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Construction of Logarithmic Spiral-like Curve Using G^2 Quadratic Spline with Self Similarity

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Abstract

In this paper, we construct an logarithmic spiral-like curve using curvature-continuous quadratic spline and quadratic rational spline. The quadratic (rational) spline has self-similarity. We present some properties of the quadratic spline. Also using this G^2 quadratic spline, an approximation of logarithmic spiral is proposed and error analysis is obtained.

Key words: Logarithmic Spiral Curve, Curvature-continuous Spline, Quadratic Spline, Quadratic Rational Spline, Quadratic Bézier Curve

1. Introduction

Logarithmic spiral is a beautiful curve with some special properties, for example, monotone curvature and self similarity. But the curve cannot be expressed by Bézier or spline curve, which are widely used in the fields of CAD/CAM or Computer Vision[1].

Mineur *et al.*[2] proposed an approximation of the family of Bézier curves whose control polygons are defined by a matrix with rotation and magnification. These curves are called by Class-A Bézier curves[3] and have the similar properties as spiral curves. Amati[4] presented a wavelet-based method to transform non-Class-A curves into much smoother Class-A. Cao[5] found an weak condition for the Class-A Bézier to have monotone curvature.

Inspiring these Bézier curves we construct a logarithmic spiral-like quadratic (rational) spline. This spline is constructed by the polygon using the matrix multiplication with rotation and magnification and is curvature(G^2) continuous. We find some properties of our quadratic (rational) spline.

Previously Schaback[6] constructed the G^2 continuous quadratic spline and Farin[7] presented an approximation method by G^2 conic spline for offset curve. Baumgarten and Farin[8] gave the G^2 cubic rational approximation of spiral curve, and Dietz and Piper[9] developed the interpolation of cubic spiral curves. But these curves are not considered the self similarity or have the degree greater than two. We construct the G^2 spiral-like quadratic (rational) spline with self similarity, which has some advantages, for example, easy calculation for error analysis and construction of control polygon.

In the next section, definitions and properties of logarithmic spiral and quadratic (rational) Bézier curves are reintroduced. In section 3, we construct G^2 spiral-like quadratic (rational) spline with self similarity and find the limit point of the spline. In section 4, we present the approximation of the logarithmic spline by the G^2 quadratic (rational) spline with self similarity, and summary our works in section 5.

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2. Prior Works

In this section we remind the definition and properties of the logarithmic spiral and quadratic (rational) Bézier curve. We use the expression of point in the plane by the Cartesian coordinates (x, y) or complex number $x+iy$, where i is the imaginary unit.

Definition 2.1 Logarithmic spiral is defined by

$$z(t)=\alpha e^{\beta t}e^{it}=\alpha e^{\beta t}(\cos t+i\sin t)$$

for $t\geq 0$ for real constant numbers α,β. If $\beta>0$, $\beta<0$

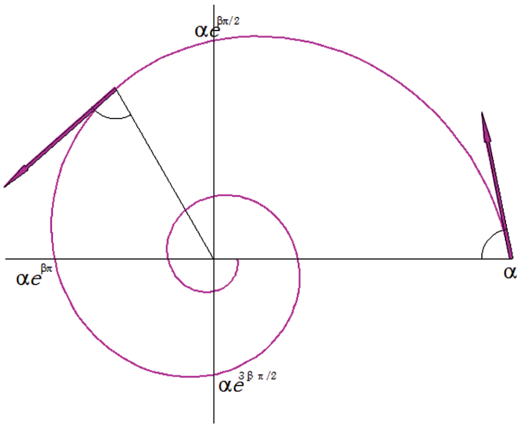

Fig. 1. Logarithmic spiral curve $z(t)=\alpha e^{\beta t}e^{it}$.

or $\beta=0$, then $z(t)$ is getting larger, smaller or rounding on circle as $t\to\infty$. Its radius of curvature is $\rho(t)=\alpha\sqrt{\beta^2+1}\,e^{\beta t}$.[4,10]

Remark 2.2 The logarithmic spiral has self-similarity[4,10]. Since

$$z(t+\Delta s)=\alpha e^{\beta(t+\Delta s)}e^{i(t+\Delta s)}=e^{\beta\Delta s}e^{i\Delta s}z(t),$$

the curve $z(t+\Delta s)$ is the composition of rotation of $z(t)$ by angle Δs and magnification by $e^{\beta\Delta s}$. The similarity ratio is $z(t):z(t+\Delta s)=1:e^{\beta\Delta s}$.

