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# Chain Recurrences on Conservative Dynamics

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Abstract. Let $M$ be a manifold with a volume form $\omega$ and $f : M \to M$ be a diffeomorphism of class $\mathcal{C}^1$ that preserves $\omega$. We prove that if $M$ is almost bounded for the diffeomorphism $f$, then $M$ is chain recurrent. Moreover, we get that Lagrange stable volume-preserving manifolds are also chain recurrent.

## 1. Introduction

Our purpose of this paper is to study the chain recurrence set of volume-preserving diffeomorphisms on non-compact manifolds. We follow Conley's definitions of attractors and chain recurrences [4], and Hurley's generalized definitions [5],[6].

From Poincaré recurrence theorem, it is well-known that for any volume-preserving diffeomorphism on the compact manifolds $M$, every point of $M$ is chain recurrent. However, unfortunately, the parallel statement for the chain recurrence does not hold for the non-compact manifolds. Thus, in the non-compact case, we may impose the canonical conditions as almost boundedness and Lagrange stability. Our main theorem is as follows.

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**Theorem 2.9.** Let $M$ be a manifold with a volume form $\omega$ and $f$ be a volume-preserving diffeomorphism on $M$. If $M$ is almost bounded for $f$, then $M$ is strongly chain recurrent for $f$, i.e., every point of $M$ is strongly chain recurrent with respect to $f$.

The above theorem follows from a stronger claim, Proposition 2.10. The proposition asserts that with the assumptions in Theorem 2.9 except the Lagrange-stability assumption, almost every point in $U - A$ should have the unbounded orbit, where $A$ is an attractor and $U$ is an attractor block (weakly absorbing set) of $A$. I.e., the set of points of $U - A$ with bounded orbits is of measure 0.

The study of attractors in the volume-preserving category is meaningful only in the non-compact cases. This is because, for a compact conservative dynamics (i.e., volume-preserving or symplectic dynamics on compact manifolds), there are only trivial attractors, which is clear from the definition of conservative diffeomorphisms (volume-preserving diffeomorphisms or symplectomorphisms on compact manifolds)[3].

Hence, we attempt to understand the volume-preserving and the symplectic dynamics on non-compact spaces through the attractors and then the chain recurrence. Whilst, similar dynamical properties on compact spaces have been intensively studied with appropriate assumptions, e.g. $\mathcal{C}^1$-genericity (ref.[1]). Note that since the symplectic diffeomorphisms are automatically volume-preserving, our results in the paper is immediately applicable to the symplectic dynamics as well.

## 2. Chain Recurrences of Volume-preserving Diffeomorphisms

### 2.1. Preliminaries

We fix the notations and definitions used throughout the paper.

Let $M$ be an $n$-dimensional differentiable manifold with a metric $d$, and $f : M \to M$ be a $\mathcal{C}^1$-diffeomorphism. A *volume form* $\omega$ on $M$ is a nowhere vanishing $n$-form on $M$. A symplectic form $\omega$ on $M$ is a nowhere degenerate 2-form on $M$. Here, the non-degeneracy of $\omega$ is the same as its $(n/2)$-times wedge product $\omega^{\frac{n}{2}} = \omega \wedge \cdots \wedge \omega$ defines a volume form on $M$. Thus, when we say a symplectic form, $n$ is assumed to be even. Integration along the subsets of $M$ defines a Lebesgue measure $m$. Indeed, by the para-compactness of $M$, locally $m$ is written as a product of a $\mathcal{C}^1$-function and the standard Lebesgue measure on $\mathbb{R}^n$ (via the $\mathcal{C}^1$-transition). This clarifies a Lebesgue measurable subset of $M$, a countable union of Lebesgue measurable subsets of $\mathbb{R}^n$ (via the $\mathcal{C}^1$-transition). Thanks to the well-known theory of Lebesgue measures and Borel measures, one guarantees any compact subset of $M$ is Lebesgue measurable and is of finite measure. By the compactness, the closed balls (with finite radii) are of finite measure, as well.

