Approximation of Common Fixed Points of Mean Non-expansive Mapping in Banach Spaces

GU ZHAOHUI
Cisco School of Informatics, Guangdong University of Foreign Studies, Guangzhou, 510420, P. R. China
e-mail: zhungz@163.com

LI YONGJIN∗
Department of Mathematics, Sun Yat-Sen University, Guangzhou, 510275 P. R. China
e-mail: stslyj@mail.sysu.edu.cn

Abstract. Let $X$ be a uniformly convex Banach space, and $S, T$ be pair of mean non-expansive mappings. Some necessary and sufficient conditions are given for Ishikawa iterative sequence converge to common fixed points, and we prove that the sequence of Ishikawa iterations associated with $S$ and $T$ converges to the common fixed point of $S$ and $T$. This generalizes former results proved by Z. Gu and Y. Li [4].

1. Introduction

Let $X$ be a Banach space and $S, T$ be mappings from $X$ to $X$. In [8] the Ishikawa iteration sequence $\{x_n\}$ of $T$ was defined by

\begin{align}
(1.1) & \quad y_n = (1 - \beta_n)x_n + \beta_n Tx_n \\
(1.2) & \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n
\end{align}

Where $x_0 \in X, \alpha_n, \beta_n \in [0, 1]$.

The pair of mean non-expansive mappings was introduced by Bose in [2].

\begin{equation}
(1.3) \quad \|Sx - Ty\| \leq a\|x - y\| + b(\|x - Sx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Sx\|)
\end{equation}

* Corresponding Author.

Received March 19, 2012; accepted November 7, 2012.

2010 Mathematics Subject Classification: 26D15.

Key words and phrases: Pair of mean non-expansive mappings, Ishikawa iteration, Common fixed point.
for all $x, y \in X, a, b, c \in [0, 1], a + 2b + 2c \leq 1$.

The Ishikawa iteration sequence $\{x_n\}$ of $S$ and $T$ was defined by

\begin{align}
(1.4) & \quad y_n = (1 - \beta_n)x_n + \beta_nSx_n \\
(1.5) & \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTy_n
\end{align}

Where $x_0 \in X, \alpha_n, \beta_n \in [0, 1]$.

The problems about common fixed point for pair of mappings as an important part of the fixed point theory have been studied by many authors, see [2-4,6-14], for more details. In [2], S. C. Bose defined the pair of mean non-expansive mappings, and proved the existence of the fixed point in Banach spaces. In particular, he proved the following theorem.

**Theorem 1.1.** [2] Let $X$ be a uniformly convex Banach space and $K$ a non-empty closed convex subset of $X$, $S : K \to K$ and $T : K \to K$ are pair of mean non-expansive mappings, and $c \neq 0$, then

(i) $S$ and $T$ have a common fixed point $u$.

(ii) Further if $b \neq 0$, then

(a) $u$ is the unique common fixed point and unique as a fixed point of each $S$ and $T$.

(b) the sequence $\{x_n\}$ defined by $x_1 = Sx_0, x_2 = Tx_1, x_3 = Sx_2 \cdots$ for any $x_0 \in K$, converges strongly to $u$.

In [14], Z. Gu and Y. Li proved the following theorem.

**Theorem 1.2.** Let $X$ be a uniformly convex Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, 0 < \alpha \leq \alpha_n \leq \frac{1}{2}, 0 \leq \beta_n \leq \beta < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S$ and $T$.

It is our purpose in this paper to give some necessary and sufficient conditions for Ishikawa iterative sequence converge to common fixed points, and consider the iterative scheme, which converges to a common fixed point of the pair of mean non-expansive mappings. Our Theorem 2.1 extends and improves the corresponding results in [14].

To obtain the main results of the paper, we prove the following lemmas.

**Lemma 1.3.** [13] Let $X$ be a Banach space. Then $X$ is uniformly convex if and only if for any given number $\rho > 0$, the square norm $\| \cdot \|^2$ of $X$ in uniformly convex on $B_\rho$, the closed unit ball at the origin with radius $\rho$; namely, there exist a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that
\[ \| \alpha x + (1 - \alpha)y\| ^2 \leq \alpha \| x\|^2 + (1 - \alpha)\| y\|^2 - \alpha(1 - \alpha)\varphi(\| x - y\|), \]

for all \( x, y \in B_p, \alpha \in [0, 1]. \)

**Lemma 1.4.** Let \( X \) be a Banach space, \( S : X \to X \) and \( T : X \to X \) are pair of mean non-expansive mappings with a common fixed point, then for any \( x_0 \in X \), the Ishikawa sequence \( \{ x_n \} \) associated with \( S \) and \( T \) is bounded.

