭) respectively. He followed volumes as discussed above except that he also quoted the first problem of Chuancang (船倉) in Jiming Suanfa which is a prism, a special case of Chuancang discussed in Hong JeongHa's GuIlJib.

Nam ByeongGil (南秉吉, 1820–1869) is one of the last traditional mathematicians in Joseon dynasty with Lee SangHyeok (李尙爀, 1810–?) and Jo HuiSoon (趙羲純, 19th C.). He belonged to the noble class and kept a famous mathematical and astronomical collaboration with a JungIn (中人) mathematician Lee SangHyeok and they produced a remarkable collection of mathematical works. Further, Nam was able to import almost complete collection of Chinese mathematical works from Jiuzhang Suanshu to books published in the 19th century. He was the first mathematician in Joseon to study Jiuzhang Suanshu and wrote a book, GuJang SulHae (九章術解) presumably before 1854 [10]. Shuli Jingyun was also studied by astronomical officials and Nam served as the chief of GwanSangGam and Lee SangHyeok as its official. Their mathematics in the early period depended strongly on Shuli Jingyun. In GuJang SulHae, Nam used freely terminologies in Shuli. In the chapter SangGong (商功), their appearances are quite prominent. Further, his study of Jiuzhang was done presumably before his collaboration with Lee SangHyeok, for GuJang SulHae shows that Nam did not fully appreciate Liu Hui's commentary. His choices for terminologies in Shuli are confusing. In the second method of volumes as discussed in the above, he did not mention the added formula obtained by Liu Hui's subdivision but just quoted the part of the verification without examples with given dimensions. Nam ByeonGil and Lee SangHyeok published SanHak JeongUi (算學正義, 1867) where they included 6 problems of volumes in the section GakCheYul (各體率). The first 4 problems are just quotation of Problem 9–12 of Chapter 25 in Shuli and the last two problems deal with volumes of a sphere (圓球) and prolate spheroid (撱圓體) in the setting of $\pi = \frac{355}{113}$. This may be the first appearance of the volume, equivalent to $\frac{4\pi r^3}{3}$ of the sphere with the radius r in Joseon mathematical works. It is strange that Shuli included the traditional approximations, $3, \frac{355}{111}, \frac{22}{7}$ for π but not the Huilu (徽率) $\frac{157}{50} = 3.14$. Hong DaeYong (洪大容, 1713–1783) included Shuli Jingyun as a reference in his book JuHae SuYong (籌解需用) but still used the old formula $\frac{9d^3}{16}$ for the volume of a sphere with the diameter d in the book.

3 Conclusions

Except the theory of equations and finite series, basic structures of Chinese mathematics including East Asian mathematics were essentially established in Jiuzhang Suanshu. As is well known, commentary of Jiuzhang by Liu Hui gave the foundations for the structural approaches to its mathematics. Since then, mathematicians mostly have felt free to use the final results in Jiuzhang but not much paid their attentions to the mathematical structures contained in Jiuzhang. Since ancient times, countries in East Asia have been mostly agricultural ones, the areas of farmlands along with volumes of relating earthworks were one of the most important mathematical subjects in East Asian countries. Except the volumes of spheres, the results about volumes of solids and their verifications by Liu Hui in Jiuzhang Suanshu are completely correct and hence one can hardly find any significant development on the theory of volumes in Chinese mathematical works after Jiuzhang.

The history of volumes in Joseon mathematics has developed not far from that in Chinese mathematics but focused more on volumes of granaries than on those of earthworks. Furthermore, Jiuzhang Suanshu was brought into Joseon in the mid 19th century so that Joseon mathematicians have not known Liu's really important contributions to the mathematical structures. Joseon mathematicians before the 19th century knew him only through his approximation Huilu of π.

The oldest extant mathematical book in Joseon is MukSaJibSanBeob (默思集算法, ca. 17th C.) written by a mathematical official Gyeong SeonJing in HoJo. Gyeong has included the subdivision method to get the volume of Fangting, the truncations of pyramids with square bases and extended it to that of Yuanting. Hong JeongHa used the volumes mainly for constructing cubic equations by Tianyuanshu in his GuIlJib (九一集, 1724) whose main purpose was the theory of equations and quoted the formula of a storage in a ship in Zhiming Suanfa whose author is not known. Using the theory of volumes in Shuli Jingyun, Nam ByeongGil wrote GuJang SulHae (九章術解) where he did not appreciate Liu Hui's commentary. Nam and Lee SangHyeok included the present day formulae of spheres in their book SanHak JeongUi (算學正義, 1867) at long last. Joseon mathematicians were concerned about the volumes of solids but they were mostly satisfied with applications of formulae of volumes and disregarded their structures.

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