T-NEIGHBORHOODS IN VARIOUS CLASSES OF ANALYTIC FUNCTIONS

Saeid Shams, Ali Ebadian, Mahta Sayadiazar, and Janusz Sokół

Abstract. Let $A$ be the class of analytic functions $f$ in the open unit disk $U = \{z : |z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $\delta > 0$ are given, then the $T_\delta$-neighborhood of the function $f$ is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta \right\},$$

where $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. In the present paper we investigate some problems concerning $T_\delta$-neighborhoods of functions in various classes of analytic functions with $T = \{2^{-n/2}\}_{n=2}^{\infty}$. We also find bounds for $\delta^*_T(A, B)$ defined by

$$\delta^*_T(A, B) = \inf \{ \delta > 0 : B \subset TN_\delta(f) \text{ for all } f \in A \},$$

where $A, B$ are given subsets of $A$.

1. Introduction

Let $A$ denote the class of analytic functions $f$ in the open unit disk $U = \{z : |z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then the $T_\delta$-neighborhood of the function $f$ is defined as

$$TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta \right\},$$

where $\delta$ is a positive number and $T = \{T_n\}_{n=2}^{\infty}$ is a sequence of positive numbers. St. Ruscheweyh in [14] considered $T = \{n\}_{n=2}^{\infty}$ and showed that if $f \in C$, then $TN_{1/4}(f) \subset S^*$, where $C, S^*$ denote the well known classes of convex and starlike functions, respectively. In [4, 5, 6, 7, 10, 11, 12, 17, 18] other authors investigated some interesting results concerning neighborhoods of several classes of analytic functions. Some of the relations between the neighborhoods for a certain class of analytic functions was described by S. Shams et al. [15].
Also U. Bednarz and J. Sokół in [7] considered $T = \{\frac{1}{n^2(n-1)}\}_{n=2}^{\infty}$ and investigated $T_3$-neighborhood for various subclasses of analytic functions. Motivated by the above results, we consider in this paper $T_3$-neighborhood (1.1) with

$$T = \{2^{-n}n^{-2}\}_{n=2}^{\infty}.$$ 

We use this sequence because it is sufficiently strongly convergent to $0$, which is necessary for the series considered here to be convergent. Notice that

$$\sum_{n=1}^{\infty} 2^{-n}n^{-2} = \pi^2/12 - (\log 2)^2/2$$

and it is the value of dilogarithm at $1/2$, [13].

The convolution or Hadamard product of the functions $f$ and $g$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad |z| < 1,$$

is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

**Definition 1.1** ([2]). Let us consider the functions $f$ that are meromorphic and univalent in $U$, holomorphic at 0 and have the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. If, in addition, the complement of $f(U)$ with respect to $\mathbb{C}$ is convex, then $f$ is called a concave univalent function. The class of all concave functions is denoted by $C_0$.

It is well known [1], that if $f \in C_0$, then $|a_n| \geq 1$ for all $n > 1$ and equality holds if and only if $f(z) = z/(1 - \mu z)$, $|\mu| = 1$ (see [1, 3]). The authors in [2] considered the class $C_0(p) \subset C_0$ consisting of all concave functions that have a pole at the point $p$ and are analytic in $|z| < |p|$. They proved that if $f \in C_0(1)$, then

$$|a_n - n + \frac{1}{2}| \leq \frac{n - 1}{2} \quad \text{for } n \geq 2,$$

and equality holds only for the function $f_0$ defined by

$$f_0(z) = \frac{2z - (1 - e^{i\theta})z^2}{2(1 - z)^2}, \quad |z| < 1.$$ 

It is well known that if $f \in C_0(1)$, then the complement of $f(U)$ can be represented as the union of a set of mutually disjoint half-lines (the end point of one half-line can lie on the another half-line), so $f(U)$ is a linearly accessible domain in the strict sense (see [8, 16]).

The authors in [7] also showed that $C_0(1) \subset K$, where $K$ is the set of close-to-convex functions.

