AREA OF TRIANGLES ASSOCIATED WITH A CURVE

DONG-SOO KIM AND KYU-CHUL SHIM

Abstract. It is well known that the area $U$ of the triangle formed by three tangents to a parabola $X$ is half of the area $T$ of the triangle formed by joining their points of contact. In this article, we study some properties of $U$ and $T$ for strictly convex plane curves. As a result, we establish a characterization for parabolas.

1. Introduction

Let $X = X(s)$ be a unit speed smooth curve in the plane $\mathbb{R}^2$ with nonvanishing curvature, and let $A = X(s), A_i = X(s + h_i), i = 1, 2,$ be three distinct neighboring points on $X$. Denote by $\ell, \ell_1, \ell_2$ the tangent lines passing through the points $A, A_1, A_2$ and by $B, B_1, B_2$ the intersection points $\ell \cap \ell_2, \ell \cap \ell_1, \ell \cap \ell_2,$ respectively. It is well known that the area $U(s, h_1, h_2) = |\Delta BB_1B_2|$ of the triangle formed by three tangents to a parabola is half of the area $T(s, h_1, h_2) = |\Delta AA_1A_2|$ of the triangle formed by joining their points of contact ([1]).

The present article studies whether this property exhaustively characterizes parabolas.

Usually, a regular plane curve $X : I \to \mathbb{R}^2$ defined on an open interval is called convex if, for all $t \in I$, the trace $X(I)$ lies entirely on one side of the closed half-plane determined by the tangent line at $X(t)$ ([2]).

Hereafter, we will say that a simple convex curve $X$ in the plane $\mathbb{R}^2$ is strictly convex if the curve is smooth (that is, of class $C(3)$) and is of positive curvature $\kappa$ with respect to the unit normal $N$ pointing to the convex side. Hence, in this case we have $\kappa(s) = \langle X''(s), N(X(s)) \rangle > 0$, where $X(s)$ is an arclength parametrization of $X$.

For a smooth function $f : I \to \mathbb{R}$ defined on an open interval, we will also say that $f$ is strictly convex if the graph of $f$ has positive curvature $\kappa$ with respect to the unit normal $N$ pointing to the convex side.
to the upward unit normal $N$. This condition is equivalent to the positivity of $f''(x)$ on $I$.

Suppose that $X$ is a strictly convex curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. For a fixed point $P = A \in X$, and for a sufficiently small $h > 0$, consider the line $m$ passing through $P + hN(P)$ which is parallel to the tangent $\ell$ of $X$ at $P$. Let us denote by $A_1$ and $A_2$ the points where the line $m$ intersects the curve $X$. We denote by $L_P(h)$ the length $|A_1A_2|$ of the chord $A_1A_2$.

Let us denote by $\ell_1, \ell_2$ the tangent lines passing through the points $A_1, A_2$ and by $B, B_1, B_2$ the intersection points $\ell_1 \cap \ell_2, \ell \cap \ell_1, \ell \cap \ell_2$, respectively. We denote by $T_P(h)$, $U_P(h)$ the area $|\triangle AA_1A_2|$, $|\triangle BB_1B_2|$, of triangles, respectively. Then, obviously we have $T_P(h) = \frac{h}{2}L_P(h)$.

In this paper, first of all, in Section 2 we prove the following:

**Theorem 1.** Let $X$ denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Then we have

$$\lim_{h \to 0} \frac{T_P(h)}{h^2} = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}$$

and

$$\lim_{h \to 0} \frac{U_P(h)}{2h^2} = \frac{\sqrt{2}}{2\sqrt{\kappa(P)}}$$

Next in Section 3, using Theorem 1 we characterize parabolas as follows.

**Theorem 2.** Let $X = X(s)$ denote a strictly convex $C^{(3)}$ curve in the plane $\mathbb{R}^2$. Suppose that for all $s$ and sufficiently small $h_i, i = 1, 2$, the curve $X$ satisfies

$$U(s, h_1, h_2) = \lambda(s)T(s, h_1, h_2).$$

Then, we have $\lambda(s) = \frac{1}{2}$ and $X$ is an open part of a parabola.