**Proof.** For a common fixed point \( p \) of \( S \) and \( T \), we have
\[
\| Tx - p\| = \| Tx - Sp\| \\
\leq \alpha \| x - p\| + b(\| x - Tx\| + \| p - Sp\|) \\
+ c(\| x - Sp\| + \| p - Tx\|) \\
\leq \alpha \| x - p\| + b(\| x - p\| + \| p - T x\|) \\
+ c(\| x - Sp\| + \| p - T x\|)
\]

Let \( L = \frac{a + b + c}{1 - b - c} \), by \( a + 2b + 2c \leq 1 \), it is easy to see that \( a + b + c \leq 1 - b - c \), thus \( 0 \leq L \leq 1 \), and \( \| Tx - p\| \leq L \| x - p\| \leq \| x - p\| \).

Similarly, we have \( \| S x - p\| \leq L \| x - p\| \leq \| x - p\| \).

\[
\| x_{n+1} - p\| = \| (1 - \alpha_n)x_n + \alpha_nTy_n - p\| \\
= \| (1 - \alpha_n)(x_n - p) + \alpha_n(Ty_n - p)\| \\
\leq (1 - \alpha_n)\| x_n - p\| + \alpha_n\| Ty_n - p\| \\
\leq (1 - \alpha_n)\| x_n - p\| + \alpha_nL\| y_n - p\| \\
= (1 - \alpha_n)\| x_n - p\| + \alpha_n(1 - \beta_n)x_n + \beta_nSx_n - p\| \\
= (1 - \alpha_n)\| x_n - p\| + \alpha_n(1 - \beta_n)(x_n - p) + \beta_n(Sx_n - p)\| \\
\leq (1 - \alpha_n)\| x_n - p\| + \alpha_n(1 - \beta_n)\| x_n - p\| \\
+ \alpha_n\beta_n\| Sx_n - p\| \\
\leq (1 - \alpha_n)\| x_n - p\| + \alpha_n(1 - \beta_n)\| x_n - p\| \\
+ \alpha_n\beta_n\| x_n - p\| \\
= (1 - \alpha_n) + \alpha_n(1 - \beta_n) + \alpha_n\beta_n\| x_n - p\| \\
= \| x_n - p\|
\]

So
\[
(1.6) \quad \| x_{n+1} - p\| \leq \| x_n - p\| \leq \| x_{n-1} - p\| \leq \ldots \leq \| x_0 - p\|
\]

Thus, \( \{ x_n \} \) is bounded. \( \Box \)

2. Section 2

First, we give some necessary and sufficient conditions for Ishikawa iterative
sequence converge to common fixed points.

**Theorem 2.1.** Let $X$ be a Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, c > 0$, then the necessary and sufficient conditions that the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S$ and $T$ is:

\[(2.1) \lim_{n \to \infty} \inf \|x_n - Ty_n\| = 0.\]

**Proof.** For the first step, we will prove the sufficiency.

Let us first prove $\lim_{n \to \infty} \|x_n - Sx_n\| = 0$. Since

\[\|x_n - Sx_n\| \leq \|x_n - Ty_n\| + \|Ty_n - Sx_n\|\]

\[\leq \|x_n - Ty_n\| + a\|x_n - y_n\| + b\|x_n - Sx_n\| + \|y_n - Ty_n\|\]

\[+ c\{\|x_n - Ty_n\| + \|y_n - Sx_n\|\} = (1 + c)\|x_n - Ty_n\| + a\|x_n - y_n\| + b\|x_n - Sx_n\|\]

\[= (1 + c)\|x_n - Ty_n\| + a\|(1 - \beta_n)x_n + (1 - \beta_n)Sx_n - x_n\|\]

\[+ b\|x_n - Sx_n\| + b\|1 - \beta_n\|x_n - Sx_n + \|Ty_n - y_n\|\]

\[= (1 + c)\|x_n - Ty_n\| + a\|\|1 - \beta_n\|x_n + \beta_nSx_n - Sx_n\|\]

\[+ b\|x_n - Sx_n\| + b\|1 - \beta_n\|x_n - Sx_n + \|Ty_n - y_n\|\]

\[= (1 + c)\|x_n - Ty_n\| + a\|x_n - Sx_n\|\]

\[+ b\|x_n - Sx_n\| + a\|1 - \beta_n\|x_n - Sx_n + \|Ty_n - y_n\|\]

\[+ c\|x_n - Sx_n\|\]

we have

\[(2.2)(1 - a\beta_n - b - b\beta_n - c(1 - \beta_n))\|x_n - Sx_n\| \leq (1 + c)\|x_n - Ty_n\|\]