2. Main results

Throughout this section $T$ will always be the sequence given by

$$(2.1) \quad T = \{T_n\}_{n=2}^{\infty} = \{2^{-n}n^{-2}\}_{n=2}^{\infty},$$

unless otherwise stated.
Theorem 2.1. If \( f, g \in A \) are of the form
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \]
with \( |a_n| \leq n \) and \( |b_n| \leq n \) for \( n = 2, 3, 4, \ldots \), then \( g \in TN_{\log\{4/e\}}(f) \), where \( T \) is given in (2.1). The number \( \log\{4/e\} \) is the best possible.

Proof. A simple calculation shows that
\[ \sum_{n=1}^{\infty} \frac{z^n}{n^{2n}} = \int_0^z \sum_{n=1}^{n-1} \frac{\zeta^{n-1}}{2^n} d\zeta = \int_0^z \frac{1/2}{1 - \zeta/2} d\zeta = \log \frac{1}{1 - z/2}, \quad |z| < 2, \]
so we have
\[ \sum_{n=1}^{\infty} \frac{1}{n^{2n}} = \log 2, \]
and then
\[ \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \sum_{n=2}^{\infty} \frac{2n}{n^2 2^n} = 2 \sum_{n=2}^{\infty} \frac{1}{n^{2n}} = 2 \log 2 - 1 = \log\{4/e\}. \]

For the functions
\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z - \sum_{n=2}^{\infty} n z^n \]
we have
\[ \sum_{n=2}^{\infty} T_n |a_n - b_n| = 2 \sum_{n=2}^{\infty} \frac{1}{n^{2n}} = \log\{4/e\}. \]
Therefore, the number \( \log\{4/e\} \) cannot be replaced by a smaller one and it is the best possible. \( \square \)

It is well known that \( C \subset S^* \subset K \subset S \) (see [9]), where \( S, S^*, C \) and \( K \) denote the classes of univalent, starlike, convex and close-to-convex functions, respectively. Also, if \( f \in S^* \), then \( |a_n| \leq n, n = 2, 3, \ldots \), while if \( f \in C \), then \( |a_n| \leq 1, n = 2, 3, \ldots \).

Therefore we obtain the following corollary.

Corollary 2.2. If \( f \in S \), then we have
\[ S \subset TN_{\log\{4/e\}}(f), \]
where \( T \) is given in (2.1).

The constant \( \log\{4/e\} \approx 0.386 \) seems not to be the best possible. An interesting open problem is to find the smallest constant \( \rho \) such that for each \( f \in S \)
\[ S \subset TN_{\rho}(f), \]
where $T$ is given in (2.1). For the Koebe function $f(z) = z/(1 - z)^2$ and $g(z) = -f(-z)$ we have $f, g \in S$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = z + \sum_{n=2}^{\infty} n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z + \sum_{n=2}^{\infty} (-1)^{n-1} n z^n$$

so by (2.2)

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| = \sum_{k=1}^{\infty} \frac{4k}{(2k)^2 2^k} = \log\{4/3\}.$$ 

Therefore, the number $\varrho$ cannot be smaller than $\log\{4/3\}$. We conjecture that $\varrho = \log\{4/3\} = 0.28768 \cdots$.

**Corollary 2.3.** Let $f \in C$. Then $S \subset TN_{\beta}(f)$ with

$$\beta = \log\{2/e\} + \frac{\pi^2}{12} - \frac{(\log 2)^2}{2} = 0.275 \cdots.$$ 

**Proof.** At first, note that

$$f_2(x) = -\int_1^x \frac{\log t}{t - 1} \text{dt}, \quad x \in [0, 2],$$

is the dilogarithm. From the tables of dilogarithms we have

$$f_2(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^k}{k^2}, \quad x \in [0, 2],$$

(2.6) $f_2(x) + f_2(1-x) = -\log \{x\} \cdot \log \{1-x\} + \pi^2/6$,

(2.7) $f_2(1+x) - f_2(x) = -\log \{x\} \cdot \log \{x+1\} - \pi^2/12 - f_2(x^2)/2$.

