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# On a Class of $\gamma^*$-pre-open Sets in Topological Spaces

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Abstract. In this paper, a new class of open sets, namely $\gamma^*$-pre-open sets was introduced and its basic properties were studied. Moreover a new type of topology $\tau_{\gamma p^*}$ was generated using $\gamma^*$-pre-open sets and characterized the resultant topological space $(X, \tau_{\gamma p^*})$ as $\gamma^*$-pre-$T_{\frac{1}{2}}$ space.

## 1. Introduction

The concepts of pre-open sets and semi-pre-open sets were introduced respectively by Mashhour et al.[6] and Andrijevic[1]. Andrijevic[1] introduced a new class of topology generated by pre-open sets and corresponding closure and interior operators. Kasahara[3] defined the concept of an operation on topological spaces and introduced the concept of $\alpha$-closed graphs of an operation. Ogata[7] called the operation $\alpha$ (respectively $\alpha$-closed set) as $\gamma$-operation (respectively $\gamma$-closed set) and introduced the notion of $\tau_\gamma$ which is the collection of $\gamma$-open sets in a topological space. Further, he defined the concept of $\gamma$-closure and $\tau_\gamma$-closure operators and

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investigated the relation between them. Moreover, he introduced the notation of $\gamma$-$T_i$ ($i = 0, \frac{1}{2}, 1, 2$) and characterized $\gamma$-$T_i$ spaces using the notion of $\gamma$-closed and $\gamma$-open sets. Sai Sundara Krishnan et al.[9] introduced the concept of $\gamma$-pre-open sets and studied various basic properties. If $A$ is a subset of $X$, throughout this paper $X \setminus A$ denotes complement of $A$.

In this paper in section 2, we introduced the concept of $\gamma^*$-pre-open sets, which is analogous to pre-open sets and introduced the notion $PO_{\gamma^*}(X)$ which is the set of all $\gamma^*$-pre-open sets in a topological space $(X, \tau)$. Further, we introduced the concepts of $\gamma^*$-pre-closure and $\gamma^*$-pre-interior operators and studied some of their fundamental properties. In section 3, we introduced the concept of $\gamma^*$-semi-pre-open sets in a topological space $(X, \tau)$ together with $\gamma^*$-semi-pre-closure and $\gamma^*$-semi-pre-interior operators and investigated their basic properties. In section 4, we generated a new topology $\tau_{\gamma p^*}$ on $X$ using the notion of $\gamma^*$-pre-open sets. In section 5, we introduced the notion of $\gamma^*$-pre-$T_i$ spaces ($i = 0, \frac{1}{2}, 1, 2$) and characterized $\gamma^*$-pre-$T_i$ spaces using $\gamma^*$-pre-closed and $\gamma^*$-pre-open sets. Finally, we proved that $(X,\ \tau_{\gamma p^*})$ space is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space.

## 2. $\gamma^*$-pre-open Sets

In this section, we introduce the concept of $\gamma^*$-pre-open sets and study some of their basic properties.

**Definition 2.1.** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then $A$ is said to be

(i) pre-open[6] if $A \subseteq int(cl(A))$. $PO(X)$ denotes the family of pre-open sets in $(X, \tau)$;

(ii) semi-pre-open[1] if and only if there exists a pre-open set $U$ such that $U \subseteq A \subseteq cl(U)$. $SPO(X)$ denotes the family of semi-pre-open sets in $(X, \tau)$.

**Definition 2.2.** Let $(X, \tau)$ be a topological space and $A \subseteq X$. Then

(i) pre-interior[6] (resp. semi-pre-interior[1]) of $A$ is defined by union of all pre-open (resp. semi-pre-open) sets contained in $A$ and it is denoted by $pint(A)$ (resp. $spint(A)$);

(ii) pre-closure[6] (resp. semi-pre-closure[1]) of $A$ is defined by intersection of all pre-closed (resp. semi-pre-closed) sets containing $A$ and it is denoted by $pcl(A)$ (resp. $spcl(A)$).

**Definition 2.3([3]).** Let $(X, \tau)$ be a topological space. An operation $\gamma$ on the topology $\tau$ is a mapping from $\tau$ into the power set $P(X)$ of $X$ such that $V \subseteq V^\gamma$ for each $V \in \tau$, where $V^\gamma$ denotes the value of $\gamma$ at $V$ . It is denoted by $\gamma : \tau \to P(X)$.

**Definition 2.4([7]).** A subset $A$ of a topological space $(X, \tau)$ is called a $\gamma$-open set of $(X, \tau)$, if for each $x \in A$, there exists an open neighborhood $U$ such that $x \in U$

and $U^\gamma \subseteq A$. $\tau_\gamma$ denotes set of all $\gamma$-open sets in $(X, \tau)$.

**Definition 2.5([7]).** Let $(X,\ \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$ . Then for any subset $A$ of $X$,
(i) $\tau_\gamma\text{-}cl(A) = \cap\{F : A \subseteq F \text{ and } X \setminus F \in \tau_\gamma\}$;
(ii) $\tau_\gamma\text{-}int(A) = \cup\{G : G \subseteq A \text{ and } G \in \tau_\gamma\}$.

**Definition 2.6([7]).** (i) Let $A \subseteq X$. A point $x \in A$ is said to be a $\gamma$-interior point of $A$ if and only if there exists an open neighborhood $N$ of $x$ such that $N^\gamma \subseteq A$ and we denote the set of all such points by $int_\gamma(A)$.

That is $int_\gamma(A) = \{x \in A : x \in N \in \tau \text{ and } N^\gamma \subseteq A \text{ for some } N\}$;

(ii) A point $x \in X$ is called a $\gamma$-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \emptyset$, for each open neighborhood $U$ of $x$. The set of all $\gamma$-closure points of $A$ is called the $\gamma$-closure of $A$ and is denoted by $cl_\gamma(A)$.

That is $cl_\gamma(A) = \{x \in X : x \in U \in \tau \text{ and } U^\gamma \cap A \neq \emptyset \text{ for all } U\}$.

**Remark 2.1.** (i) A subset $A$ of $X$ is called $\gamma$-open[7] if and only if $A = int_\gamma(A)$. A set $A$ is called $\gamma$-closed[7] if and only if $X \setminus A$ is $\gamma$-open;

(ii) A subset $A$ of $X$ is called $\gamma$-closed[7], if $cl_\gamma(A) \subseteq A$.

**Definition 2.7([7]).** An operation $\gamma$ on $\tau$ is said to be
(i) regular, if for any open neighborhoods $U$, $V$ of each $x \in X$, there exists an open neighborhood $W$ of $x$ such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$;
(ii) open, if for every neighborhood $U$ of each $x \in X$, there exists a $\gamma$-open set B such that $x \in B$ and $U^\gamma \supseteq B$.

**Definition 2.8([7]).** Let $(X,\ \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $(X,\ \tau)$ is said to be $\gamma$-regular, if for each $x \in X$ and for each open neighborhood $V$ of $x$, there exists an open neighborhood $U$ of $x$ such that $U^\gamma \subseteq V$.