Remark 2.3 The logarithmic spiral has a special direction at every point. Since

$$z'(t)=\alpha(\beta+i)e^{\beta t}e^{it}=(\beta+i)z(t)$$

we have $\arg(z'(t))=\arg(z(t))+\arg(\beta+i)$. Thus the direction of $z'(t)$ is the rotation of direction of $z(t)$ by the angle $\arg(\beta+i)$, as shown in Fig. 1.

The quadratic Bézier curve and rational Bézier curve are well-known curves in the fields of CAD/CAM or CAGD[1].

Definition 2.4 Quadratic Bézier curve is

$$\boldsymbol{b}(t)=\sum_{i=0}^{2}B_i^2(t)\boldsymbol{b}_i$$

and quadratic rational Bézier curve is

$$\boldsymbol{r}(t)=\frac{\sum_{i=0}^{2}w_iB_i^2(t)\boldsymbol{b}_i}{\sum_{i=0}^{2}w_iB_i^2(t)}$$

where $\boldsymbol{b}_i$, $i=0,1,2$, is control point, $w_i>0$ is weight, and $B_i^2(t)=\binom{2}{i}t^i(1-t)^{2-i}$ is the quadratic Bernstein polynomial.

If $w_0=w_1=w_2$, then rational curve $\boldsymbol{r}(t)$ is a quadratic Bézier curve. Every quadratic rational Bézier $\boldsymbol{r}(t)$ can be expressed, without changing its shape, in form of

$$\boldsymbol{r}(t)=\frac{B_0^2(t)\boldsymbol{b}_0+wB_1^2(t)\boldsymbol{b}_1+B_2^2(t)\boldsymbol{b}_2}{B_0^2(t)+wB_1^2(t)+B_2^2(t)}$$

which is called standard form. The quadratic rational Bézier curve in standard form has the following facts[11,7] that for $i=0,1$

$$\boldsymbol{r}'(i)=2w\Delta\boldsymbol{b}_i \tag{2.1}$$

$$\kappa(i)=\frac{Area(\triangle\boldsymbol{b}_0\boldsymbol{b}_1\boldsymbol{b}_2)}{w^2\|\Delta\boldsymbol{b}_i\|^3} \tag{2.2}$$

where $\kappa(t)$ is the curvature of $\boldsymbol{r}(t)$.

3. Logarithmic Spiral-like Quadratic Spline

In this section we construct G^2 quadratic spline which is a logarithmic spiral-like curve. Put $\Delta\boldsymbol{b}_k=\boldsymbol{b}_{k+1}-\boldsymbol{b}_k$. Let the control polygon $\boldsymbol{b}_0,\boldsymbol{b}_1,\cdots,\boldsymbol{b}_{2n}$ satisfy

$$\Delta\boldsymbol{b}_{2k+1}=rR_\phi\Delta\boldsymbol{b}_{2k} \tag{3.1}$$

for $k=0,\cdots,n-1$, and

$$\Delta\boldsymbol{b}_{2k+2}=r^2\Delta\boldsymbol{b}_{2k+1} \tag{3.2}$$

for $k=0,\cdots,n-2$, where R_ϕ is the rotation operator by the angle $\phi\in(-\pi,\pi)$, as shown in Fig. 2. If $r>1$ or $r<1$, then the control polygon diverges or converges as $n\to\infty$, respectively. If $\phi>0$ or $\phi<0$, the control polygon is turning to the left or right, respectively.

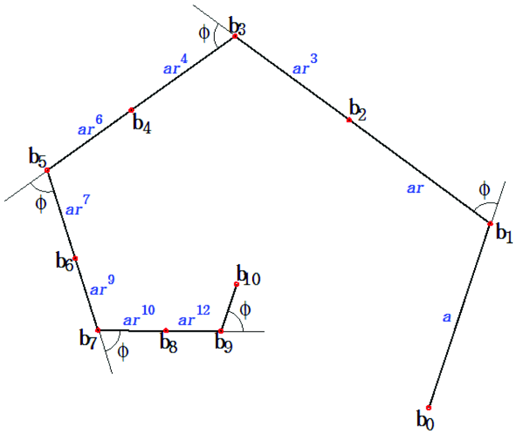

Fig. 2. Control polygon of quadratic spiral spline.