In [8], Krawczyk showed that for a strictly convex $C^{(4)}$ curve $X = X(s)$ in the plane $\mathbb{R}^2$, the following holds:

$$\lim_{h_1, h_2 \to 0} \frac{T(s, h_1, h_2)}{U(s, h_1, h_2)} = 2.$$  

His application of (1.4) states that if a strictly convex $C^{(4)}$ curve $X = X(s)$ in the plane $\mathbb{R}^2$ satisfies (1.3), then $\lambda(s) = \frac{1}{2}$ and $X$ is an open part of the graph of a quadratic polynomial.

But, for example, consider a function $f(x)$ given by

$$y = \frac{2\sqrt{acx} + 1 - \sqrt{4\sqrt{acx} + 1}}{2c^2},$$

where $a, c > 0$. Then, the function $f$ is defined on $I = (-\frac{1}{4\sqrt{ac}}, \infty)$. Its graph $X$ is strictly convex and satisfies (1.3) with $\lambda = \frac{1}{2}$. Note that $X$ is not the
graph of a quadratic polynomial, but an open part of the parabola given in (3.16) in Section 3.

In [6], the first author and Y. H. Kim established five characterizations of parabolas, which are the converses of well-known properties of parabolas originally due to Archimedes ([10]). In [4] and [5], they also proved the higher dimensional analogues of some results in [6].

For a few characterizations of parabolas or conic sections by some properties of tangent lines, see [3] and [7].

Among the graphs of functions, B. Richmond and T. Richmond established a dozen characterizations of parabolas using elementary techniques ([9]). In [9], parabola means the graph of a quadratic polynomial in one variable.

Finally, we pose a question as follows.

**Question 3.** Let $X$ be a strictly convex $C^3$ plane curve. Suppose that for each $P \in X$ there exists a positive number $\epsilon = \epsilon(P) > 0$ such that $U_P(h) = T_P(h)/2$ for all $h$ with $0 < h < \epsilon(P)$. Then, is it an open part of a parabola?

Throughout this article, all curves are of class $C^3$ and connected, unless otherwise mentioned.

### 2. Preliminaries and Theorem 1

In order to prove Theorem 1, we need the following lemma ([6]).

**Lemma 4.** Suppose that $X$ is a strictly convex $C^3$ curve in the plane $\mathbb{R}^2$ with the unit normal $N$ pointing to the convex side. Then we have

$$\lim_{h \to 0} \frac{1}{h^{\sqrt{h}}} L_P(h) = \frac{2\sqrt{2}}{\sqrt{\kappa(P)}},$$

where $\kappa(P)$ is the curvature of $X$ at $P$ with respect to the unit normal $N$.

First of all, we give a proof of (1.1) in Theorem 1. Since $T_P(h) = \frac{h}{2} L_P(h)$, it follows from Lemma 4 that the following holds:

$$\lim_{h \to 0} \frac{1}{h^{\sqrt{h}}} T_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}.$$

In order to prove (1.2) in Theorem 1, we fix an arbitrary point $P$ on $X$. Then, we may take a coordinate system $(x, y)$ of $\mathbb{R}^2$: $P$ is taken to be the origin $(0, 0)$ and $x$-axis is the tangent line $\ell$ of $X$ at $P$. Furthermore, we may regard $X$ to be locally the graph of a non-negative strictly convex function $f: \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$. Then $N$ is the upward unit normal.

Since the curve $X$ is of class $C^3$, the Taylor’s formula of $f(x)$ is given by