**Definition 2.9([9]).** Let $(X,\ \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. A subset $A$ of $X$ is said to be
(i) $\gamma$-pre-open if $A \subseteq \tau_\gamma\text{-}int(\tau_\gamma\text{-}cl(A))$. $\tau_\gamma\text{-}PO(X)$ denotes the set of all $\gamma$-pre-open sets in $(X,\ \tau)$;
(ii) $\gamma$-pre-closed in $(X,\ \tau)$ if and only if $X \setminus A$ is $\gamma$-pre-open, equivalently a subset $A$ of $X$ is $\gamma$-pre-closed if and only if $\tau_\gamma\text{-}cl(\tau_\gamma\text{-}int(A)) \subseteq A$. $\tau_\gamma\text{-}PC(X)$ denotes set of all $\gamma$-pre-closed sets in $(X,\ \tau)$.

**Definition 2.10([9]).** Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then for any subset $A$ of $X$,
(i) $\tau_\gamma\text{-}pcl(A) = \cap\{F : X \setminus F \in \tau_\gamma\text{-}PO(X) \text{ and } A \subseteq F\}$;

(ii) $\tau_\gamma$-$pint(A) = \cup\{G : G \in \tau_\gamma\text{-}PO(X) \text{ and } G \subseteq A\}$.

**Definition 2.11.** Let $(X,\ \tau)$ be a topological space and $\gamma :\ \tau\ \rightarrow P(X)$ be an operation on $\tau$. A subset $A$ of $X$ is said to be a $\gamma^*$-pre-open set, if $A \subseteq int_\gamma(cl_\gamma(A))$. The set of all $\gamma^*$-pre-open sets is denoted by $PO_{\gamma^*}(X)$.

**Example 2.1.** Let $X = \{a,\ b,\ c\}$, $\tau = \{\emptyset,\ X,\ \{a\},\ \{b\},\ \{a,b\},\ \{a,c\}\}$ and $\gamma : \tau \rightarrow P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} A \cup \{b\} & \text{if } A = \{a\} \\ cl(A) & \text{if } A \neq \{a\} \end{cases} \text{ for every } A \in \tau.$$

Then $PO_{\gamma^*}(X) = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.

**Theorem 2.1.** *Let $\{A_\alpha :\ \alpha\ \in\ J\}$ be the collection of $\gamma^*$-pre-open sets in a topological space $(X,\ \tau)$. Then $\bigcup_{\alpha\in J} A_\alpha$ is also a $\gamma^*$-pre-open set in $(X,\ \tau)$.*

*Proof.* Since $A_\alpha$ is $\gamma^*$-pre-open, then $A_\alpha \subseteq int_\gamma(cl_\gamma(A_\alpha))$. This implies that $\bigcup_{\alpha\in J} A_\alpha \subseteq \bigcup_{\alpha\in J}(int_\gamma(cl_\gamma(A_\alpha))) \subseteq int_\gamma(cl_\gamma(\bigcup_{\alpha\in J} A_\alpha))$. Hence $\bigcup_{\alpha\in J} A_\alpha$ is a $\gamma^*$-pre-open set in $(X,\ \tau)$. □

**Remark 2.2.** If $A$ and $B$ are two $\gamma^*$-pre-open sets in $(X,\ \tau)$, then $A \cap B$ need not be $\gamma^*$-pre-open in $(X,\ \tau)$.

Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a,c\}, \{b,c\}\}$ and $\gamma : \tau \rightarrow P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} A & \text{if } b \in A \\ cl(A) & \text{if } b \notin A \end{cases} \text{ for every } A \in \tau.$$

Then $A = \{a,b\}$ and $B = \{a,c\}$ are $\gamma^*$-pre-open sets in $(X,\ \tau)$ but $A \cap B = \{a\}$ is not $\gamma^*$-pre-open in $(X,\ \tau)$.

**Theorem 2.2.** *Let $(X,\ \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \rightarrow P(X)$ be an operation on $\tau$. If $A$ is a $\gamma$-open set in $(X,\ \tau)$, then $A$ is $\gamma^*$-pre-open.*

*Proof.* Let $x \in A$. Then $x \in cl_\gamma(A)$. Since $A$ is $\gamma$-open, there exists an open neighborhood $U$ such that $x \in U$ and $U^\gamma \subseteq A$. This implies that $U^\gamma \subseteq cl_\gamma(A)$. Thus $x$ is a $\gamma$-interior point of $cl_\gamma(A)$. Hence $x \in int_\gamma(cl_\gamma(A))$. This shows that $A$ is a $\gamma^*$-pre-open set in $(X,\ \tau)$. □

**Example 2.2.** The following example shows that the converse of the above theorem need not be true.

Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$ and $\gamma : \tau \rightarrow P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} A \cup \{c\} & \text{if } A = \{a\} \text{ } or \text{ } \{b\} \\ A & \text{if } A \neq \{a\} \text{ } and \text{ } \{b\} \end{cases} \text{ for every } A \in \tau.$$

Then $\{b\}$ and $\{a,c\}$ are $\gamma^*$-pre-open sets in $(X,\tau)$ but not $\gamma$-open in $(X,\tau)$.

**Remark 2.3.** By Theorem 2.2 and Example 2.2, we have that $\tau_\gamma \subseteq PO_{\gamma^*}(X)$.

**Remark 2.4.** The concepts of $\gamma^*$-pre-open and pre-open are independent.
Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{b\},\{c\},\{a,b\},\{b,c\}\}$ and $\gamma:\tau\to P(X)$ be an operation on $\tau$ such that

$$\gamma(A)=\begin{cases} cl(A) & \text{if } A=\{b\} \\ A\cup\{c\} & \text{if } A\neq\{b\}\end{cases} \quad \text{for every } A\in\tau.$$

Then $PO(X)=\{\emptyset,X,\{b\},\{c\},\{a,b\},\{b,c\}\}$ and $PO_{\gamma^*}(X)=\{\emptyset,X,\{b\},\{c\},\{a,c\},\{b,c\}\}$. Thus $\{a,\ b\}$ is a pre-open set in $(X,\ \tau)$ but not $\gamma^*$-pre-open in $(X,\ \tau)$. Similarly the set $\{a,c\}$ is a $\gamma^*$-pre-open set in $(X,\tau)$ but not pre-open in $(X,\tau)$.

**Theorem 2.3.** *Let $(X,\tau)$ be a $\gamma$-regular space. Then the concepts of $\gamma^*$-pre-open and pre-open are coincide. That is $PO_{\gamma^*}(X)=PO(X)$.*

*Proof.* Follows from the Definition 2.8 and Theorem 3.6(ii)[7]. □

**Lemma 2.1.** *Let $(X,\tau)$ be a topological space and $\gamma:\tau\to P(X)$ be an operation on $\tau$. If $A$ and $B$ are two subsets of $X$, then the following are hold:*
*(i) If $A\subseteq B$, then $int_\gamma(A)\subseteq int_\gamma(B)$;*
*(ii) $int_\gamma(A)\cup int_\gamma(B)=int_\gamma(A\cup B)$;*
*(iii) If $\gamma$ is regular, then $int_\gamma(A)\cap int_\gamma(B)=int_\gamma(A\cap B)$ and $cl_\gamma(A)\cup cl_\gamma(B)=cl_\gamma(A\cup B)$.*

*Proof.* Follows from the Definitions 2.6, 2.7 and Lemma 3.10[7]. □

**Remark 2.5.** The concepts of $\gamma^*$-pre-open and $\gamma$-pre-open are independent.

(i) Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\},\{a,b,d\}\}$ and $\gamma:\tau\to P(X)$ be an operation on $\tau$ such that

$$\gamma(A)=\begin{cases} int(cl(A)) & \text{if } A=\{a\} \\ cl(A) & \text{if } A\neq\{a\}\end{cases} \quad \text{for every } A\in\tau.$$

Then $\tau_\gamma$-$PO(X)=\{\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$ and $PO_{\gamma^*}(X)=\{\emptyset,X,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{a,b,c\},\{a,b,d\},\{a,c,d\}\}$. Thus $\{b\}$ and $\{b,c\}$ are $\gamma^*$-pre-open sets in $(X,\tau)$ but not $\gamma$-pre-open in $(X,\tau)$.