Therefore, using (2.5) and (2.6) we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} = f_2(1/2) = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}.$$ 

If

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in C, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in S,$$

then $|a_n| \leq 1, \ |b_n| \leq n$ and by (2.3), (2.8) we have

$$\sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \sum_{n=2}^{\infty} \frac{n + 1}{n^2 2^n} = \sum_{n=2}^{\infty} \frac{1}{n^2 2^n} + \sum_{n=2}^{\infty} \frac{1}{n^2 2^n} = \log \{2/e\} + f_2(1/2) = 0.275 \cdots.$$ 

In a similar way as in Corollary 2.2, the constant $0.275 \cdots$ given in Corollary 2.3 is also not sharp but if the class $S$ is replaced by the much larger class of all normalized analytic functions $f$ such that $|a_n(f)| \leq n$ for $n \geq 2$, then (2.4)
becomes sharp. The best possible constant in the case $f \in \mathcal{S}$ is not known. We conjecture that the sharp constant is attained by the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n = \frac{z}{(1-z)^n} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n = \frac{z}{1+z}.$$  

It is clear that $f \in \mathcal{S}$ and $g \in \mathcal{C}$. Moreover,

$$\sum_{n=2}^{\infty} T_n|a_n - b_n| = \sum_{n=2}^{\infty} \frac{n+1}{2^n n^2} - \sum_{n=2}^{\infty} \frac{1+(-1)^{n-1}}{2^n n^2}  
= \log \{2/e\} + f_2(1/2) - \sum_{k=1}^{\infty} \frac{2}{2^{2k+1}(2k+1)^2}. \quad (2.9)$$

From the tables of dilogarithms we have

$$\sum_{k=1}^{\infty} \frac{2}{2^{2k+1}(2k+1)^2} = \int_0^{1/2} \frac{1}{t} \log \frac{1+t}{1-t} dt - 1 = f_2(1/2) - f_2(3/2) - 1.$$  

By (2.7) we have

$$f_2(1/2) - f_2(3/2) = \frac{f_2(1/4)}{2} + \frac{\pi^2}{12} - \log \{2\} \cdot \log \{3/2\}.$$  

Applying this in (2.9) we further get,

$$\sum_{n=2}^{\infty} T_n|a_n - b_n|  
= \log \{2/e\} + f_2(1/2) - \left\{ \frac{f_2(1/4)}{2} + \frac{\pi^2}{12} - \log \{2\} \cdot \log \{3/2\} - 1 \right\}  
= \log \{2\} \cdot \log \{3e/(2\sqrt{2})\} - \frac{f_2(1/4)}{2} = 0.24473 \cdots,$$

because $f_2(1/4) = 0.978469393 \cdots$. Therefore, the smallest constant $\beta$ such that $\mathcal{S} \subset TN_\beta(f)$ for each $f \in \mathcal{C}$ lies between 0.24473 \cdots and 0.275 \cdots. We conjecture that it is the first number.

**Theorem 2.4.** Let $f, g_1, g_2$ be of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g_1(z) = z + \sum_{n=2}^{\infty} c_n z^n, \quad g_2(z) = z + \sum_{n=2}^{\infty} d_n z^n,$$

where $|a_n| \leq n, \quad |c_n| \leq n, \quad |d_n| \leq n, \quad n = 2, 3, \ldots$. Then

$$g_1 \ast g_2 \in T_{N \log 2}(f).$$

The number $\log 2$ is the best possible.

**Proof.** Since

$$(g_1 \ast g_2)(z) = z + \sum_{n=2}^{\infty} c_n d_n z^n,$$
then we have
\[ \sum_{n=2}^{\infty} \frac{1}{n^2 2^n} |c_n d_n - a_n| \leq \sum_{n=2}^{\infty} \frac{n^2 + n}{n^2 2^n} = \log 2. \]

The functions
\[ f(z) = z - \sum_{n=2}^{\infty} nz^n, \quad g_1(z) = g_2(z) = z + \sum_{n=2}^{\infty} nz^n \]
show that the number \( \log\{2\} \) is the best possible. Therefore the proof is completed. \( \square \)

**Definition 2.5 ([7]).** Let \( A \) and \( B \) be arbitrary subsets of the \( \mathbb{A} \), and let \( T \) be a sequence of positive number, then \( \delta^*_T(A, B) \) is defined by
\[ \delta^*_T(A, B) = \inf \{ \delta > 0 : B \subset T \delta(f) \text{ for all } f \in A \}. \]