$$f(x) = ax^2 + g(x),$$

where $2a = f'''(0)$ and $g(x)$ is an $O(|x|^3)$ function. From $\kappa(P) = f'''(0) > 0$, we see that $a$ is positive.
For a sufficiently small \( h > 0 \), we denote by \( A_1(s, f(s)) \) and \( A_2(t, f(t)) \) the points where the line \( m : y = h \) meets the curve \( X \) with \( s < 0 < t \). Then \( f(s) = f(t) = h \) and we get \( B_1(s - h/f'(s), 0) \), \( B_2(t - h/f'(t), 0) \) and \( B(x_0, y_0) \) with
\[
(2.3) \quad x_0 = \frac{tf'(t) - sf'(s)}{f'(t) - f'(s)} \quad \text{and} \quad y_0 = \frac{(t - s)f'(t)f'(s) + h(f'(t) - f'(s))}{f'(t) - f'(s)}.
\]
Noting that \( L_P(h) = t - s \), one obtains
\[
(2.5) \quad 2U_P(h) = \{t - s - \frac{h}{f'(t)} + \frac{h}{f'(s)}\}(-y_0) = h^2\frac{(f'(t) - f'(s))}{-f'(s)f'(t)} - 2hL_P(h) + \frac{-f'(s)f'(t)}{f'(t) - f'(s)}L_P(h)^2.
\]
It follows from (2.5) that
\[
(2.6) \quad \frac{2U_P(h)}{h\sqrt{h}} = \alpha_P(h) - 2\frac{L_P(h)}{\sqrt{h}} + \frac{1}{\alpha_P(h)}:\left(\frac{L_P(h)}{\sqrt{h}}\right)^2,
\]
where we denote
\[
(2.7) \quad \alpha_P(h) = \sqrt{h}\frac{(f'(t) - f'(s))}{-f'(s)f'(t)}.
\]
Finally, we prove a lemma, which together with (2.6) and Lemma 4, completes the proof of (2) in Theorem 1.

**Lemma 5.** We have the following.
\[
(2.8) \quad \lim_{h \to 0} \alpha_P(h) = \frac{\sqrt{2}}{\sqrt{\kappa(P)}}.
\]
*Proof.* Note that
\[
(2.9) \quad \alpha_P(h) = \frac{\beta_P(h)}{\gamma_P(h)},
\]
where we denote
\[
(2.10) \quad \beta_P(h) = \frac{f'(t) - f'(s)}{t - s}
\]
and
\[
(2.11) \quad \gamma_P(h) = \frac{-f'(s)f'(t)}{\sqrt{h}(t - s)}.
\]
Applying mean value theorem to the derivative \( f'(x) \) of \( f(x) \) shows that as \( h \) tends to 0, \( \beta_P(h) \) goes to \( f''(0) = \kappa(P) \). To get the limit of \( \gamma_P(h) \), we put
\[
(2.12) \quad \delta_P(h) = \frac{f'(s)f'(t)}{st}
\]
and
\begin{equation}
\eta_P(h) = \frac{-st}{\sqrt{h(t-s)}}.
\end{equation}

Then, we have
\begin{equation}
\gamma_P(h) = \delta_P(h)\eta_P(h).
\end{equation}

Note that
\begin{equation}
\lim_{h \to 0} \delta_P(h) = \kappa(P)^2.
\end{equation}

If we use \( L_P(h) = t - s \), \( \eta_P(h) \) can be written as
\begin{equation}
\eta_P(h) = \left(-\frac{st}{h}\right)/\left(\frac{L_P(h)}{\sqrt{h}}\right)
\end{equation}
and the numerator of (2.16) can be decomposed as
\begin{equation}
\frac{-st}{h} = \left(\frac{L_P(h)}{\sqrt{h}} - \frac{t}{\sqrt{h}}\right)\frac{t}{\sqrt{h}}.
\end{equation}

Now, to obtain the limit of \( \frac{t}{\sqrt{h}} \), we use (2.2). Recalling that \( \kappa(P) = f''(0) = 2a \), we have
\begin{equation}
\frac{t}{\sqrt{h}} = \frac{t}{\sqrt{at^2 + g(t)}}.
\end{equation}

Since \( g(x) \) is an \( O(|x|^3) \) function, (2.18) implies that \( \lim_{h \to 0} \frac{t}{\sqrt{h}} = 1/\sqrt{a} \).

Hence, together with (2.17), Lemma 4 shows that \( \lim_{h \to 0} (-st)/h = 1/a \), and hence from (2.16) we get
\begin{equation}
\lim_{h \to 0} \eta_P(h) = \frac{1}{2\sqrt{a}}.
\end{equation}

Thus, it follows from (2.14) and (2.15) that
\begin{equation}
\lim_{h \to 0} \gamma_P(h) = 2a\sqrt{a}.
\end{equation}

Using \( \kappa(P) = 2a \), together with (2.9) and (2.10), (2.20) completes the proof of Lemma 5. \( \square \)

### 3. Proof of Theorem 2

In this section, we prove Theorem 2.