(ii) Let $X=\{a,b,c,d\}$, $\tau=\{\emptyset,X,\{a\},\{b\},\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\}\}$ and $\gamma:\tau\to P(X)$ be an operation on $\tau$ such that

$$\gamma(A)=\begin{cases} A\cup\{c\} & \text{if } A=\{a\} \\ A\cup\{d\} & \text{if } A=\{b\} \\ int(cl(A)) & \text{if } A\neq\{a\} \textit{ and } \{b\}\end{cases} \quad \text{for every } A\in\tau.$$

Then $\tau_\gamma$-$PO(X)=\{\emptyset,X,\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{b,d\},\{c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}\}$ and $PO_{\gamma^*}(X)=\{\emptyset,X,\{b\},\{a,b\},\{b,c\},\{c,d\},\{a,b,c\},\{a,b,d\},$

$\{a, c, d\}, \{b, c, d\}\}$. Thus $\{c\}, \{a, c\}$ and $\{b, d\}$ are $\gamma$-pre-open sets in $(X, \tau)$ but not $\gamma^*$-pre-open in $(X, \tau)$.

**Theorem 2.4.** *Let $(X, \tau)$ be a $\gamma$-regular space and $\gamma : \tau \to P(X)$ be an open operation on $\tau$. Then the concepts of $\gamma^*$-pre-open and $\gamma$-pre-open are coincide. That is $PO_{\gamma^*}(X) = \tau_\gamma\text{-}PO(X)$.*

*Proof.* Follows from Definitions 2.7, 2.8 and Theorem 3.6(iii)[7]. □

**Lemma 2.2.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. If $A$ is a subset of $X$, then*
*(i) $cl_\gamma(A) = X \setminus int_\gamma(X \setminus A)$;*
*(ii) $int_\gamma(A) = X \setminus cl_\gamma(X \setminus A)$.*

*Proof.* (i) Let $x \notin cl_\gamma(A)$. Then there exists an open set $U$ such that $x \in U$ and $U^\gamma \cap A = \emptyset$. This implies that $U^\gamma \subseteq X \setminus A$ and hence we have that $x \in int_\gamma(X \setminus A)$. Therefore $x \notin X \setminus int_\gamma(X \setminus A)$, implies that $cl_\gamma(A) \supseteq X \setminus int_\gamma(X \setminus A)$. Conversely, suppose that $x \notin X \setminus int_\gamma(X \setminus A)$. This implies that there exists an open set $N$ of $x$ such that $N^\gamma \subseteq X \setminus A$. Therefore $N^\gamma \cap A = \emptyset$ and hence $x \notin cl_\gamma(A)$. Thus $X \setminus int_\gamma(X \setminus A) \supseteq cl_\gamma(A)$.

(ii) From (i), we have that $cl_\gamma(X \setminus A) = X \setminus int_\gamma(X \setminus (X \setminus A)) = X \setminus int_\gamma(A)$. This implies that $X \setminus cl_\gamma(X \setminus A) = int_\gamma(A)$. □

**Lemma 2.3.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. If $A$ is a subset of $X$, then*
*(i) for every $\gamma$-open set $G$ of $X$, we have that $cl_\gamma(A) \cap G \subseteq cl_\gamma(A \cap G)$;*
*(ii) for every $\gamma$-closed set $F$ of $X$, we have that $int_\gamma(A \cup F) \subseteq int_\gamma(A) \cup F$.*

*Proof.* (i) Let $x \in cl_\gamma(A) \cap G$ and let $U$ be an open set containing $x$. Since $x \in cl_\gamma(A)$, implies that $U^\gamma \cap A \neq \emptyset$. Since $G$ is a $\gamma$-open set, there exists an open set $V$ of $x$ such that $V^\gamma \subseteq G$. Thus $(U \cap V)^\gamma \cap A \neq \emptyset$. This implies that $U^\gamma \cap (A \cap G) \neq \emptyset$ and hence $x \in cl_\gamma(A \cap G)$. Therefore $cl_\gamma(A) \cap G \subseteq cl_\gamma(A \cap G)$.
(ii) Follows from (i) and Lemma 2.2(ii). □

**Theorem 2.5.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be a regular operation on $\tau$. Let $A$ be a $\gamma^*$-pre-open set and $U$ be a $\gamma$-open subset of $X$. Then $A \cap U$ is also a $\gamma^*$-pre-open set.*

*Proof.* Let $x \in A \cap U$. Since $A$ is $\gamma^*$-pre-open, we have that $x \in int_\gamma(cl_\gamma(A))$. This implies that there exists an open set $N$ of $x$ such that $N^\gamma \subseteq cl_\gamma(A)$. Since $U$ is a $\gamma$-open set, there exists an open set $V$ of $x$ such that $V^\gamma \subseteq U$. Since $\gamma$ is regular, there exists an open set $W$ such that $W^\gamma \subseteq N^\gamma \cap V^\gamma$. This implies that $W^\gamma \subseteq cl_\gamma(A) \cap U \subseteq cl_\gamma(A \cap U)$ (by Lemma 2.3(i)). Thus $x \in int_\gamma(cl_\gamma(A \cap U))$. □

**Definition 2.12.** Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then a subset $A$ of $X$ is said to be
(i) $\gamma^*$-dense set if $cl_\gamma(A) = X$;

(ii) $\gamma^*$-nowhere dense set if $int_\gamma(cl_\gamma(A)) = \emptyset$.

**Theorem 2.6.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and an open operation on $\tau$. Then a subset $N$ of $X$ is $\gamma^*$-nowhere dense set if and only if any one of the following condition hold:*
*(i) $cl_\gamma(X \setminus cl_\gamma(N)) = X$;*
*(ii) $N \subseteq cl_\gamma(X \setminus cl_\gamma(N))$;*
*(iii) Every non empty $\gamma$-open set $U$ contains a non empty $\gamma$-open set $A$ disjoint with $N$.*

*Proof.* (i) $int_\gamma(cl_\gamma(N)) = \emptyset$ if and only if $X \setminus cl_\gamma(X \setminus cl_\gamma(N)) = \emptyset$ (by Lemma 2.2(ii)) if and only if $X \subseteq cl_\gamma(X \setminus cl_\gamma(N))$ if and only if $X = cl_\gamma(X \setminus cl_\gamma(N))$.

(ii) $N \subseteq X = cl_\gamma(X \setminus cl_\gamma(N))$ (by (i)). Conversely, $N \subseteq cl_\gamma(X \setminus cl_\gamma(N))$, implies that $cl_\gamma(N) \subseteq cl_\gamma(X \setminus cl_\gamma(N))$. Since $X = cl_\gamma(N) \cup (X \setminus cl_\gamma(N))$, implies that $X \subseteq cl_\gamma(X \setminus cl_\gamma(N)) \cup (X \setminus cl_\gamma(N)) = cl_\gamma(X \setminus cl_\gamma(N))$. Hence $X = cl_\gamma(X \setminus cl_\gamma(N))$.