Let $M_1 = 1 - a\beta_n - b - b\beta_n - c(1 - \beta_n)$, then

\[M_1 = 1 - a\beta_n - b - b\beta_n - c + c\beta_n\]

\[= 1 - b - c - (a + b - c)\beta_n\]

\[\geq a + b + c - (a + b - c)\beta_n\]

\[> 0\]

so

\[(2.3) \|x_n - Sx_n\| \leq \frac{1 + b + c}{M_1} \|x_n - Ty_n\|\]
if \( \lim_{n \to \infty} \inf \|x_n - Ty_n\| = 0 \), there exists subsequence \( \|x_{n_k} - Ty_{n_k}\| \to 0 \) when \( n_k \to \infty \), by (2.3) we have \( \lim_{n_k \to \infty} \|x_{n_k} - Sx_{n_k}\| = 0 \).

Next we will show that sequence \( \{Sx_{n_k}\} \) is a Cauchy sequence. For any \( n_{k_1}, n_{k_2} \), we have

\[
\|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| \\
\leq \|Sx_{n_{k_1}} - Ty_{n_{k_2}}\| + \|Sx_{n_{k_2}} - Ty_{n_{k_2}}\| \\
\leq a\|x_{n_{k_1}} - y_{n_{k_2}}\| + b\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|y_{n_{k_2}} - Ty_{n_{k_2}}\|\} \\
+ c\{\|x_{n_{k_1}} - Ty_{n_{k_2}}\| + \|y_{n_{k_2}} - Sx_{n_{k_2}}\|\} \\
\leq a\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| + \|y_{n_{k_2}} - Ty_{n_{k_2}}\|\} \\
+ b\{\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + \|y_{n_{k_2}} - Ty_{n_{k_2}}\|\} \\
+ c\{\|x_{n_{k_1}} - Ty_{n_{k_1}}\| + \|y_{n_{k_2}} - Sx_{n_{k_2}}\|\} \\
+ \|Sx_{n_{k_2}} - Ty_{n_{k_2}}\| + \|Sx_{n_{k_2}} - Ty_{n_{k_2}}\|
\]

Since \( b > 0 \), thus obviously we have \( 1 - a - 2c > 0 \). Simplify, then we get

\[
\|Sx_{n_{k_1}} - Sx_{n_{k_2}}\| \leq R_1\|x_{n_{k_1}} - Sx_{n_{k_1}}\| + R_2\|y_{n_{k_2}} - Ty_{n_{k_2}}\| \\
+ R_3\|y_{n_{k_2}} - Sx_{n_{k_2}}\| + R_4\|Sx_{n_{k_2}} - Ty_{n_{k_2}}\|
\]

Where \( R_1 = \frac{a + b + c}{1 - a - 2c} \geq 0 \), \( R_2 = \frac{b}{1 - a - 2c} \geq 0 \), \( R_3 = \frac{a + c}{1 - a - 2c} \geq 0 \), and \( R_4 = \frac{1 + c}{1 - a - 2c} \geq 0 \).

According to Ishikawa iteration

\[
y_n = (1 - \beta_n)x_n + \beta_n Sx_n \\
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n
\]

It is easy to see that \( \|x_{n_k} - y_{n_k}\| = \beta_{n_k}\|x_{n_k} - Sx_{n_k}\| \) and \( \|Sx_{n_k} - y_{n_k}\| = (1 - \beta_{n_k})\|x_{n_k} - Sx_{n_k}\| \). Then we consider the sequence \( \|Ty_{n_k} - Sx_{n_k}\| \),

\[
\|y_{n_k} - Ty_{n_k}\| = \|(1 - \beta_{n_k})x_{n_k} + \beta_{n_k} Sx_{n_k} - Ty_{n_k}\| \\
\leq (1 - \beta_{n_k})\|x_{n_k} - Ty_{n_k}\| + \beta_{n_k}\|Sx_{n_k} - Ty_{n_k}\|
\]
From the definition of the mean nonexpansive mappings, we obtain
\[
\|Ty_{n_k} - Sx_{n_k}\| \leq a\|x_{n_k} - y_{n_k}\| + b\{\|x_{n_k} - Sx_{n_k}\| + \|y_{n_k} - Ty_{n_k}\|\} \\
+ c\{\|x_{n_k} - Ty_{n_k}\| + \|y_{n_k} - Sx_{n_k}\|\} \\
\leq a\beta_{n_k}\|x_{n_k} - Sx_{n_k}\| + b\|x_{n_k} - Sx_{n_k}\| \\
+ b(1 - \beta_{n_k})\|x_{n_k} - Ty_{n_k}\| + b\beta_{n_k}\|Sx_{n_k} - Ty_{n_k}\| \\
+ c\|x_{n_k} - Ty_{n_k}\| + c(1 - \beta_{n_k})\|x_{n_k} - Sx_{n_k}\| \\
= (a\beta_{n_k} + b + c(1 - \beta_{n_k}))\|x_{n_k} - Sx_{n_k}\| \\
+ (b(1 - \beta_{n_k}) + c)\|x_{n_k} - Ty_{n_k}\| + b\beta_{n_k}\|Sx_{n_k} - Ty_{n_k}\| \\
\]

Since
\[
\|x_{n_k} - Sx_{n_k}\| \leq \|Sx_{n_k} - Ty_{n_k}\| + \|x_{n_k} - Ty_{n_k}\| \\
\]
we can get the following inequality
\[
\|Ty_{n_k} - Sx_{n_k}\| \leq (b(1 - \beta_{n_k}) + c + a\beta_{n_k} + b + c(1 - \beta_{n_k}))\|x_{n_k} - Ty_{n_k}\| \\
+ (b\beta_{n_k} + a\beta_{n_k} + b + c(1 - \beta_{n_k}))\|Sx_{n_k} - Ty_{n_k}\| \\
\]

Therefore,
\[
(1 - b - c - (a + b - c)\beta_{n_k})\|Ty_{n_k} - Sx_{n_k}\| \\
\leq (b(1 - \beta_{n_k}) + c + a\beta_{n_k} + b + c(1 - \beta_{n_k}))\|x_{n_k} - Ty_{n_k}\| \\
\]

Since \(0 \leq \beta_{n_k} \leq 1\), so we can get \(1 - b - c - (a + b - c)\beta_{n_k} \geq (a + b)(1 - \beta_{n_k}) + c(1 + \beta_{n_k}) > 0\), and it is easy to see that
\[
b(1 - \beta_{n_k}) + c + a\beta_{n_k} + b + c(1 - \beta_{n_k}) > 0 \\
\]

According to the condition \(\|x_{n_k} - Ty_{n_k}\| \to 0\), thus we get
\[
(2.4) \quad \lim_{n_k \to \infty} \|Sx_{n_k} - Ty_{n_k}\| = 0 \quad \text{and} \quad \lim_{n_k \to \infty} \|y_{n_k} - Ty_{n_k}\| = 0 \\
\]

So \(\lim_{n_k \to \infty} \|x_{n_k} - y_{n_k}\| = \lim_{n_k \to \infty} \beta_{n_k}\|Sx_{n_k} - x_{n_k}\| = 0\). We know
\[
(2.5) \quad \|x_{n_k} - Sx_{n_k}\| \to 0, \quad \|y_{n_k} - Ty_{n_k}\| \to 0, \quad \|Sx_{n_k} - Ty_{n_k}\| \to 0 \\
\]

It is easy to see that \(\|y_{n_k} - Sx_{n_k}\| \to 0\), in particular, \(\|x_{n_k} - Sx_{n_k}\| \to 0\), \(\|y_{n_k} - Sx_{n_k}\| \to 0\), \(\|Sx_{n_k} - Ty_{n_k}\| \to 0\) thus \(\|Sx_{n_k} - Sx_{n_k}\| \to 0\), it is also say that \(\{Sx_{n_k}\}\) is a Cauchy sequence. We may assume that \(p = \lim_{n_k \to \infty} Sx_{n_k}\), so we have \(\lim_{n_k \to \infty} x_{n_k} = p\) using (1.4), we can get \(\lim_{n \to \infty} x_n = p\). By (1.1), we have
\[
\|Sx_{n_k} -Tp\| \leq a\|x_{n_k} - p\| + b\{\|x_{n_k} - Sx_{n_k}\| + \|p - Tp\|\} \\
+ c\{\|x_{n_k} - Tp\| + \|p - Sx_{n_k}\|\} \\
\]
Let $n_k \to \infty$, we get

\[ \|p - Tp\| \leq (b + c)\|p - Tp\| \]

Since $b + c < 1$, that means $\|p - Tp\| = 0$, that is $Tp = p$, $p$ is a fixed point of $T$. Similarly, we can prove that $Sp = p$, thus, $\{x_n\}$ converges to the common fixed point of $S$ and $T$.