Suppose that \( X = X(s) \) denote a strictly convex \( C^3 \) curve in the plane \( \mathbb{R}^2 \) which satisfies for all \( s \) and sufficiently small \( h, i = 1, 2 \),
\begin{equation}
U(s, h_1, h_2) = \lambda(s)T(s, h_1, h_2).
\end{equation}
Then, in particular, for all \( P = X(s) \) and sufficiently small \( h > 0 \) the curve \( X \) satisfies
\begin{equation}
U_P(h) = \lambda(P)T_P(h).
\end{equation}
Hence, Theorem 1 implies that \( \lambda(P) = \frac{1}{2} \).
In order to prove the remaining part of Theorem 2, first, we fix an arbitrary point $A$ on $X$. As in Section 1, we take a coordinate system $(x, y)$ of $\mathbb{R}^2$: $A$ is taken to be the origin $(0, 0)$ and $x$-axis is the tangent line $\ell$ of $X$ at $A$. Furthermore, we may regard $X$ to be locally the graph of a non-negative strictly convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = f'(0) = 0$ and $2a = f''(0) > 0$.

For sufficiently small $|s|$ and $|t|$ with $0 < s < t$ or $t < s < 0$, we let $A_1 = (s, f(s)), A_2 = (t, f(t))$ be two neighboring points of $A$ on $X$. Then, the area $T(s, t) of the triangle $\triangle AA_1A_2$ is given by

$$T(s, t) = (sf(t) - tf(s)),$$

where $\epsilon = 1$ if $0 < s < t$ and $\epsilon = -1$ if $t < s < 0$.

Denote by $\ell, \ell_1, \ell_2$ the tangent lines passing through the points $A, A_1, A_2$ and by $B, B_1, B_2$ the intersection points $\ell_1 \cap \ell_2, \ell \cap \ell_1, \ell \cap \ell_2$, respectively. Then we have $B_1(s - f(s)/f'(s), 0), B_2(t - f(t)/f'(t), 0)$ and $B(x_0, y_0)$ with

$$y_0 = \frac{(t - s)f'(t)f'(s) + f(s)f'(t) - f'(s)f(t)}{f'(t) - f'(s)}.$$

Hence the area $U(s, t)$ of the triangle $\triangle BB_1B_2$ is given by

$$2\epsilon U(s, t) = \frac{(t - s - f(t)f'(s) + f(s)f'(t) - f'(s)f(t))}{f'(s)f'(t)(f'(t) - f'(s))}.$$

Second, we prove:

**Lemma 6.** The function $f$ satisfies the following:

$$f(t)f'(t)^2 = 4a(f'(t) - f(t))^2,$$

where $a$ is given by $f''(0) = 2a$.

**Proof.** Since the curve $X$ satisfies (1.3) with $\lambda = 1/2$, we get $2U(s, t) = T(s, t)$.

By letting $s \to 0$, from (3.2) we get

$$\epsilon \lim_{s \to 0} \frac{T(s, t)}{s} = \frac{f(t)}{2},$$

where we use $f'(0) = 0$. From (3.4) we also get

$$2\epsilon \lim_{s \to 0} \frac{U(s, t)}{s} = \frac{f''(0)(tf'(t) - f'(0)f(t))^2}{f''(0)f'(t)^2}$$

$$= 2a \frac{(tf'(t) - f(t))^2}{f'(t)^2},$$

where we use $f'(0) = 0$ and $f''(0) = 2a > 0$. Together with (3.6), (3.7) completes the proof. \qed

Third, we prove:
Lemma 7. The function $f$ satisfies the following:

\[(3.8) \quad 2f(t)^2f''(t) = f'(t)^2\{tf'(t) - f(t)\}.\]

Proof. By letting $s \to t$, we get from (3.2)

\[(3.9) \quad \lim_{s \to t} \frac{T(s, t)}{s - t} = \frac{1}{2} \lim_{s \to t} \frac{sf(t) - tf(s)}{s - t} = \frac{1}{2}(f(t) - tf'(t)).\]

On the other hand, from (3.4) we get

\[(3.10) \quad 2\lim_{s \to t} \frac{U(s, t)}{s - t} = \lim_{s \to t} \frac{(t - s)f'(t)f'(s) + f(s)f'(t) - f'(s)f(t)}{(s - t)f'(s)f'(t)(f'(t) - f'(s))} = -\frac{f(t)^2f''(t)}{f'(t)^2}.\]