(iii) Given $N$ is a $\gamma^*$-nowhere dense subset of $X$ follows that $int_\gamma(cl_\gamma(N)) = \emptyset$. This implies that $cl_\gamma(N)$ does not contain any non empty $\gamma$-open set. Hence for any non empty $\gamma$-open set $U$, $U \cap (X \setminus cl_\gamma(N)) \neq \emptyset$. Thus by Proposition 2.9(ii)[7] $A = U \cap (X \setminus cl_\gamma(N))$ is a non empty $\gamma$-open set contained in $U$ and disjoint with $N$. Conversely, If for any given non empty $\gamma$-open set $U$, there exists a non empty $\gamma$-open set $A$ such that $A \subseteq U$ and $A \cap N = \emptyset$, then $N \subseteq X \setminus A$, implies that $cl_\gamma(N) \subseteq cl_\gamma(X \setminus A) = X \setminus A$. Therefore $U \setminus cl_\gamma(N) \supseteq U \setminus (X \setminus A) = A \neq \emptyset$. Thus $cl_\gamma(N)$ does not contain any non empty $\gamma$-open set. This implies that $int_\gamma(cl_\gamma(N)) = \emptyset$. Hence $N$ is a $\gamma^*$-nowhere dense set in $X$. □

**Theorem 2.7.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then every singleton set $\{x\}$ is either a $\gamma^*$-pre-open set or a $\gamma^*$-nowhere dense set.*

*Proof.* Suppose $\{x\}$ is not $\gamma^*$-pre-open. Then $int_\gamma(cl_\gamma(\{x\})) = \emptyset$. This implies that $\{x\}$ is a $\gamma^*$-nowhere dense set in $X$. □

**Definition 2.13.** Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $(X, \tau)$ is said to be $\gamma^*$-submaximal if every $\gamma^*$-dense subset of $X$ is $\gamma$-open.

**Theorem 2.8.** *Let $(X, \tau)$ be a topological space in which every $\gamma^*$-pre-open set is a $\gamma$-open set. Then $(X, \tau)$ is $\gamma^*$-submaximal.*

*Proof.* Let $A$ be a $\gamma^*$-dense subset of $(X, \tau)$. Then $A \subseteq int_\gamma(cl_\gamma(A))$. This implies that $A$ is a $\gamma^*$-pre-open set and hence it follows from the assumption that $A$ is $\gamma$-open. Therefore $(X, \tau)$ is $\gamma^*$-submaximal. □

**Definition 2.14.** Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. A subset $A$ of a space $(X, \tau)$ is called $\gamma^*$-pre-closed if and only if $X \setminus A$ is $\gamma^*$-pre-open, equivalently a subset $A$ of $X$ is $\gamma^*$-pre-closed if and only if

$cl_\gamma(int_\gamma(A)) \subseteq A$.

**Lemma 2.4.** *Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then*
*(i) $cl_\gamma(int_\gamma(A))$ is $\gamma^*$-pre-closed;*
*(ii) $int_\gamma(cl_\gamma(A))$ is $\gamma^*$-pre-open.*

*Proof.* (i) $cl_\gamma(int_\gamma(cl_\gamma(int_\gamma(A)))) \subseteq cl_\gamma(cl_\gamma(int_\gamma(A))) = cl_\gamma(int_\gamma(A))$. Hence $cl_\gamma(int_\gamma(A))$ is $\gamma^*$-pre-closed.

(ii) Follows from (i) and Lemma 2.2(ii). □

**Definition 2.15.** Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $\gamma^*$-pre-interior of $A$ is defined as union of all $\gamma^*$-pre-open sets contained in $A$.

Thus $pint_{\gamma^*}(A) = \cup\{U : U \in PO_{\gamma^*}(X) \text{ and } U \subseteq A\}$.

**Definition 2.16.** Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $\gamma^*$-pre-closure of $A$ is defined as intersection of all $\gamma^*$-pre-closed sets containing $A$.

Thus $pcl_{\gamma^*}(A) = \cap\{F : X \setminus F \in PO_{\gamma^*}(X) \text{ and } A \subseteq F\}$.

**Theorem 2.9.** *Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then*
*(i) $pint_{\gamma^*}(A)$ is a $\gamma^*$-pre-open set contained in $A$;*
*(ii) $pcl_{\gamma^*}(A)$ is a $\gamma^*$-pre-closed set containing $A$;*
*(iii) $A$ is $\gamma^*$-pre-closed if and only if $pcl_{\gamma^*}(A) = A$;*
*(iv) $A$ is $\gamma^*$-pre-open if and only if $pint_{\gamma^*}(A) = A$;*
*(v) $pint_{\gamma^*}(A) = X \setminus pcl_{\gamma^*}(X \setminus A)$;*
*(vi) $pcl_{\gamma^*}(A) = X \setminus pint_{\gamma^*}(X \setminus A)$.*

*Proof.* (i) Follows from Definition 2.15 and Theorem 2.1.
(ii) Follows from Definition 2.16 and Theorem 2.1.
(iii) and (iv) Follows from Definition 2.16, (ii) and Definition 2.15, (i) respectively.
(v) and (vi) Follows from Definitions 2.14, 2.15 and 2.16. □

**Theorem 2.10.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. If $A$ and $B$ are two subsets of $X$, then the following are hold:*
*(i) If $A \subseteq B$, then $pint_{\gamma^*}(A) \subseteq pint_{\gamma^*}(B)$;*
*(ii) $pint_{\gamma^*}(A \cup B) = pint_{\gamma^*}(A) \cup pint_{\gamma^*}(B)$;*
*(iii) $pint_{\gamma^*}(A \cap B) \subseteq pint_{\gamma^*}(A) \cap pint_{\gamma^*}(B)$.*

*Proof.* (i) Follows from Definition 2.15.
(ii) Follows from (i) and Theorem 2.1.
(iii) Follows from (i). □

**Theorem 2.11.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and*

*an open operation on $\tau$. If $A$ is a subset of $X$, then*
*(i) $pcl_{\gamma^*}(A) = A \cup cl_\gamma(int_\gamma(A))$;*
*(ii) $pint_{\gamma^*}(A) = A \cap int_\gamma(cl_\gamma(A))$.*

*Proof.* (i) $cl_\gamma(int_\gamma(A \cup cl_\gamma(int_\gamma(A)))) \subseteq cl_\gamma(int_\gamma(A) \cup cl_\gamma(int_\gamma(A)))$ (by Lemma 2.3(ii)) $= cl_\gamma(int_\gamma(A)) \cup cl_\gamma(cl_\gamma(int_\gamma(A)))$ (by Lemma 2.1(iii)) $= cl_\gamma(int_\gamma(A)) \cup cl_\gamma(int_\gamma(A)) = cl_\gamma(int_\gamma(A)) \subseteq A \cup cl_\gamma(int_\gamma(A))$. Hence $A \cup cl_\gamma(int_\gamma(A))$ is a $\gamma^*$-pre-closed set containing $A$. By Theorem 2.9(ii) $pcl_{\gamma^*}(A) \subseteq A \cup cl_\gamma(int_\gamma(A))$. Conversely, since $A \cup cl_\gamma(int_\gamma(A)) \subseteq A \cup cl_\gamma(int_\gamma(pcl_{\gamma^*}(A))) \subseteq pcl_{\gamma^*}(A)$ (by Theorem 2.9(ii)). Therefore $pcl_{\gamma^*}(A) = A \cup cl_\gamma(int_\gamma(A))$.
(ii) Follows from (i) and Theorem 2.9(i). □

**Corollary 2.1.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and an open operation on $\tau$. If $A$ is a subset of$X$, then*
*(i) $pint_{\gamma^*}(cl_\gamma(A)) = int_\gamma(cl_\gamma(A))$;*
*(ii) $pcl_{\gamma^*}(int_\gamma(A)) = cl_\gamma(int_\gamma(A))$;*
*(iii) $int_\gamma(pcl_{\gamma^*}(A)) = int_\gamma(cl_\gamma(int_\gamma(A)))$;*
*(iv) $cl_\gamma(pint_{\gamma^*}(A)) = cl_\gamma(int_\gamma(cl_\gamma(A)))$.*

*Proof.* (i) By Theorem 2.11(ii) it follows that $pint_{\gamma^*}(cl_\gamma(A)) = cl_\gamma(A) \cap int_\gamma(cl_\gamma(cl_\gamma(A))) = cl_\gamma(A) \cap int_\gamma(cl_\gamma(A)) = int_\gamma(cl_\gamma(A))$.
(ii) By Theorem 2.11(i) it follows that $pcl_{\gamma^*}(int_\gamma(A)) = int_\gamma(A) \cup cl_\gamma(int_\gamma(int_\gamma(A))) = int_\gamma(A) \cup cl_\gamma(int_\gamma(A)) = cl_\gamma(int_\gamma(A))$.
(iii) Follows from (i) and Theorem 2.11(i).
(iv) Follows from (ii) and Theorem 2.11(ii). □

**Theorem 2.12.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and an open operation on $\tau$. If $A$ is a subset of $X$, then $pcl_{\gamma^*}(pint_{\gamma^*}(A)) = pint_{\gamma^*}(A) \cup cl_\gamma(int_\gamma(A))$.*

*Proof.* Since $\tau_\gamma \subseteq PO_{\gamma^*}(X)$, we have that $int_\gamma(A) \subseteq pint_{\gamma^*}(A) \subseteq A$ and hence $int_\gamma(pint_{\gamma^*}(A)) = int_\gamma(A)$. By Theorem 2.11(i) $pcl_{\gamma^*}(pint_{\gamma^*}(A)) = pint_{\gamma^*}(A) \cup cl_\gamma(int_\gamma(pint_{\gamma^*}(A))) = pint_{\gamma^*}(A) \cup cl_\gamma(int_\gamma(A))$. □

## 3. $\gamma^*$-semi-pre-open Sets

In this section, we introduce the concept of $\gamma^*$-semi-pre-open sets and study some of their basic properties.

**Definition 3.1.** A subset $A$ of a topological space $(X, \tau)$ is $\gamma^*$-semi-pre-open if and only if there exists a $\gamma^*$-pre-open set $U$ in $X$ such that $U \subseteq A \subseteq cl_\gamma(U)$. The family of all $\gamma^*$-semi-pre-open sets in $(X, \tau)$ is denoted by $PO_{\gamma^*}(X)$.

**Remark 3.1.** If $A$ is a $\gamma^*$-pre-open set in $(X, \tau)$, then $A$ is $\gamma^*$-semi-pre-open. But the converse need not be true.

Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} cl(A) & \text{if } A = \{b\} \\ A \cup \{c\} & \text{if } A \neq \{b\} \end{cases} \quad \text{for every } A \in \tau.$$

Then $PO_{\gamma^*}(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $SPO_{\gamma^*}(X) = \{\emptyset, X, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Thus $\{a, b\}$ is a $\gamma^*$-semi-pre-open set in $(X, \tau)$ but not $\gamma^*$-pre-open in $(X, \tau)$.

**Theorem 3.1.** *Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$.*
*(i) If $A$ is a $\gamma^*$-semi-pre-open set in $(X, \tau)$, then $A \subseteq cl_\gamma(int_\gamma(cl_\gamma(A)))$.*
*(ii) If $\gamma$ is a regular, open operation on $\tau$ and $A \subseteq cl_\gamma(int_\gamma(cl_\gamma(A)))$, then $A$ is a $\gamma^*$-semi-pre-open set in $(X, \tau)$.*

*Proof.* (i) Given $A$ is a $\gamma^*$-semi-pre-open set, there exists a $\gamma^*$-pre-open set $U$ such that $U \subseteq A \subseteq cl_\gamma(U)$. Hence $A \subseteq cl_\gamma(U) \subseteq cl_\gamma(int_\gamma(cl_\gamma(U))) \subseteq cl_\gamma(int_\gamma(cl_\gamma(A)))$.
(ii) Let $U = A \cap int_\gamma(cl_\gamma(A))$. Then by Theorem 2.11(ii) $U = pint_{\gamma^*}(A)$ and therefore $U$ is $\gamma^*$-pre-open. This implies that $U \subseteq A \subseteq cl_\gamma(A) \subseteq cl_\gamma(cl_\gamma(int_\gamma(cl_\gamma(A)))) = cl_\gamma(int_\gamma(cl_\gamma(A))) = cl_\gamma(pint_{\gamma^*}(A))$ (by Corollary 2.1(iv)) $= cl_\gamma(U)$. Hence $A$ is a $\gamma^*$-semi-pre-open set. □

**Theorem 3.2.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. If $\{A_\alpha : \alpha \in J\}$ is a set of all $\gamma^*$-semi-pre-open sets in $(X, \tau)$, then $\bigcup_{\alpha \in J} A_\alpha$ is also a $\gamma^*$-semi-pre-open set.*

*Proof.* Since each $A_\alpha$ is $\gamma^*$-semi-pre-open set, implies that there exists a $\gamma^*$-pre-open set $U_\alpha$ such that $U_\alpha \subseteq A_\alpha \subseteq cl_\gamma(U_\alpha)$. Hence $\bigcup_{\alpha \in J} U_\alpha \subseteq \bigcup_{\alpha \in J} A_\alpha \subseteq \bigcup_{\alpha \in J} cl_\gamma(U_\alpha) \subseteq cl_\gamma(\bigcup_{\alpha \in J} U_\alpha)$. Then by Theorem 2.1 it follows that $\bigcup_{\alpha \in J} A_\alpha$ is a $\gamma^*$-semi-pre-open set. □

**Remark 3.2.** If $A$ and $B$ are two $\gamma^*$-semi-pre-open sets in a topological space $(X, \tau)$, then $A \cap B$ need not be a $\gamma^*$-semi-pre-open set.

*Proof.* From Remark 3.1 both $A = \{a, b\}$ and $B = \{a, c\}$ are $\gamma^*$-semi-pre-open sets but $A \cap B = \{a\}$ is not $\gamma^*$-semi-pre-open. □

**Theorem 3.3.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and an open operation on $\tau$. Let $V$ be a $\gamma$-open set and $A$ be a $\gamma^*$-semi-pre-open set. Then $V \cap A$ is also a $\gamma^*$-semi-pre-open set.*

*Proof.* By Theorem 3.1 and Lemma 2.3(i) it follows that $V \cap A \subseteq V \cap cl_\gamma(int_\gamma(cl_\gamma(A))) \subseteq cl_\gamma(V \cap int_\gamma(cl_\gamma(A))) = cl_\gamma(int_\gamma(V) \cap int_\gamma(cl_\gamma(A))) = cl_\gamma(int_\gamma(V \cap cl_\gamma(A))) \subseteq cl_\gamma(int_\gamma(cl_\gamma(V \cap A)))$. Therefore $V \cap A$ is a $\gamma^*$-semi-pre-open set. □

**Definition 3.2.** Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then $A$ is said to be a $\gamma^*$-semi-pre-closed set if and only if $X \setminus A$ is a $\gamma^*$-semi-pre-open set.

