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The Structure of Maximal Ideal Space of Certain Banach Algebras of Vector-valued Functions

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Abstract. Let X be a compact metric space, B be a unital commutative Banach algebra and $\alpha \in (0,1]$. In this paper, we first define the vector-valued (B-valued) α-Lipschitz operator algebra $\mathrm{Lip}_\alpha(X,B)$ and then study its structure and characterize of its maximal ideal space.

1. Introduction

Let (X,d) be a compact metric space with at least two elements and $(B,\|\,.\,\|)$ be a Banach space over the scaler field F $(= R$ or $C)$. For a constant $0 < \alpha \leq 1$ and an operator $f : X \to B$, set

$$p_\alpha(f) := \sup_{s \neq t} \frac{\| f(t) - f(s) \|}{d^\alpha(s,t)}; \quad (s,t \in X),$$

which is called the Lipschitz constant of f. Define

$$\mathtt{Lip}_\alpha(X,B) := \{f : X \to B : \quad p_\alpha(f) < \infty\},$$

and for $0 < \alpha < 1$
$\mathtt{lip}_\alpha(X,B) := \{f : X \to B : \quad \frac{\|f(t)-f(s)\|}{d^\alpha(s,t)} \to 0 \quad as \quad d(s,t) \to 0,\ \ s,t \in X,\ \ s \neq t\}.$
The elements of $\mathrm{Lip}_\alpha(X,B)$ and $\mathrm{lip}_\alpha(X,B)$ are called big and little α-Lipschitz operators, respectively [1]. Let $C(X,B)$ be the set of all continuous operators from

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X into B and for each $f \in C(X,B)$, define

$$\| f \|_\infty := \sup_{x\in X} \| f(x) \| .$$

For f, g in $C(X,B)$ and λ in F, define

$$(f+g)(x) := f(x)+g(x), \quad (\lambda f)(x) := \lambda f(x), \quad (x \in X).$$

It is easy to see that $(C(X,B), \| \,.\, \|_\infty)$ becomes a Banach space over F and $\mathrm{Lip}_\alpha(X,B)$ is a linear subspace of $C(X,B)$. For each element f of $\mathrm{Lip}_\alpha(X,B)$, define

$$\| f \|_\alpha := \| f \|_\infty + p_\alpha(f).$$

When $(B, \| \,.\, \|)$ is a Banach space, Cao, Zhang and Xu [6] proved that $\big(\mathrm{Lip}_\alpha(X,B), \| \,.\, \|_\alpha \big)$ is a Banach space over F and $\mathrm{lip}_\alpha(X,B)$ is a closed linear subspace of $\big(\mathrm{Lip}_\alpha(X,B), \| \,.\, \|_\alpha \big)$, and when $(B, \| \,.\, \|)$ is a unital commutative Banach algebra, A. Ebadian and A.A. Shokri [1] proved that $(\mathrm{Lip}_\alpha(X,B), \| \,.\, \|_\alpha)$ is a Banach algebra over F under pointwise multiplication and $\mathrm{lip}_\alpha(X,B)$ is a closed linear subalgebra of $(\mathrm{Lip}_\alpha(X,B), \| \,.\, \|_\alpha)$. Furthermore, Sherbert [4,5], Weaver [7,8], Honary and Mahyar [9], Johnson [3], Cao, Zhang and Xu [6], Ebadian [2], Bade, Curtis and Dales [11] and etc studied some properties of Lipschitz algebras.

Finally, in this paper, we will study the maxima ideal space of $\mathrm{Lip}_\alpha(X,B)$.

2. Maximal Ideal Space of $\mathrm{Lip}_\alpha(X,B)$

In this section, let us use (X,d) to denote a compact metric space in C which has at least two elements, $(B, \| \,.\, \|)$ to denote a unital bounded commutative Banach algebra with unit $\mathbf{e}$ over the scalar field $F(= R$ or $C)$, $\mathrm{Lip}_\alpha(X) = \mathrm{Lip}_\alpha(X,C)$ and $0<\alpha<1$. Let E_1 and E_2 be Banach spaces with dual spaces E_1^* and E_2^*. Then we define for $X \in E_1 \otimes E_2$

$$\| X \|_\varepsilon = \sup \big\{ \mid \langle X,\ \phi_1 \otimes \phi_2 \rangle \mid :\ \phi_j \in B_1[0, E_j^*] \ for\ j=1,2 \big\},$$

where

$$X = \sum_{k=1}^{m} x_1^{(k)} \otimes x_2^{(k)},\ (m \in N,\ x_1^{(k)} \in E_1,\ x_2^{(k)} \in E_2\ ,\ 1 \le k \le m),$$

and

$$\langle X, \phi_1 \otimes \phi_2 \rangle = \langle \sum_{k=1}^{m} x_1^{(k)} \ \otimes\ x_2^{(k)}, \phi_1 \otimes \phi_2 \rangle = \sum_{k=1}^{m} \phi_1(x_1^{(k)}) \phi_2(x_2^{(k)}),$$

and $B_1[0, E_j^*]$ is called ball in E_j^* with radius 1 centered at 0 for $j=1,2$. We call $\| \,.\, \|_\varepsilon$ the injective norm on $E_1 \otimes E_2$ [6]. The injective tensor product $E_1 \check{\otimes} E_2$ is the completion of $E_1 \otimes E_2$ with respect to $\| \,.\, \|_\varepsilon$ [10].

Theorem 2.1. $\big(\mathrm{Lip}_\alpha(X,B), \|\ .\ \|_\alpha\ \big)$ is isometrically isomorphic to $\big(\mathrm{Lip}_\alpha(X)\check{\otimes}B,$ $\|\ .\ \|_\varepsilon\ \big)$.

Proof. See [1]. □

Lemma 2.2. Let $\alpha \in (0,1)$, $f \in\mathrm{Lip}_\alpha(X,B)$ and

$$\varphi(x) :=\| f(x) \|^{1/2}, \quad (x \in X).$$

Then $\varphi \in\mathrm{Lip}_\alpha(X)$.

Proof. Firstly, we show that $\varphi \in C(X)$. For this purpose, suppose that $x \in X$ and $\{x_n\} \subset X$ is a sequence such that $x_n \longrightarrow x$ (in X). Let $f \in\mathrm{Lip}_\alpha(X,B)$. Then $f \in C(X,B)$, and so $f(x_n) \longrightarrow f(x)$ (with $\|\ .\ \|$). