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A Cyclic Subnormal Completion of Complex Data

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Abstract. For a finite subset Λ of $\mathbb{N}_0 \times \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of nonnegative integers, we say that a complex data $\gamma_\Lambda := \{\gamma_{ij}\}_{(i,j)\in\Lambda}$ in the unit disc $\mathbf{D}$ of complex numbers has a cyclic subnormal completion if there exists a Hilbert space $\mathcal{H}$ and a cyclic subnormal operator S on $\mathcal{H}$ with a unit cyclic vector $x_0 \in \mathcal{H}$ such that $\langle S^{*i}S^j x_0, x_0\rangle = \gamma_{ij}$ for all $i, j \in \mathbb{N}_0$. In this note, we obtain some sufficient conditions for a cyclic subnormal completion of γ_Λ, where Λ is a finite subset of $\mathbb{N}_0 \times \mathbb{N}_0$.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is *subnormal* if there exists a normal operator N on a Hilbert space $\mathcal{K}$ with $\mathcal{K} \supset \mathcal{H}$ such that $N|_{\mathcal{H}} = T$. Given a finite sequence $\alpha := \{\alpha_i\}_{i=0}^n \subset \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of positive real numbers, one says that α has a *subnormal completion* if there exists a sequence $\widehat{\alpha} = \{\widehat{\alpha}_i\}_{i=0}^\infty \subset \mathbb{R}_+$ with $\widehat{\alpha}_i = \alpha_i$ for $0 \le i \le n$ such that the associated weighted shift $W_{\widehat{\alpha}}$ with weight sequence $\widehat{\alpha}$ is subnormal. For three initial segment of positive weights $\alpha = \{\alpha_i\}_{i=0}^2$ with $\alpha_0 \le \alpha_1 \le \alpha_2 \le 1$, it is well-known that α has

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a subnormal completion, but not the case of an initial segment of positive weights $\{\alpha_i\}_{i=0}^n$ for $n \geq 3$ ([11]). This notion of subnormal completion can be extended to the truncated complex data and applied to the truncated complex moment problem approached by complex moment matrices $M(n)$ ([4],[5]). In [7], they obtained some results that the matrices $M(n)$ [or $E(n)$] come from the Bram-Halmos' [or Embry's, resp.] characterization ([1],[6],[8],[9],[10]). These matrices are closely related to the cyclic subnormal completion which is the main topic of this note.

The organization of this paper is as follows. In Section 2, we recall some notation and terminologies about moment matrices of $M(n,s)$ which were generalized by $M(n)$ in [7]. In Section 3, we obtain some sufficient conditions for a cyclic subnormal completion of γ_Λ for a finite subset Λ of $\mathbb{N}_0 \times \mathbb{N}_0$. Denote a pentagonal set by

$$\Gamma_{n,s} = \{(i,j) : 0 \leq i+j \leq 2n,\ |i-j| \leq s,\ 0 \leq s \leq 2n\}.$$

Let $\Lambda = \Gamma_{n,n}$ and $\gamma_\Lambda \subset \mathbf{D}$. We prove that if $M(n,n) \geq 0$ has a flat extension $M(n+1,n+1)$, then γ_Λ has the unique cyclic subnormal completion. In addition, we consider a pentagonal set $\Lambda = \Gamma_{n,s}$ and $\gamma_\Lambda \subset \mathbf{D}$. In this case, we also prove that if $M(n,s) \geq 0$ has a double flat extension $M(n+2,s)$ for $0 \leq s < n$, then γ_Λ has a unique cyclic subnormal completion.

2. Preliminaries and Notation

Let S be a cyclic operator in $\mathcal{L}(\mathcal{H})$ with a unit cyclic vector x_0 in $\mathcal{H}$. Then we may construct a double sequence $\{\delta_{ij}\}_{(i,j)\in\mathbb{N}_0\times\mathbb{N}_0}$ in $\mathbb{C}$ defined by $\delta_{ij} := \langle S^{*i}S^j x_0, x_0\rangle$ for $i,j \in \mathbb{N}_0$. Obviously $\delta_{00} = 1$ and $\overline{\delta_{ij}} = \delta_{ji}$. We recall some notation from [4] and [7] below. Consider the following lexicographic order on the rows and columns of an infinite matrix $(\delta_{ij})_{0\leq i,j<\infty}$:

$$\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, ...; \tag{2.1}$$

e.g., the first column is labeled $\mathit{1}$, the second column is labeled Z, the third $\bar{Z}$, the fourth Z^2, etc. For $m,n \in \mathbb{N}_0$, let $M_{m,n}$ be the $(m+1)\times(n+1)$ block of Toeplitz form whose first row has entries given by $\gamma_{m,n}, \gamma_{m+1,n-1}, ..., \gamma_{m+n,0}$ and whose first column has entries given by $\gamma_{m,n}, \gamma_{m-1,n+1}, ..., \gamma_{0,n+m}$, and the infinite matrix can be constructed as following:

$$M := \begin{pmatrix} M_{0,0} & M_{0,1} & M_{0,2} & \cdots \\ M_{1,0} & M_{1,1} & M_{1,2} & \cdots \\ M_{2,0} & M_{2,1} & M_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.2}$$

Next we introduce matrices $M(n,s)$ whose definition was defined in [7]. If $n = 2k$, let

$$\eta_{n,s} = \begin{cases} (k+1)^2 + 2mk - m(m-1) & \text{if } s = 2m, \\ (k+1)^2 + (2m-1)k - (m-1)^2 & \text{if } s = 2m-1, \end{cases}$$

and if $n = 2k+1$, let

$$\eta_{n,s} = \begin{cases} (k+1)(k+3) + 2mk - m(m-1) & \text{if } s = 2m+1, \\ (k+1)(k+3) + (2m-1)k - (m-1)^2 & \text{if } s = 2m. \end{cases}$$

Let $\mathcal{M}_k(\mathbb{C})$ be the set of all complex $k \times k$ matrices. For $A \in \mathcal{M}_{\eta_{n,s}}(\mathbb{C})$, we give the order on the rows and columns of A as in (2.1). For example, if $n = 4$ and $s = 3$, i.e., $\eta_{4,3} = 14$, then the order of columns of such A in $\mathcal{M}_{\eta_{4,3}}(\mathbb{C})$ is as follows:

$$\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3, Z^4, \bar{Z}Z^3, \bar{Z}^2Z^2, \bar{Z}^3Z.$$

Let

$$\Lambda_{n,s} = \{(i,j) : 0 \le i+j \le n, \max\{i-s, 0\} \le j,\ 0 \le s \le n\}.$$

And we define the moment matrix $M(n,s)$ for an $\eta_{n,s} \times \eta_{n,s}$ matrix that the entry in row $\bar{Z}^k Z^l$ and column $\bar{Z}^i Z^j$ is $M(n,s)_{(k,l)(i,j)} = \gamma_{l+i,j+k}$, where $(k,l),(i,j) \in \Lambda_{n,s}$. For example, if $n = 2, s = 1$, i.e., for

$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{13}, \gamma_{22}, \gamma_{31},$$

the associated moment matrix is

$$M(2,1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In particular, $M(2,s)$ is referred to as the *quartic moment matrix* here.

3. Cyclic Subnormal Completion

We first recall that if S is a cyclic contractive subnormal operator in $\mathcal{L}(\mathcal{H})$, then there exists a compactly supported Borel measure μ on $\mathbf{D}$ and unitary transformation $V : \mathcal{H} \to H^2(\mu)$ such that $S = V^{-1}S_\mu V$ and $Vx_0 = 1$, where S_μ is a multiplication operator on $H^2(\mu)$ defined by $S_\mu f(z) = zf(z)$.

We now begin this section with the following definition.

Definition 3.1. Let Λ be a finite subset of $\mathbb{N}_0 \times \mathbb{N}_0$. A complex data $\gamma_\Lambda := \{\gamma_{ij}\}_{(i,j)\in\Lambda}$ in $\mathbf{D}$ has a *cyclic subnormal completion* if there exists a Hilbert space $\mathcal{H}$ and a cyclic subnormal operator S on $\mathcal{H}$ with a unit cyclic vector $x_0 \in \mathcal{H}$ such that $\langle S^{*i}S^j x_0, x_0\rangle = \gamma_{ij}$ for all $i,j \in \mathbb{N}_0$. This bounded operator S is called a *cyclic subnormal completion* of γ_Λ. In this case, we denote $\widehat{\gamma}_{ij} := \langle S^{*i}S^j x_0, x_0\rangle$ for all $i,j \in \mathbb{N}_0$, and $\widehat{\gamma}_\Lambda := \{\widehat{\gamma}_{ij}\}_{0\le i,j<\infty}$. And $\widehat{\gamma}_\Lambda$ is called the *matrix completion* of γ_Λ with respect to S.