**Theorem 3.4.** *Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. If $\{B_\alpha : \alpha \in J\}$ is the set of all $\gamma^*$-semi-pre-closed sets in $(X, \tau)$,* *then* $\bigcap_{\alpha\in J} B_\alpha$ *is also a* $\gamma^*$*-semi-pre-closed set.*

*Proof.* Follows from Theorem 3.2. □

**Theorem 3.5.** *Let* $(X, \tau)$ *be a topological space,* $\gamma : \tau \to P(X)$ *be a regular and an open operation on* $\tau$*. Then*
*(i) any subset* $B$ *of* $X$ *is* $\gamma^*$*-semi-pre-closed if and only if* $int_\gamma(cl_\gamma(int_\gamma(B))) \subseteq B$*;*
*(ii) if* $F$ *is* $\gamma$*-closed and* $B$ *is* $\gamma^*$*-semi-pre-closed, then* $F \cup B$ *is also* $\gamma^*$*-semi-pre-closed.*

*Proof.* (i) Follows from Theorem 3.1(i).
(ii) Follows from Theorem 3.3. □

**Theorem 3.6.** *Let* $(X, \tau)$ *be a topological space,* $\gamma : \tau \to P(X)$ *be a regular and an open operation on* $\tau$*. If* $A$ *is a subset of* $X$*, then*
*(i)* $int_\gamma(cl_\gamma(int_\gamma(A)))$ *is a* $\gamma^*$*-semi-pre-closed set;*
*(ii)* $cl_\gamma(int_\gamma(cl_\gamma(A)))$ *is a* $\gamma^*$*-semi-pre-open set.*

*Proof.* (i) Follows from Lemma 2.4(i) and Theorem 3.5(i).
(ii) Follows from (i) and Theorem 3.1(ii). □

**Definition 3.3.** Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $\gamma^*$-semi-pre-closure of $A$ and $\gamma^*$-semi-pre-interior of $A$ are defined as
$spcl_{\gamma^*}(A) = \cap\{F : A \subseteq F \text{ and } X \setminus F \in SPO_{\gamma^*}(X)\}$ and
$spint_{\gamma^*}(A) = \cup\{U : U \subseteq A \text{ and } U \in SPO_{\gamma^*}(X)\}$ respectively.

**Remark 3.3.** Let $(X, \tau)$ be a topological space and $\gamma : \tau \to P(X)$ be an operation on $\tau$. If $A$ is a subset of $X$, then
(i) $spcl_{\gamma^*}(A)$ is a $\gamma^*$-semi-pre-closed set containing $A$;
(ii) $spint_{\gamma^*}(A)$ is a $\gamma^*$-semi-pre-open set contained in $A$.

*Proof.* (i) Follows from the Definition 3.3 and Theorem 3.4.
(ii) Follows from the Definition 3.3 and Theorem 3.2. □

**Theorem 3.7.** *Let* $(X, \tau)$ *be a topological space,* $\gamma : \tau \to P(X)$ *be a regular and an open operation on* $\tau$*. If* $A$ *is a subset of* $X$*, then*
*(i)* $spcl_{\gamma^*}(A) = A \cup int_\gamma(cl_\gamma(int_\gamma(A)))$*;*
*(ii)* $spint_{\gamma^*}(A) = A \cap cl_\gamma(int_\gamma(cl_\gamma(A)))$*.*

*Proof.* (i) We have that $int_\gamma[cl_\gamma(int_\gamma(A \cup int_\gamma(cl_\gamma(int_\gamma(A)))))] = int_\gamma[cl_\gamma(int_\gamma(A) \cup int_\gamma(cl_\gamma(int_\gamma(A))))] = int_\gamma[cl_\gamma(int_\gamma(A)) \cup cl_\gamma(int_\gamma(cl_\gamma(int_\gamma(A))))]$ (by Lemma 2.1(iii)) $\subseteq int_\gamma[cl_\gamma(int_\gamma(A)) \cup cl_\gamma(int_\gamma(A))]$ (by Lemma 2.4(i)) $\subseteq A \cup int_\gamma(cl_\gamma(int_\gamma(A)))$. Therefore from Theorem 3.5(i) $A \cup int_\gamma(cl_\gamma(int_\gamma(A)))$ is a $\gamma^*$-semi-pre-closed set and hence by Remark 3.3(i) $spcl_{\gamma^*}(A) \subseteq A \cup int_\gamma(cl_\gamma(int_\gamma(A)))$. Conversely, since $spcl_{\gamma^*}(A)$ is a $\gamma^*$-semi-pre-closed set, it follows from Theorem 3.5(i) that $A \cup int_\gamma(cl_\gamma(int_\gamma(A))) \subseteq A \cup int_\gamma(cl_\gamma(int_\gamma(spcl_{\gamma^*}(A)))) \subseteq A \cup spcl_{\gamma^*}(A) = spcl_{\gamma^*}(A)$. Hence $A \cup int_\gamma(cl_\gamma(int_\gamma(A))) \subseteq spcl_{\gamma^*}(A)$.
(ii) Follows from (i), Theorem 3.1(ii) and Remark 3.3(ii). □

## 4. Topology Generated by $\gamma^*$-pre-open Sets

Now, we study some properties of the topology generated by $\gamma^*$-pre-open sets.

**Definition 4.1.** Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $A$ is said to be
(i) $\tau_\gamma$-$p^*$-open if $A \cap B \in PO_{\gamma^*}(X)$ for every $B \in PO_{\gamma^*}(X)$. The set of all $\tau_\gamma$-$p^*$-open sets in a topological space $(X, \tau)$ is denoted by $\tau_{\gamma p^*}$;
(ii) $\tau_\gamma$-$p^*$-closed if and only if $X \setminus A \in \tau_{\gamma p^*}$.

**Remark 4.1.** $\tau_{\gamma p^*} \subseteq PO_{\gamma^*}(X)$, for any $\tau$ on $X$.

**Definition 4.2.** Let $(X, \tau)$ be a topological space, $A$ be a subset of $X$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then $\tau_\gamma$-$p^*$-interior of $A$ and $\tau_\gamma$-$p^*$-closure of $A$ are defined as
$\tau_\gamma$-$p^*$-$int(A) = \cup\{U : U \in \tau_{\gamma p^*}$ and $U \subseteq A\}$ and
$\tau_\gamma$-$p^*$-$cl(A) = \cap\{F : F \in X \setminus \tau_{\gamma p^*}$ and $A \subseteq F\}$ respectively.