Since $T = 2U$, together with (3.9), (3.10) completes the proof. □

By eliminating $tf'(t) - f(t)$ from (3.5) and (3.8), we get

\[(3.11) \quad f''(t) = \frac{1}{4\sqrt{a}} \frac{f'(t)^3}{f(t)^{3/2}}.\]

Letting $y = f(t)$, a standard method of ordinary differential equations using the substitution $w = dy/dt$ and $y''(t) = w(dw/dy)$ leads to

\[(3.12) \quad dt = \left(\frac{1}{2\sqrt{ay}} + c\right)dy,\]

where $c$ is a constant. Since $f(0) = 0$, we obtain from (3.12)

\[(3.13) \quad t = \frac{1}{\sqrt{a}}(\sqrt{y} + cy).\]

After replacing $t$ by $x$, we have for $y = f(x)$

\[(3.14) \quad y = \begin{cases} \frac{2\sqrt{a}x + 1}{\sqrt{a}^2}, & \text{if } c \neq 0, \\ \frac{2c}{ax^2}, & \text{if } c = 0. \end{cases}\]

Note that

\[(3.15) \quad f(0) = f'(0) = 0, \quad f''(0) = 2a \quad \text{and} \quad f'''(0) = -12\sqrt{a}c \quad \text{or} \quad 0.\]

It follows from (3.14) that the curve $X$ around an arbitrary point $A$ is an open part of the parabola defined by

\[(3.16) \quad ax^2 - 2\sqrt{ac}xy + c^2y^2 - y = 0.\]

Finally using (3.15), in the same manner as in [6], we can show that the curve $X$ is globally an open part of a parabola. This completes the proof of Theorem 2.
4. Corollaries and examples

In this section, we give some corollaries and examples.

Suppose that \( X(s) \) is a strictly convex \( C^3 \) curve in the plane \( \mathbb{R}^2 \) which satisfies for all \( s \) and sufficiently small \( h_i, i = 1, 2 \),

\[
U(s, h_1, h_2) = \lambda(s)T(s, h_1, h_2)\mu(s),
\]

where \( \lambda(s) \) and \( \mu(s) \) are some functions. Then, in particular, for all \( P = X(s) \) and sufficiently small \( h > 0 \) the curve \( X \) satisfies

\[
U_P(h) = \lambda(P)T_P(h)\mu(P).
\]

Using Theorem 1, by letting \( h \to 0 \) we see that \( \mu(P) = 1 \). Hence, Theorem 1 again implies that \( \lambda(P) = \frac{1}{2} \).

Thus, from Theorem 2 we get:

**Corollary 8.** Let \( X \) denote a strictly convex curve in the plane \( \mathbb{R}^2 \). Then, the following are equivalent.

1) \( X \) satisfies (4.1) for some functions \( \lambda(s) \) and \( \mu(s) \).
2) \( X \) satisfies (4.1) with \( \lambda = \frac{1}{2} \) and \( \mu = 1 \).
3) \( X \) is an open part of a parabola.

Finally, we give an example of a convex curve which satisfies

\[
U_P(h) = \frac{1}{2}T_P(h)
\]

for sufficiently small \( h > 0 \) at every point \( P \in X \), but it is not a parabola. Note that the example is not of class \( C^2 \), and hence it is not strictly convex either.

**Example 9.** Consider the graph \( X \) of a function \( f : \mathbb{R} \to \mathbb{R} \) which is given by

\[
f(x) = \begin{cases} 
ax^2, & \text{if } x < 0, \\
bx^2, & \text{if } x \geq 0.
\end{cases}
\]

It is straightforward to show that if \( P \) is the origin, then for all \( h \) we have

\[
U_P(h) = \frac{1}{2}T_P(h).
\]

Hence \( X \) satisfies \( U_P(h) = T_P(h)/2 \) at the origin for all \( h > 0 \). If \( P \in X \) is not the origin, then there exists a positive number \( \varepsilon(P) \) such that for every positive number \( h \) with \( h < \varepsilon(P) \), \( X \) satisfies \( U_P(h) = T_P(h)/2 \).

Thus, \( X \) satisfies \( U_P(h) = T_P(h)/2 \) for sufficiently small \( h > 0 \) at every point \( P \in X \). But it is not a parabola.
References


Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: dosokim@chonnam.ac.kr

Kyu-Chul Shim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: mathtsim@naver.com