Suppose that $\gamma_\Lambda = \{\gamma_{ij}\}_{(i,j)\in\Lambda}$ has a cyclic subnormal completion S. Let $\widehat{\gamma}_\Lambda = \{\widehat{\gamma}_{ij}\}_{0\le i,j<\infty}$ be a matrix completion of γ_Λ with respect to S. Then we have

(3.1)

$$\widehat{\gamma}_{ij} = \left\langle S^{*i}S^j x_0, x_0\right\rangle_{\mathcal{H}} = \left\langle S_\mu^{*i}S_\mu^j 1, 1\right\rangle_{H^2(\mu)} = \left\langle z^j, z^i\right\rangle_{H^2(\mu)} = \int_{\mathbf{D}} \bar{z}^i z^j \, d\mu, \quad \forall i,j \ge 0.$$

Since $\widehat{\gamma}_{00} = \gamma_{00} = 1$, the total mass of μ is 1. And, in general, cyclic subnormal completion is not unique; see Example 3.2 below.

Example 3.2. Let $\Lambda = \{(0,0),(0,1)\}$ and let γ_{01} be a nonzero complex number in $\mathbf{D}$. Then we can choose numbers γ_{11}, γ_{20} in $\mathbf{D}$ such that $\gamma_{11} \le \gamma_{00} = 1$, $\gamma_{02} = \bar{\gamma}_{20}$ and

$$M(1,1) := \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix} > 0.$$

By [4, Th. 1.8], there exists an associated representing Borel measure μ supported on $\mathbf{D}$ such that

$$\gamma_{ij} = \int_{\mathbf{D}} \bar{z}^i z^j \, d\mu, \qquad (i,j) \in \Gamma_{1,1}.$$

By [4, Prop 6.4], $M(1,1)$ has a flat extension $M(2,2)$. Hence γ_Λ admits a unique positive extension $\widehat{\gamma}_\Lambda = \{\widehat{\gamma}_{ij}\}_{0\le i,j<\infty}$ such that $\widehat{\gamma}_{ij} = \int_{\mathbf{D}} \bar{z}^i z^j \, d\mu$, $i,j \in \mathbb{N}_0$. We may define a multiplication function S_μ on $H^2(\mu)$ so that

$$\widehat{\gamma}_{ij} = \int_{\mathbf{D}} \bar{z}^i z^j \, d\mu = \left\langle z^j, z^i\right\rangle_{H^2(\mu)} = \left\langle S_\mu^{*i}S_\mu^j 1, 1\right\rangle_{H^2(\mu)}, \quad i,j \in \mathbb{N}_0.$$

This cyclic operator S_μ is a cyclic subnormal completion of γ_Λ. Because μ depends on the choice of γ_{11} and γ_{20} (cf. [4, Prop 6.4]), the representing measure μ supported on $\mathbf{D}$ is not unique. In fact, the corresponding cyclic subnormal completion S_μ can be chosen infinitely many.

We give some conditions for the uniqueness on the cyclic subnormal completion below.

Theorem 3.3. *Under the same notation as in* (2.5)*, we have the following assertions.*

(i) *Suppose* $\Lambda = \Gamma_{n,n}$ *and* $\gamma_\Lambda \subset \mathbf{D}$. *If* $M(n,n) \ge 0$ *has a flat extension* $M(n+1, n+1)$, *i.e.,* $\mathrm{rank} M(n,n) = \mathrm{rank} M(n+1,n+1)$, *then* γ_Λ *has the unique cyclic subnormal completion.*

(ii) *Suppose* $\Lambda = \Gamma_{n,s}$ *and* $\gamma_\Lambda \subset \mathbf{D}$. *If* $M(n,s) \ge 0$ *has a double flat extension* $M(n+2,s)$ *for* $0 \le s < n$, *i.e.,* $\mathrm{rank} M(n,s) = \mathrm{rank} M(n+2,s)$, *then* γ_Λ *has a unique cyclic subnormal completion. In particular, if* n *is even, then the condition of "double flat" in the above assertion can be replaced by "flat".*