If one says $f$ preserves $\omega$, this means $f^*\omega = \omega$. When $\omega$ is a symplectic form, the $\omega$-preservation implies the volume-preservation. The volume-preservation of $f$ amounts to the measure-preservation. In the case, for a Lebesgue measurable subset $N \subset M$, we have $m(N) = m(f(N))$.

We fix a manifold with metric $(X, d)$ and a homeomorphism $f : X \to X$. We define

$$\mathcal{P} = \text{the set of } \mathbb{R}^+\text{-valued continuous functions on } X.$$

**Definition 2.1.** A nonempty open subset $U$ of $X$ is an *attractor block* for $f$ if the closure of $f(U)$ is contained in $U$. When $U$ is an attractor block, the set

$$A = \bigcap_{n \geq 0} \overline{f^n(U)}$$

is called the *weak attractor determined by* $U$.

**Definition 2.2.** If $\varepsilon \in \mathcal{P}$, then $x_0, x_1, \cdots, x_n$ is an $\varepsilon$*-chain* if $d(f(x_j), x_{j+1}) < \varepsilon(f(x_j))$ for $0 \leq j < n - 1$. The number $n$ is called the *length* of the $\varepsilon(x)$-chain. A point $p$ is *strongly chain recurrent* for $f$ if for every $\varepsilon \in \mathcal{P}$, there exists an $\varepsilon(x)$-chain of length at least 1 that begins and ends at $p$. We denote by

$$CR^+(f) = \text{the set of all strong chain recurrence points of } f\ .$$

Note that if $M$ is a compact manifold, then the strong chain recurrence point coincides with the usual chain recurrence point.

**Definition 2.3.** Let $U$ be an attractor block for $f$ and $A$ be the associated weak attractor. We define the *basin of a weak attractor A relative to U, B(A;U)* as the open set $\cup_{n \geq 0} f^{-n}(U)$.

Every point of $B(A; U)$ has the omega-limit sets contained in $A$. When $X$ is a compact space, $B(A; U)$ is independent of $U$ while it is not true for non-compact manifolds. Therefore, we define the *extended basin* $B(A)$ of $A$ by the union of the sets $B(A; U)$ as $U$ runs over all the absorbing sets that determine $A$.

### 2.2. Strong chain recurrences of volume-preserving diffeomorphisms

The chain recurrence theorem on compact manifolds with volume-preserving diffeomorphism is almost direct to prove. Our focus is non-compact manifolds. The simple examples below exhibit the failure of the chain recurrence theorem in the volume-preserving dynamics over non-compact manifolds.

**Example 2.4.** Let $M = \mathbb{R}$ and $f : M \to M$ given by $f(x) = x + 1$. Then, $f$ preserves a differential form and no point of $M$ is a (strong) chain recurrence for $f$. While, let $U = (0, \infty)$, then we have that $U$ is an attractor block, with associated empty weak attractor.

**Example 2.5.** Let $M = \mathbb{R}^2$ and $f : M \to M$ given by $f(x, y) = (x + 1, y)$. Let $\omega$ be a volume form (equivalently, a symplectic form) by $\omega = dx \wedge dy$. Then, it is clear that $f$ preserves $\omega$. Let

$$U_n = \{(x, y) \in M \mid y < \frac{-1}{x - n},\ x < n\} \cup \{(x, y) \in M \mid x \geq n\}.$$

Since $f(U_n) = U_{n+1}$, we can easily check that $U_0$ is an attractor block for the translation $f$ and

$$A = \{(x, y) \mid y \leq 0\}$$

is the weak attractor determined by $U$. Whilst, no point of $M$ is a (strong) chain recurrence for $f$.

The following theorem by Hurley is a generalized version of Conley's theorem.

**Theorem 2.6.**[5, 6] If $X$ is a locally compact metric space and $f : X \to X$ is continuous, then the strong chain recurrence set $CR^+(f)$ of $f$ is the complement of the union of the set $B(A) - A$, as $A$ runs over the collection of weak attractors of $f$. I.e.,

$$X - CR^+(f) = \bigcup_{A: weak\ attractor} (B(A) - A). \tag{2.1}$$

Here, the strong chain recurrence and weak attractors are defined with respect to a continuous map $f$ with a suitable adaptation of the definitions in the previous subsection.