For the second step, we will prove the necessity, if $\{x_n\}$ converges to the common fixed point of $S$ and $T$, we assume that $\lim_{n \to \infty} x_n = p$,

\[
\|x_n - Ty_n\| \leq \|x_n - p\| + \|Ty_n - p\| \\
\leq \|x_n - p\| + \|y_n - p\| \\
\leq \|x_n - p\| + \|(1 - \beta_n)x_n + \beta_n Sx_n - p\| \\
\leq \|x_n - p\| + (1 - \beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \\
= (1 + 1 - \beta_n + \beta_n)\|x_n - p\| \\
= 2\|x_n - p\| \\
\]

Since $\lim_{n \to \infty} x_n = p$, thus $\lim_{n \to \infty} \|x_n - Ty_n\| = 0$, that is $\lim_{n \to \infty} \inf \|x_n - Ty_n\| = 0$.

**Corollary 2.2.** Let $X$ be a Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0$, $c > 0$, then the necessary and sufficient conditions that the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S$ and $T$ is:

\[
\lim_{n \to \infty} \|x_n - Ty_n\| = 0.
\]

**Proof.** If $\lim_{n \to \infty} \|x_n - Ty_n\| = 0$, then $\lim_{n \to \infty} \inf \|x_n - Ty_n\| = 0$. By Theorem 2.1, we get $\lim_{n \to \infty} x_n = p$.

For the converse, we assume that $\lim_{n \to \infty} x_n = p$, then

\[
\|x_n - Ty_n\| \leq \|x_n - p\| + \|Ty_n - p\| \\
\leq \|x_n - p\| + \|y_n - p\| \\
\leq \|x_n - p\| + \|(1 - \beta_n)x_n + \beta_n Sx_n - p\| \\
\leq \|x_n - p\| + (1 - \beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \\
\leq (1 + 1 - \beta_n + \beta_n)\|x_n - p\| \\
= 2\|x_n - p\| \\
\]

Since $\lim_{n \to \infty} x_n = p$, thus $\lim_{n \to \infty} \|x_n - Ty_n\| = 0$. 

The following theorem is a strong convergence theorem for Ishikawa iteration in uniformly convex Banach spaces.
Theorem 2.3. Let $X$ be a uniformly convex Banach space, $S : X \to X$ and $T : X \to X$ are pair of mean non-expansive with a nonempty common fixed points set, if $b > 0, c > 0, 0 < \alpha \leq \alpha_n \leq \alpha' < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the common fixed point of $S$ and $T$.

Proof. Let $p$ is a common fixed point of $S$ and $T$, by Lemma 1.3, we have

$$\|x_{n+1} - p\|^2 = \|(1 - \alpha_n)x_n + \alpha_nTy_n - p\|^2 \leq \alpha_n\|Ty_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|)$$

However

$$\|y_n - p\| = \|(1 - \beta_n)x_n + \beta_nSx_n - p\| \leq (1 - \beta_n)\|x_n - p\| + \beta_n\|Sx_n - p\| \leq (1 - \beta_n + \beta_n)\|x_n - p\| = \|x_n - p\|$$

Therefore

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - \alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|)$$

We can get

$$\alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2$$

Since $0 < \alpha \leq \alpha_n \leq \alpha' < 1$, thus $\alpha_n(1 - \alpha_n) > \alpha(1 - \alpha') > 0$. By Lemma 1.4, $\|x_n - p\|^2$ is a real decreasing bounded sequence, so $\|x_n - p\|^2$ converges.

Hence for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that whenever $n > n_0$, we have

$$\alpha_n(1 - \alpha_n)\varphi(\|Ty_n - x_n\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 < \varepsilon$$

thus $\varphi(\|Ty_n - x_n\|) \to 0$, as $n \to \infty$, and hence $\|Ty_n - x_n\| \to 0$, by the continuity and strictly increasing nature of $\varphi$. By Theorem 2.1, we get that $\{x_n\}$ converges to the common fixed point of $S$ and $T$, so that the conclusion of the theorem follows. \qed

Between the pair of mappings, if $S = T$, then

$$\|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|)$$

the mapping $T$ is called mean nonexpansive mapping, we obtain the following.

Corollary 2.4. Let $X$ be a Banach space, $T : X \to X$ is a mean non-expansive with a nonempty fixed points set, if $b > 0, c > 0, 0 < \alpha \leq \alpha_n \leq \alpha' < 1$, then the Ishikawa sequence $\{x_n\}$ converges to the fixed point of $T$. 
References