Proof. (i) By [4, Cor. 5.12], there exists uniquely the representing measure μ of $M(n,n)$ such that μ induces $\widehat{\gamma}_\Lambda = \{\widehat{\gamma}_{ij}\}_{0\le i,j<\infty}$. Since $\widehat{\gamma}_{ij} = \int_{\mathbf{D}} \bar{z}^i z^j \, d\mu$, by (3.1) we obtain a cyclic subnormal completion S_μ of $\widehat{\gamma}_\Lambda$. To show the uniqueness property, we suppose γ_Λ has a cyclic subnormal completion S. Obviously S is unitarily equivalent to a multiplication S_ν on $H^2(\nu)$. According to (3.1), we have that

$$\int_{\mathbf{D}} p(\bar{z}, z)d\mu = \int_{\mathbf{D}} p(\bar{z}, z)d\nu, \quad \forall p(\bar{z}, z) \in \mathbb{C}[\bar{z}, z].$$

And hence $\mu = \nu$.

(ii) By [7, Propositions 3.2 and 3.4], similarly we have this assertion. $\square$

We may restate Theorem 3.3 as follows.

Theorem 3.4. *Let Λ be a finite subset of $\mathbb{N}_0 \times \mathbb{N}_0$ and let $\gamma_\Lambda \subset \mathbf{D}$. Let $\Gamma_{n,s}$ be the smallest pentagonal set containing Λ. Suppose there exists a finite sequence $\gamma_{\Gamma_{n,s}}$ in $\mathbf{D}$ with $\gamma_{\Gamma_{n,s}} \supset \gamma_\Lambda$ such that one of the following conditions* (i) *and* (ii) *holds*:

(i) *$M(n, s) \geq 0$ has a double flat extension $M(n + 2, s)$ for $0 \leq s < n$,*

(ii) *$M(n, n) \geq 0$ has a flat extension $M(n + 1, n + 1)$ for $s = n$.*

Then γ_Λ has a unique cyclic subnormal completion.

The following remark is interesting as its own purpose.

Remark 3.5. Recall that if $T \in \mathcal{L}(\mathcal{H})$ is a cyclic operator with cyclic vector x_0 in $\mathcal{H}$, by Bram-Halmos' characterization, one of the well-known equivalent conditions for subnormality of T is that

$$\sum_{0\leq i,j\leq n} \langle T^{*i}T^j p_i(T)x_0, p_j(T)x_0\rangle \geq 0$$

for all $p_i(z) \in \mathbf{P}[z]$, $0 \leq i \leq n$, $n \in \mathbb{N}_0$. An operator T is called a *cyclic n-hyponormal operator of order s* if T satisfies the following two conditions:

(i) T is a cyclic operator with a cyclic vector x_0;

(ii) it holds that

$$\sum_{0\leq i,j\leq n} \langle T^{*i+\max\{j-s,0\}}T^{j+\max\{i-s,0\}} p_i(T)x_0, p_j(T)x_0\rangle \geq 0,$$

for all $p_i(z) \in \mathbf{P}[z]$ with $\deg p_i(z) \leq n - i - \max\{i - s, 0\}$ $(1 \leq i \leq n)$. Then it follows from [7, Prop. 2.2] that a cyclic operator T with cyclic vector x_0 is a cyclic n-hyponormal operator order s if and only if $M(n, s) \geq 0$.

We now recapture the real case of the subnormal completion which was introduced by J. Stampfli ([11]). Before doing this, we recall that, for a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of positive real numbers, we denote $\gamma_0 = 1$, $\gamma_n = \alpha_1^2 \cdots \alpha_{n-1}^2$ $(n \geq 1)$, which is called *moments* sometimes (see [2]).