The following proposition (and its corollary) shows the invariance of the weak attractors and the boundaries.

**Proposition 2.7.** Let $f$ be a homeomorphism on a metric space $X$, $U$ be an attractor block, and $A$ be an associated weak attractor. If a point $x$ is in $U - A$, then the intersection of the (positive) $f$-orbit of $x$ and $A$ is empty.

*Proof.* Let $O_f^+(x)$ be the (positive) $f$-orbit of $x$, i.e., $O_f^+(p) = \{f^n(p) \mid n \geq 0\}$. Suppose $O_f^+(x) \cap A \neq \varnothing$, Then there exists a nonnegative integer $k$ such that $f^k(x) \in A$, that is, $f^k(x) \in \cap_{n\geq 0}\overline{f^n(U)}$. Note that, $f(\overline{U}) = \overline{f(U)}$. Thus $x \in f^{n-k}(U)$ for all $n \geq 1$ and so $x \in A$ by the shrinking property. This is a contradiction, which completes the proof. □

When $f : X \to X$ is continuous, by the definition, it is easily shown that an attractor is positively $f$-invariant. If $f$ is a homeomorphism, an attractor $A$ is $f$-invariant, i.e., $f(A) = A$. Indeed, if $f(A) \neq A$ then there is an element $x$ in $A - f(A)$. From the definitions, $f^{-1}(x) \in U - A$, where $U$ is an associated attractor block. Then, Proposition 2.7, we must meet a contradiction. Hence an attractor is invariant.

**Corollry 2.8.** Let $f$ be a homeomorphism on a locally compact metric space $M$. Then the boundary of every weak attractor is positively $f$-invariant, that is, $f(\partial A) \subseteq \partial A$ for every weak attractor $A$.

*Proof.* Suppose the contrary of the conclusion. Then by the above statement, we may assume that there exists a boundary point $x$ satisfying $f(x)$ is in the interior of $A$. From the local compactness, we can choose compact neighborhood $C$ of $f(x)$ such that $f^{-1}(C)$ is also a compact neighborhood of $x$. Then, we are able to pick a point in $U - A$ where $U$ is an associated attractor block which determines $A$. By Proposition 2.7, it is a contradiction. $\square$

Now we embark on the main proposition and the main theorem for the (strong) chain recurrences on the non-compact manifolds. The proposition tells us that the points near an attractor with bounded orbits form a measure 0 set, in the volume-preserving dynamics. Recall that in Example , every orbit is unbounded thus the proposition trivially holds.

For $p \in M$, we denote $K^+(p) := \overline{O_f^+(p)}$. We call $M$ *almost bounded* for $f$, if for almost everywhere $p \in M$, $K^+(p)$ is compact. Since we are working on a metric space, the compactness of $K^+(p)$ amounts to the boundedness of $O_f^+(p)$.

**Theorem 2.9.** Let $M$ be a manifold with a volume form $\omega$, and $f$ be a volume-preserving diffeomorphism on $M$. If $M$ is almost bounded for $f$, then $M$ is strongly chain recurrent for $f$, i.e., every point of $M$ is strongly chain recurrent with respect to $f$.

*Proof.* We use Hurley's theorem (Theorem 2.6) for locally compact spaces. To prove our theorem, the nonexistence of weak attractors should be guaranteed. On the contrary, suppose that a nonempty proper weak attractor $A$ exists. Let $U$ be an associated attractor block of $A$ (so that $A \subsetneq U$). We will prove that the complement in $U - A$ of the set of points of $U - A$ with unbounded orbits is of measure 0 in the following proposition. $\square$

**Proposition 2.10.** Let $M$ be a manifold (not necessarily compact) with a volume form $\omega$. Let $f$ be a volume-preserving diffeomorphism on $M$. Let $A$ be any weak attractor and $U$ be an associated attractor block with $A \subsetneq U$. Then, the complement in $U - A$ of the set of points $p \in U - A$ with unbounded orbits is of measure 0. That is, $m\{p \in (U - A) \mid O_f^+(p) \text{ is unbounded}\}^c = 0$, here $m$ is a measure induced by the volume form.