Theorem 3.1 The quadratic spline $b(t)$ defined by

$$b(t)=\begin{cases} r_0(t) & (\text{for } t\in[0,1]) \\ r_k(t-k) & (\text{for } t\in(k,k+1], 1\le k\le n-1) \end{cases}$$

is G^2 continuous, where

$$r_k(t)=\frac{B_0^2(t)b_{2k}+wB_1^2(t)b_{2k+1}+B_2^2(t)b_{2k+2}}{B_0^2(t)+wB_1^2(t)+B_2^2(t)}$$

for $k=0,\cdots,n-1$.

proof. By Equation (2.1)
$b'(k^+)=r_k'(0^+)=2w\Delta b_{2k}$
$b'(k^-)=r_{k-1}'(1^-)=2w\Delta b_{2k-1}$

and since $\Delta b_{2k}=r^2\Delta b_{2k-1}$ at each junction point b_{2k}, $k=1,\cdots,n-1$, we have

$$b'(k^+)=r^2 b'(k^-)$$

and so $b(t)$ is at least G^1 continuous.

By Equations (3.1)-(3.2), all control polygons $b_{2k}b_{2k+1}b_{2k+2}$, $k=0,\cdots,n-1$, are similar and the similarity ratio is

$$b_{2k}b_{2k+1}b_{2k+2}:b_{2k+2}b_{2k+3}b_{2k+4}=1:r^3.$$

Thus the quadratic rational Bézier curves $r_k(t)$ with the control polygon $b_{2k}b_{2k+1}b_{2k+2}$ are also similar and have the same similarity ratios $r_k:r_{k+1}=1:r^3$.

At each junction point b_{2k}, $k=1,\cdots,n-1$, we will show that the curvature $\kappa(t)$ of $b(t)$ is continuous. Let $\kappa_k(t)$ be the curvature of $r_k(t)$. By Equation (2.2), for $k=0,\cdots,n-1$

$$\kappa_k(0^+)=\frac{Area(\triangle b_{2k}b_{2k+1}b_{2k+2})}{w^2\|\Delta b_{2k}\|^3}$$

$$\kappa_k(1^-)=\frac{Area(\triangle b_{2k}b_{2k+1}b_{2k+2})}{w^2\|\Delta b_{2k+1}\|^3}.$$

By the similarity of r_k and r_{k+1},

$$\begin{aligned}\kappa_{k+1}(0^+)&=\frac{1}{r^3}\kappa_k(0^+)\\ &=\frac{Area(\triangle b_{2k}b_{2k+1}b_{2k+2})}{r^3w^2\|\Delta b_{2k}\|^3}\\ &=\frac{Area(\triangle b_{2k}b_{2k+1}b_{2k+2})}{w^2\|\Delta b_{2k+1}\|^3}=\kappa_k(1^-)\end{aligned}$$

and so $\kappa(k+1^+)=\kappa(k+1^-)$. Hence the curvature is continuous at each junction point b_{2k}, $k=1,\cdots,n-1$, so that the quadratic spline $b(t)$ is G^2-continuous. □

We call the spline in Theorem 3.1 by quadratic (rational) spiral spline with self similarity. The limit point of the quadratic spline is obtained as follows.

Theorem 3.2 The limit point of the quadratic spiral spline with self similarity for $r<1$ is

$$b_\infty=b_0+\left\{1+\frac{(r+r^3)(R_\phi-r^3)}{1-2r^3\cos(\phi)+r^6}\right\}\Delta b_0$$

as $n\to\infty$.

proof. By Equations (3.1)-(3.2)
$\Delta b_1=rR_\phi\Delta b_0$
$\Delta b_2=r^2\Delta b_1=r^3R_\phi\Delta b_0$
$\cdots$
$\Delta b_{2k-1}=rR_\phi\Delta b_{2k-2}=r^{3k-2}R_{k\phi}\Delta b_0$
$\Delta b_{2k}=r^{3k}R_{k\phi}\Delta b_0$