**Theorem 4.1.** *Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then $A$ is $\tau_\gamma$-$p^*$-closed in $(X, \tau_{\gamma p^*})$ if and only if $A \cup B$ is $\gamma^*$-pre-closed for every $\gamma^*$-pre-closed set $B$ in $(X, \tau)$.*

*Proof.* Given $B$ is a $\gamma^*$-pre-closed set in $(X, \tau)$, implies that $(X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B) \in PO_{\gamma^*}(X)$ and hence $A \cup B$ is $\gamma^*$-pre-closed. Conversely, if $A \cup B$ is $\gamma^*$-pre-closed for every $\gamma^*$-pre-closed set $B$, then $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$ is $\gamma^*$-pre-open. This implies that $A$ is $\tau_\gamma$-$p^*$-closed in $(X, \tau_{\gamma p^*})$. $\square$

**Theorem 4.2.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and an open operation on $\tau$. If $A$ is a subset of $X$, then*
*(i) $\tau_\gamma$-$p^*$-$int(cl_\gamma(A)) = int_\gamma(cl_\gamma(A))$;*
*(ii) $\tau_\gamma$-$p^*$-$cl(int_\gamma(A)) = cl_\gamma(int_\gamma(A))$.*

*Proof.* (i) It follows from Definition of $\tau_{\gamma p^*}$ and Theorem 2.5 that $\tau_\gamma \subseteq \tau_{\gamma p^*}$, hence $int_\gamma(cl_\gamma(A)) \subseteq \tau_\gamma$-$p^*$-$int(cl_\gamma(A))$. Therefore by Remark 4.1 and Corollary 2.1(i), we have that $\tau_\gamma$-$p^*$-$int(cl_\gamma(A)) \subseteq pint_{\gamma^*}(cl_\gamma(A)) = int_\gamma(cl_\gamma(A))$ and hence we have that $\tau_\gamma$-$p^*$-$int(cl_\gamma(A)) = int_\gamma(cl_\gamma(A))$.
(ii) Proof follows from Remark 4.1 and Corollary 2.1(ii). $\square$

**Theorem 4.3.** *$\tau_{\gamma p^*}$ is a topology on $X$.*

*Proof.* It is obvious that $\emptyset \in \tau_{\gamma p^*}$ and $X \in \tau_{\gamma p^*}$. Let $\{A_\alpha : \alpha \in J\}$ be a collection of $\tau_\gamma$-$p^*$-open sets in $(X, \tau)$. Then $A_\alpha \cap B \in PO_{\gamma^*}(X)$ for all $B \in PO_{\gamma^*}(X)$ and every $\alpha \in J$. Hence $(\cup(A_\alpha)) \cap B \in PO_{\gamma^*}(X)$. This implies that $\cup(A_\alpha) \in \tau_{\gamma p^*}$. If $C, D \in \tau_{\gamma p^*}$, then $(C \cap D) \cap B = C \cap (D \cap B) \in PO_{\gamma^*}(X)$ for all $B \in PO_{\gamma^*}(X)$. This implies that $C \cap D \in \tau_{\gamma p^*}$. Hence $\tau_{\gamma p^*}$ is a topology on $X$. $\square$

## 5. Separation Axioms

In this section, we investigate general operator approaches on $T_i$ spaces, where

$i = 0, \frac{1}{2}, 1, 2$, an operation $\gamma : \tau \to P(X)$ on topology $\tau$. Also, we prove some properties.

**Definition 5.1.** A topological space $(X, \tau)$ is called a $\gamma^*$-pre-$T_0$ space if for each pair of distinct points $x, y \in X$, there exists a $\gamma^*$-pre-open set $U$ such that either $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.

**Definition 5.2.** A topological space $(X, \tau)$ is called a $\gamma^*$-pre-$T_1$ space if for each pair of distinct points $x, y \in X$, there exists a $\gamma^*$-pre-open sets $U$ and $V$ contain $x$ and $y$ respectively such that $y \notin U$ and $x \notin V$.

**Definition 5.3.** A topological space $(X, \tau)$ is called a $\gamma^*$-pre-$T_2$ space if for each pair of distinct points $x, y \in X$, there exists a $\gamma^*$-pre-open sets $U$ and $V$ such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

**Definition 5.4.** Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then $A$ is called a $\gamma^*$-pre-generalized closed (briefly $\gamma^*$-*pg*.closed) set if $pcl_{\gamma^*}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\gamma^*$-pre-open set in $(X, \tau)$.

**Remark 5.1.** From Definition 5.4, every $\gamma^*$-pre-closed set is $\gamma^*$-*pg*.closed set. But, the converse need not be true.

**Definition 5.5.** A topological space $(X, \tau)$ is called a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space if each $\gamma^*$-*pg*.closed set of $(X, \tau)$ is $\gamma^*$-pre-closed.

**Theorem 5.1.** *Let $(X, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be an operation on $\tau$. Then for a point $x \in X$, $x \in pcl_{\gamma^*}(A)$ if and only if $V \cap A \neq \emptyset$ for any $V \in PO_{\gamma^*}(X)$ such that $x \in V$.*

*Proof.* Let $F_0$ be the set of all $y \in X$ such that $V \cap A \neq \emptyset$ for any $V \in PO_{\gamma^*}(X)$ and $y \in V$. Now, we prove that $pcl_{\gamma^*}(A) = F_0$. Let us assume $x \in pcl_{\gamma^*}(A)$ and $x \notin F_0$. Then there exists a $\gamma^*$-pre-open set $U$ of $x$ such that $U \cap A = \emptyset$. This implies that $A \subseteq X \setminus U$. Therefore $pcl_{\gamma^*}(A) \subseteq X \setminus U$. Hence $x \notin pcl_{\gamma^*}(A)$. This is a contradiction. Hence $pcl_{\gamma^*}(A) \subseteq F_0$. Conversely, let F be a set such that $A \subseteq F$ and $X \setminus F \in PO_{\gamma^*}(X)$. Let $x \notin F$. Then we have that $x \in X \setminus F$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin F_0$. Therefore $F_0 \subseteq F$. Hence $F_0 \subseteq pcl_{\gamma^*}(A)$. □

**Theorem 5.2.** *Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then $A$ is $\gamma^*$-pg.closed if and only if $pcl_{\gamma^*}(\{x\}) \cap A \neq \emptyset$ holds for every $x \in pcl_{\gamma^*}(A)$.*

*Proof.* Let $U$ be any $\gamma^*$-pre-open set in $(X, \tau)$ such that $A \subseteq U$. Let $x \in pcl_{\gamma^*}(A)$. By assumption there exists a point $z \in pcl_{\gamma^*}(\{x\})$ and $z \in A \subseteq U$. Therefore from Theorem 5.1, we have that $U \cap \{x\} \neq \emptyset$. This implies that $x \in U$. Hence $A$ is a $\gamma^*$-*pg*.closed set in $X$. Conversely, suppose there exists a point $x \in pcl_{\gamma^*}(A)$ such that $pcl_{\gamma^*}(\{x\}) \cap A = \emptyset$. Since $pcl_{\gamma^*}(\{x\})$ is a $\gamma^*$-pre-closed set implies that $X \setminus pcl_{\gamma^*}(\{x\})$ is a $\gamma^*$-pre-open set. Since $A \subseteq X \setminus pcl_{\gamma^*}(\{x\})$ and $A$ is $\gamma^*$-*pg*.closed set, implies that $pcl_{\gamma^*}(A) \subseteq X \setminus pcl_{\gamma^*}(\{x\})$. Hence $x \notin pcl_{\gamma^*}(A)$. This is a contradiction. □