Let us denote
(2.10) \[ T(f, g) = \sum_{n=2}^{\infty} T_n |a_n - b_n|. \]

Therefore, we can write
\[ \delta^*_T(A, B) = \inf \{ \delta : \exists T(f, g) < \delta \text{ for all } f \in A, g \in B \} \]
\[ = \sup \{ T(f, g) : f \in A, g \in B \}, \]
where the condition \( T(f, g) < \delta \) means that the series \( T(f, g) \) is convergent and its sum is less than \( \delta \). Therefore, we see that \( \delta^*_T(A, B) = \delta^*_T(B, A) \), and we will say that \( \delta^*_T(A, B) \) is the \( T \)-factor with respect to the classes \( A \) and \( B \).

Making use of the above definition, Corollary 2.2 and the consideration below Corollary 2.2, we can state next corollary where \( T = \{ T_n \}_{n=2}^{\infty} \) is again of the form (2.1).

**Corollary 2.6.** The \( T \)-factor with respect to the classes \( S \) and \( S \) satisfies the following inequality
(2.11) \[ 0.287 \cdots \leq \delta^*_T(S, S) \leq \log\{4/\sqrt{e}\} = 0.386 \cdots. \]

It is well known that the Koebe function and all its rotations belong to each of the classes \( S, S^* \) and \( K \) (univalent, starlike and close-to-convex functions respectively), then Corollary 2.6 follows the next corollary.

**Corollary 2.7.** Let \( A \) and \( B \) be one of the classes \( S, S^* \) or \( K \). Then
\[ \log\{4/3\} \leq \delta^*_T(A, B) \leq \log\{4/\sqrt{e}\}. \]

In the same way as above, we can express Corollary 2.3 in terms \( T \)-factor. It is done in the next result.

**Corollary 2.8.** The \( T \)-factor with respect to the classes \( C \) of convex functions and \( S \) satisfies the following inequality
\[ 0.24473 \cdots \leq \delta^*_T(C, S) \leq 0.275 \cdots. \]
Remark 2.9. Now we consider the “central” function with respect to coefficient in the class $Co(1)$ which is denoted by $f_c(z)$ and defined by

\[
(2.12) \quad f_c(z) = \frac{1}{2} \left\{ \frac{z}{1-z} + \frac{z}{(1-z)^2} \right\} = z + \sum_{n=1}^{\infty} \frac{n+1}{2} z^n, \quad |z| < 1.
\]

In [7] the authors showed that $f_c \in Co(1)$.

Theorem 2.10. The following inclusion relation holds

\[
Co(1) \subset TN_\delta(f_c),
\]

where $\delta = \log \sqrt{2/e} + \frac{\pi^2}{24} - \left( \log 2 \right)^2/4 = 0.13769 \cdots$.

Proof. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Co(1)$, then from (1.2), and using (2.3) and (2.4) with $x = -1/2$, we obtain

\[
\sum_{n=2}^{\infty} T_n \left| a_n - \frac{n+1}{2} \right| \leq \frac{1}{2} \sum_{n=2}^{\infty} \frac{n-1}{n^2 2^n}
\]

\[
\leq \frac{1}{2} \left\{ \log 2 - \frac{1}{2} + f_2(1/2) - \frac{1}{2} \right\}
\]

\[
= \frac{1}{2} \left\{ \log 2 - 1 + \frac{\pi^2}{12} - \left( \log 2 \right)^2/2 \right\}
\]

\[
= 0.13769 \cdots = \delta.
\]

Acknowledgment. The authors would like to express their sincerest thanks to the referees for a careful reading and various suggestions made for the improvement of the paper.

References


Saeid Shams
Department of Mathematics
University of Urmia
Urmia, Iran
E-mail address: s.shams@urmia.ac.ir

Ali Ebadian
Department of Mathematics
Payame Noor University
Tehran, Iran
E-mail address: a.ebadian@urmia.ac.ir

Mahta Sayadiazar
Department of Mathematics
University of Urmia
Urmia, Iran
E-mail address: m.sayadiazar@yahoo.com

Janusz Sokół
Department of Mathematics
Rzeszów University of Technology
Al. Powstańców Warszawy 12, 35-959 Rzeszów, Poland
E-mail address: jsokol@prz.edu.pl