*Proof.* Let $p \in U - A$ and $K \subset U - A$ be a compact neighborhood of $p$ with a finite measure $c > 0$. Let us fix any point $x_0 \in M$. Let $B_r(x_0)$ be the closed ball of the radius $r$ centered at $x_0 \in M$ (where $r \in \mathbb{Z}_+$). Let us define

$$K_r = \{q \in K | f^k(q) \notin B_r(x_0) \text{ for some positive integer } k\}. \tag{2}$$

Note that

$$K_1 \supset K_2 \supset K_3 \supset \dots.$$

and that

$$L = \bigcap_{r \in \mathbb{Z}_+} K_r \tag{3}$$

is the set of points of $K$ with unbounded orbits and $m(L) = m(K)$ implies the claim of the proposition.

Now, to prove the proposition, it suffices to show that $L$ is of measure $c$. Let us observe

$$\begin{aligned} K - K_r &= \{q \in K | f^k(q) \in B_r(x_0) \text{ for all } k \in \mathbb{Z}_+\} \\ &= \bigcap_{k\in\mathbb{Z}_+} \{q \in K | f^k(q) \in B_r(x_0)\} \end{aligned}$$

and thus $K - K_r$ is measurable as $\{q \in K \mid f^k(q) \in B_r(x_0)\}$ is measurable for each $k \in \mathbb{Z}+$. Therefore, $L$ is measurable as well.

We claim that

$$m((f^k(U) - f^k(A)) \cap B_r(x_0)) \to 0 \tag{4}$$

as $k \to \infty$. Indeed, Lebesgue's dominated convergence theorem assures it from the following:

(a) the definition of attractors (i.e., the descending sequence $U \supset f(U) \supset f^2(U) \supset ... \supset A = \cap_k f^k(U)$ ),

(b) the $f$-invariance of $A$,

(c) $f^k(U) - f^k(A)$ and $B_r(x_0)$ are measurable and their intersection is of finite measure.

Note that $\{q \in K|f^k(q) \in B_r(x_0)\} = f^{-k}(B_r(x_0)) \cap K$. Thus, we have

$$\begin{aligned} m(\{q \in K|f^k(q) \in B_r(x_0)\}) &= m(f^{-k}(B_r(x_0)) \cap K) \\ &= m((B_r(x_0)) \cap f^k(K)) \end{aligned}$$

where the latter equality is due to the measure-preservation of $f$. Because of the inclusion $f^k(K) \subset f^k(U) - f^k(A)$ and (4), we obtain

$$m(\{q \in K|f^k(q) \in B_r(x_0)\}) \to 0$$

as $k \to \infty$ by Lebesgue's dominated convergence theorem. Therefore, for each $r$, we have $m(K - K_r) = 0$, equivalently, $m(K_r) = m(K) - m(K - K_r) = c$. By applying Lebesgue's dominated convergence theorem to (3), we obtain $m(L) = c$, as desired. □

Let us continue the proof of Theorem 2.9. By Proposition 2.10, almost every point of $U - A$ has an unbounded orbit. This contradicts to our assumption of the almost boundedness of $M$ with respect to $f$. This finishes the proof of Theorem 2.9. □

Recall that a riemannian manifold $M$ is said to be *Lagrange-stable* for a diffeomorphism $f$ if every closure of an orbit is compact, i.e., for each $p \in M$, $K^+(p)$ is a compact subset of $M$. Since the Lagrange-stability is a stronger condition than the almost-boundedness, we obtain the corollary.

**Corollry 2.11.** Let $M$ be a manifold with a volume form $\omega$, and $f$ be a Lagrange-stable volume-preserving diffeomorphism on $M$. Then, $M$ is strongly chain recurrent for $f$, that is, each point of $M$ is strongly chain recurrent with respect to $f$.

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