Thus $b_{2n+1}-b_1=\Delta b_1+\cdots+\Delta b_{2n}$ is

$$(r+r^3)R_\phi\{1+r^3R_\phi+\cdots+r^{3(n-1)}R_{(n-1)\phi}\}\Delta b_0.$$

Putting Δb_0 be the complex number of the point Δb_0,

$$(r+r^3)e^{i\phi}\{1+r^3e^{i\phi}+\cdots+r^{3(n-1)}e^{i(n-1)\phi}\}\Delta b_0$$
$$=(r+r^3)e^{i\phi}\frac{\{1-r^{3n}e^{in\phi}\}}{1-r^3e^{i\phi}}\Delta b_0$$

By letting $n\rightarrow\infty$, it converges to

$$\frac{(r+r^3)e^{i\phi}\Delta b_0}{1-r^3e^{i\phi}}=\frac{(r+r^3)e^{i\phi}(1-r^3e^{-i\phi})\Delta b_0}{(1-r^3e^{i\phi})(1-r^3e^{-i\phi})}$$
$$=\frac{(r+r^3)(e^{i\phi}-r^3)\Delta b_0}{1-2r^3\cos(\phi)+r^6}.$$

Thus we have

$$b_\infty-b_1=\frac{(r+r^3)(R_\phi-r^3)}{1-2r^3\cos(\phi)+r^6}\Delta b_0$$

and so

$$b_\infty=b_0+\left\{1+\frac{(r+r^3)(R_\phi-r^3)}{1-2r^3\cos(\phi)+r^6}\right\}\Delta b_0. \qquad \square$$

4. Approximation of Logarithmic Spiral by G^2 Quadratic Spline with Self Similarity

In this section we present the approximation of logarithmic spiral by quadratic spline with self similarity which is curvature continuous.

Consider the segment of spiral, $z(t)$, $0\le t\le \Delta s$, and its approximate quadratic Bézier and rational Bézier curve whose control points are $b_0=z(0)$, b_1, and $b_2=z(\Delta s)$. By self-similarity of spiral and the quadratic spiral spline,

$$b_{2k}=e^{k\beta\Delta s}R_{k\Delta s}b_0=z(k\Delta s)$$
$$b_{2k+1}=e^{k\beta\Delta s}R_{k\Delta s}b_1.$$

The middle control point b_1 should be chosen such that the quadratic (rational) spline $b(t)$ are tangent and curvature continuous.

Matching the tangent continuity at b_2,

$$\angle b_1b_2b_0+\angle b_0b_2o+\angle ob_2b_4+\angle b_4b_2b_3=\pi$$

yields $\angle b_1b_0b_2+\angle b_1b_2b_0=\Delta s$ since $\angle b_0b_2o=\pi-\Delta s-\angle ob_0b_2$ and $\angle ob_0b_2=\angle ob_2b_4$, where o is the origin. Thus b_1 should be on the circle

$$\angle b_0b_1b_2=\pi-\Delta s \tag{4.1}$$

Matching the continuity of curvature at b_2, $\kappa(1-)=\kappa(1+)=\frac{\kappa(0)}{e^{\beta\Delta s}}$ yields

$$\frac{Area(\triangle b_0b_1b_2)}{w^2\|\Delta b_1\|^3}=\frac{Area(\triangle b_0b_1b_2)}{w^2\|\Delta b_0\|^3e^{\beta\Delta s}}$$

so that b_1 should be on another circle

$$\|\Delta b_1\|=\|\Delta b_0\|e^{\frac{\beta\Delta s}{3}} \tag{4.2}$$

Solving two equations (4.1)-(4.2) which are equations of circles, we get the control point b_1 satisfying $b(t)$ is G^2 continuous at the point b_2.

Since the polygons $b_0b_1b_2b_3b_4$ and $b_2b_3b_4b_5b_6$ are similar, $b(t)$ is also G^2 continuous at the point b_4. By the same reason, $b(t)$ is also G^2 continuous at all junction points b_{2k}, so that $b(t)$ is G^2 continuous.