**Theorem 5.3.** *Let $(X, \tau)$ be a topological space and $A$ be the $\gamma^*$-pg.closed set in $(X, \tau)$. Then $pcl_{\gamma^*}(A) \setminus A$ does not contain a non empty $\gamma^*$-pre-closed set.*

*Proof.* Suppose there exists a non empty $\gamma^*$-pre-closed set $F$ such that $F \subseteq pcl_{\gamma^*}(A) \setminus A$. Let $x \in F$. Then $x \in pcl_{\gamma^*}(A)$, implies that $F \cap A = pcl_{\gamma^*}(A) \cap A \supseteq pcl_{\gamma^*}(\{x\}) \cap A \neq \emptyset$ and hence $F \cap A \neq \emptyset$. This is a contradiction. □

**Theorem 5.4.** *For each $x \in X$, $\{x\}$ is $\gamma^*$-pre-closed or $X \setminus \{x\}$ is $\gamma^*$-pg.closed.*

*Proof.* Suppose that $\{x\}$ is not $\gamma^*$-pre-closed. Then $X \setminus \{x\}$ is not $\gamma^*$-pre-open. This implies that $X$ is the only $\gamma^*$-pre-open set containing $X \setminus \{x\}$ and hence $X \setminus \{x\}$ is $\gamma^*$-*pg*.closed. □

**Theorem 5.5.** *A topological space $(X, \tau)$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space if and only if for each $x \in X$, $\{x\}$ is $\gamma^*$-pre-open or $\gamma^*$-pre-closed.*

*Proof.* Suppose that $\{x\}$ is not $\gamma^*$-pre-closed. Then it follows from the assumption and Theorem 5.4, $\{x\}$ is $\gamma^*$-pre-open. Conversely, Let $F$ be a $\gamma^*$-*pg*.closed set in $(X, \tau)$. Let $x \in pcl_{\gamma^*}(F)$. Then by the assumption $\{x\}$ is either $\gamma^*$-pre-open or $\gamma^*$-pre-closed.

Case(i): Suppose that $\{x\}$ is $\gamma^*$-pre-open. Then by Theorem 5.1, $\{x\} \cap F \neq \emptyset$. This implies that $pcl_{\gamma^*}(F) = F$. Therefore $(X, \tau)$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space.

Case(ii): Suppose that $\{x\}$ is $\gamma^*$-pre-closed. Let us assume $x \notin F$. Then $x \in pcl_{\gamma^*}(F) \setminus F$. This is a contradiction. Hence $x \in F$. Therefore $(X, \tau)$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space. □

**Theorem 5.6.** *A space $(X, \tau)$ is $\gamma^*$-pre-$T_1$ if and only if for any $x \in X$, $\{x\}$ is $\gamma^*$-pre-closed.*

*Proof.* Follows from Definitions 2.14 and 5.2. □

**Remark 5.2.** (i) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} A & \text{if } A = \{c\} \\ cl(A) & \text{if } A \neq \{c\} \end{cases} \quad \text{for every } A \in \tau.$$

Then $(X, \tau)$ is both $\gamma^*$-pre-$T_0$ and $\gamma^*$-pre-$T_{\frac{1}{2}}$ space but not $\gamma^*$-pre-$T_1$.
(ii) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} A \cup \{b\} & \text{if } A = \{a\} \\ A \cup \{d\} & \text{if } A = \{b\} \\ A & \text{if } A = \{a, b\} \\ int(cl(A)) & \text{if } A \neq \{a\}, \{b\} \text{ and } \{a, b\} \end{cases} \quad \text{for every } A \in \tau.$$

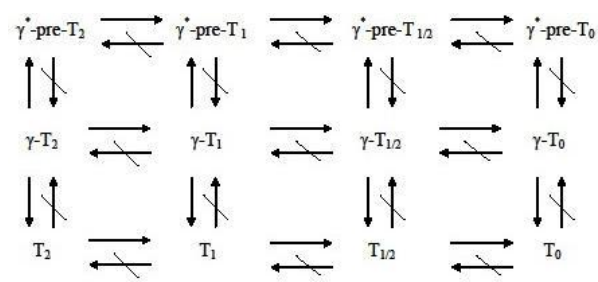
Then $(X, \tau)$ is a $\gamma^*$-pre-$T_1$ space but not $\gamma^*$-pre-$T_2$.

(iii) In Remark 2.5(i) $(X,\, \tau)$ is a $\gamma^*$-pre-$T_0$ space but not $\gamma$-$T_0$.
(iv) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\gamma : \tau \to P(X)$ be an operation on $\tau$ such that

$$\gamma(A) = \begin{cases} int(cl(A)) & \text{if } A = \{a\} \\ cl(A) & \text{if } A \neq \{a\} \end{cases} \quad \text{for every } A \in \tau.$$

Then $(X,\, \tau)$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space but not a $\gamma$-$T_{\frac{1}{2}}$.
(v) In Remark 2.5(ii) $(X,\, \tau)$ is a $\gamma^*$-pre-$T_1$ space but not $\gamma$-$T_1$.
(vi) In Example 2.1 $(X,\, \tau)$ is a $\gamma^*$-pre-$T_2$ space but not $\gamma$-$T_2$.

**Remark 5.3.** From Theorems 2.2, 5.4, 5.5, 5.6, Example 2.1, Remark 5.2 and Propositions 4.10, 4.11[7], we have that the following relationship diagram




where $A \to B$ represents $A$ implies $B$, $A \not\to B$ represents $A$ does not imply $B$ and $\gamma : \tau \to P(X)$ is a regular operation on $\tau$.

**Theorem 5.7.** *Let $(X,\, \tau)$ be a topological space, $\gamma : \tau \to P(X)$ be a regular and an open operation on $\tau$. Then the topological space $(X,\, \tau_{\gamma p^*})$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space.*

*Proof.* By Theorem 5.5, we prove $(X,\, \tau_{\gamma p^*})$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space. It is enough to prove that for every $x \in X$, $\{x\}$ is either $\gamma^*$-pre-open or $\gamma^*$-pre-closed in $(X,\, \tau_{\gamma p^*})$. Suppose $\{x\} \in \tau_{\gamma p^*}$, then by Remark 4.1 $\{x\}$ is $\gamma^*$-pre-open. Suppose $\{x\} \notin \tau_{\gamma p^*}$, then there exits a $\gamma^*$-pre-open set $A$ such that $\{x\} \cap A$ is not $\gamma^*$- pre-open. This implies that $\{x\}$ is not $\gamma^*$-pre-open and so $int_\gamma(cl_\gamma(\{x\})) = \emptyset$. This implies that $\{x\}$ is a $\gamma^*$-nowhere dense subset of $X$. This implies that $cl_\gamma(X \setminus cl_\gamma(\{x\})) = X$. Hence by Lemma 2.2(ii) $cl_\gamma(int_\gamma(X \setminus \{x\})) = X$. Since $int_\gamma(X \setminus \{x\}) \subseteq X \setminus \{x\}$, we have that $cl_\gamma(X \setminus \{x\}) \subseteq X$. Hence $X \setminus \{x\}$ is a $\gamma^*$-pre-open set in $(X,\, \tau)$. This implies that $\{x\}$ is $\gamma^*$-pre-closed. Hence $(X,\, \tau_{\gamma p^*})$ is a $\gamma^*$-pre-$T_{\frac{1}{2}}$ space. $\square$

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