Corollary 3.6. *Let $\alpha : \alpha_0, \alpha_1, \alpha_2$* with *$\alpha_0 < \alpha_1 < \alpha_2 \leq 1$ be an initial segment of positive weights. Then α has a subnormal completion.*

Proof. Let $\gamma : \gamma_0, \gamma_1, \gamma_2, \gamma_3$ be the moments induced by α. Put $\gamma_{00} = \gamma_0, \gamma_{10} = \gamma_1, \gamma_{20} = \gamma_2, \gamma_{30} = \gamma_3$. Consider $\Lambda = \{(0,0),(1,0),(2,0),(3,0)\} \subset \mathbb{N}_0 \times \mathbb{N}_0$. Taking $\gamma_{11} = \gamma_2$, $\gamma_{12} = \gamma_3$, $\gamma_{22} := a\gamma_2 + b\gamma_3$ with

$$a = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \qquad b = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}. \tag{3.2}$$

We now take $\gamma_4 := \gamma_{22}$ and $\gamma_{13} = \gamma_{04} = \gamma_4$. Then we obtain a quartic moment matrix

$$M(2,2) = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_1 & \gamma_2 & \gamma_2 & \gamma_2 \\ \gamma_1 & \gamma_2 & \gamma_2 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_1 & \gamma_2 & \gamma_2 & \gamma_3 & \gamma_3 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_3 & \gamma_4 & \gamma_4 & \gamma_4 \\ \gamma_2 & \gamma_3 & \gamma_3 & \gamma_4 & \gamma_4 & \gamma_4 \\ \gamma_2 & \gamma_3 & \gamma_3 & \gamma_4 & \gamma_4 & \gamma_4 \end{pmatrix}, \tag{3.3}$$

which implies that $M(2,2)$ is a flat extension of $M(1,1)$. Let μ be the associated moment measure for $M(2,2)$. Then by Theorem 3.3 (i), we know that γ_Λ has the unique cyclic subnormal completion. And the cyclic subnormal operator S_μ corresponding by γ_Λ is the required operator. □

By using some conditions in [2] and [3], one can extend Corollary 3.6 to the case of arbitrary finite weights. Beginning our final discussion, we introduce some notation below. For $j \geq 1$ and $\gamma_0, \cdots, \gamma_{2j+1} \in \mathbb{R}$, define Hankel matrices $A(j)$ and $B(j)$ by

$$A(j) := \begin{pmatrix} \gamma_0 & \cdots & \gamma_j \\ \vdots & \ddots & \vdots \\ \gamma_j & \cdots & \gamma_{2j} \end{pmatrix} \quad \text{and} \quad B(j) := \begin{pmatrix} \gamma_1 & \cdots & \gamma_{j+1} \\ \vdots & \ddots & \vdots \\ \gamma_{j+1} & \cdots & \gamma_{2j+1} \end{pmatrix}.$$

Then we have the following corollary.

Corollary 3.7. *Let $\alpha : \alpha_0, \cdots, \alpha_m$ $(m \geq 0)$ be an initial segment of positive weights and let $k := \left[\frac{m+1}{2}\right]$, $l := \left[\frac{m}{2}\right] + 1$. Let $\gamma : \gamma_0, ..., \gamma_{m+1}$ be moments of α. Let $\Lambda = \{(i,0) : 0 \leq i \leq m\}$. Suppose that $A(k) \geq 0$, $B(l-1) \geq 0$ and $v(k+1,k) \in \mathcal{R}(A(k))$ if m is even [$v(k+1,k-1) \in \mathcal{R}(B(l-1))$ if m is odd], where $\mathcal{R}(A(\cdot))$ is the range of $A(\cdot)$. Then γ_Λ has a cyclic subnormal completion.*

Proof. Without loss of generality, we may assume that $\gamma_0 = 1$ and $\gamma_j \in (0,1]$. Put $\gamma_{00} = 1$, $\gamma_{ij} := \gamma_{i+j}$. Consider $\Lambda = \{(i,0) : 0 \leq i \leq m\}$. By the similar method with the proof of Corollary 3.6, without difficulties we can take γ_j $(m+2 \leq j \leq 2m)$ so that we may construct

$$M(m,m) = \begin{pmatrix} M_{0,0} & M_{0,1} & \cdots & M_{0,m} \\ M_{1,0} & M_{1,1} & \cdots & M_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,0} & M_{m,1} & ... & M_{m,m} \end{pmatrix}$$

as in (2.2), where all entries of $M_{i,j}$ are γ_{i+j}; for example, see (3.3). If we correspond $M_{i,j}$ to γ_{i+j}, it follows from the hypothesis that $M(m,m)$ is the flat extension of $M(m-1,m-1)$. Hence γ_Λ has a cyclic subnormal completion. $\square$

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