All control polygons of $b(t)$ have similarity, and the similarity ratio is

$$b_{2k}b_{2k+1}b_{2k+2}:b_{2k+2}b_{2k+3}b_{2k+4}=1:e^{\beta\Delta s}.$$

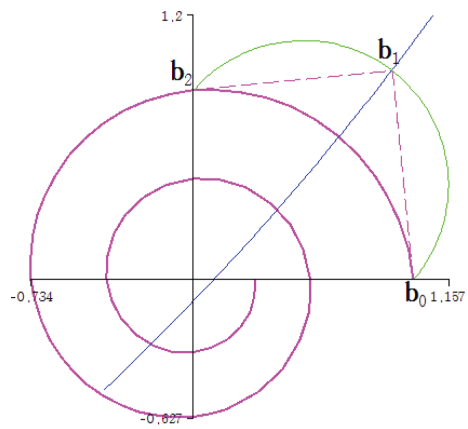

Fig. 3. Logarithmic spiral $z(t)=e^{-0.1t}e^{it}$, $\Delta s=\pi/2$, (magenta) and two circles of equations $\angle b_0b_1b_2=\pi/2$ (green) and $\|\Delta b_1\|=\|\Delta b_0\|\exp(-\pi/60)$ (blue).

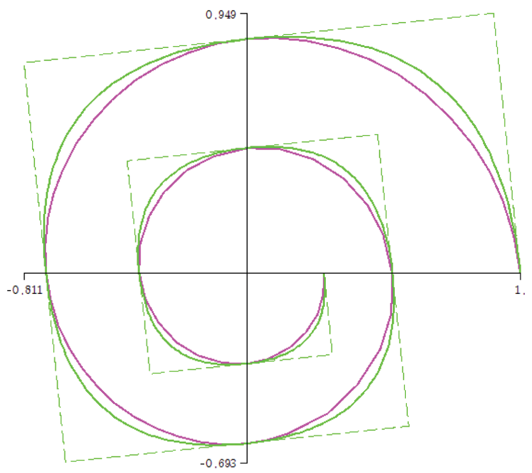

Fig. 4. Exponential spiral $z(t) = e^{-0.1t}e^{it}$, (magenta) $t \in [0, 4\pi]$ and the quadratic spiral spline(green) with $w = 1$.

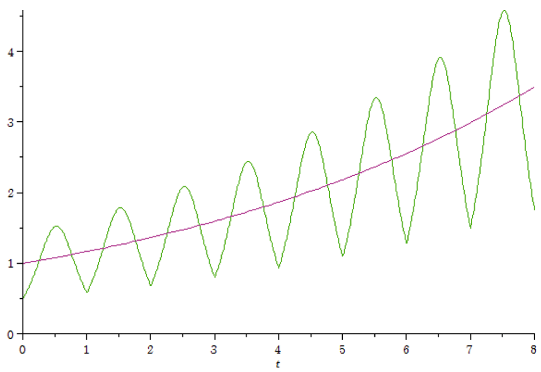

Fig. 5. Curvature of quadratic spiral spline (green) and logarithmic spiral (magenta) $z(\pi t/2)$.

By Equation (4.2), $r = e^{\frac{\beta \Delta s}{3}}$. Thus the control polygon satisfy Theorem 3.1 and the approximate quadratic spline is our quadratic spiral spline with self similarity.

Fig. 3 shows an example of the approximation of the logarithmic spiral $z(t) = e^{-0.1t}e^{it}$ with $\Delta s = \frac{\pi}{2}$. The control points $b_0 = (1, 0)$, $b_2 = (0, e^{-0.05\pi})$, and b_1 satisfy two equations (4.1)-(4.2), which are blue and green circle, respectively, in Fig. 3. The point $b_1 = (0.901, 0.949)$ is slightly different from the point $(0.905, 0.945)$ on the both tangency (dash-lines) of the spiral at b_0 and b_2.

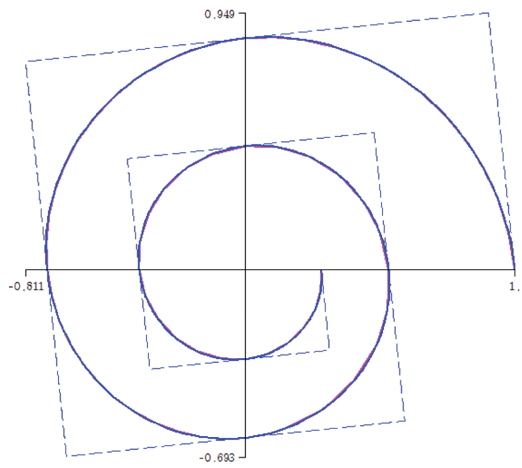

Fig. 6. Logarithmic spiral $z(t) = e^{-0.1t}e^{it}$, (magenta) $t \in [0, 4\pi]$ and the quadratic rational spiral spline(green) with $w = 1/\sqrt{2}$.

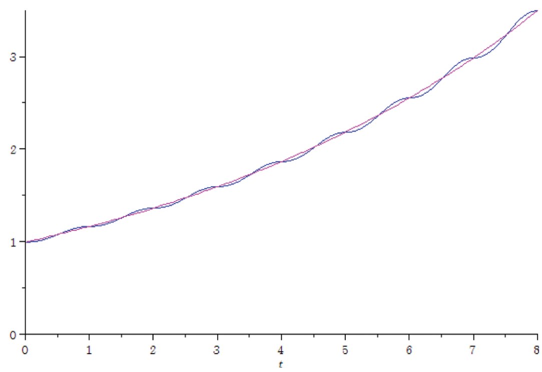

Fig. 7. Curvature of the quadratic rational spiral spline with $w = 1/\sqrt{2}$ (blue) and logarithmic spiral (magenta) $z(\pi t/2)$.

Fig. 4 shows the logarithmic spiral (magenta) $z(t) = e^{-0.1t}e^{it}$, $t \in [0, 4\pi]$ and the quadratic spiral spline (green) $b(t)$ with $w = 1$, which is a composite curve of quadratic Bézier curves. The (green) dashed lines are control polygon of the quadratic spiral spline. The curvature is continuous as shown in Fig. 5.

The logarithmic spiral is an circular arc if $\beta = 0$, and it can be expressed by quadratic rational Bézier curve with $w = \cos(\Delta s/2)$. For the logarithmic spiral(magenta) $z(t) = e^{-0.1t}e^{it}$, $t \in [0, 4\pi]$, the quadratic rational spiral spline (blue) $b(t)$ with $w = \cos(\frac{\Delta s}{2})$ is a good approximation, as shown in Fig. 6. The (blue) dashed lines are control polygon of the quadratic spiral spline. The curvature is continuous and almost monotone as shown in Fig. 7.

5. Conclusion

In this paper, we construct G^2 quadratic (rational) spline with self similarity and present some properties of this spline. This spline has two advantages. One is curvature-continuity of quadratic spline and the other is self similarity which makes error analysis and construction of control polygon easy. Since all segment of the logarithmic spiral and our quadratic spiral spline with self similarity have similar, if the error for one segment is measured, then the whole error for the approximation can be obtained.

We will find the necessary and sufficient condition for the quadratic (rational) spline with self similarity to have monotone curvature in the future.

Acknowledgements

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References

[1] G. Farin, J. Hoschek, and M.-S. Kim, "Handbook of computer aided geometric design", *Amsterdam, Netherlands*, Elsevier, 2002.

[2] Y. Mineur, T. Lichah, J. M. Castelain, and H. Giaume, "A shape controled fitting method for Bézier curves", *Comp. Aided Desi. Appl*., Vol. 15, pp. 879-891, 1998.

[3] G. Farin, "Class a Bézier curves", *Comp. Aided Geom. Desi*., Vol. 23, pp. 573-581, 2006.

[4] C. Baumgarten and G. Farin, "Approximation of logarithmic spirals", *Comp. Aided Geom. Desi*., Vol. 14, pp. 515-532, 1997.

[5] J. Cao and G. Wang. "A note on Class A Bézier curves", *Comput. Aided. Geom. D*, Vol. 25, pp. 523-528, 2008.

[6] R. Schaback. "Interpolation with piecewise quadratic visually C2 Bézier polynomials", *Comp. Aided Geom. Desi*., Vol. 6, pp. 219-233, 1989.

[7] G. Farin, "Curvature continuity and offsets for piecewise conics", *ACM Trans. Graph*., Vol. 8, pp. 89-99, 1989.

[8] G. Amati, A. Liverani, and G. Caligiana. "From spline to Class-A curves through multi-scale analysis filtering", *Comp. Grap*., Vol. 30, pp. 345-352, 2006.

[9] D. A. Dietz and B. Piper. "Interpolation with cubic spirals", *Comp. Aided Geom. Desi*., Vol. 21, pp. 165-180, 2004.

[10] K. Miura, "A general equation of aesthetic curves and its self-affinity", *Comp. Aided Desi. Appl*., Vol. 3, pp. 457-464, 2006.

[11] Y. J. Ahn and H. O. Kim, "Curvatures of the quadratic rational Bézier curves", *Comp. Math. Appl*., Vol. 36, pp. 71-83, 1998.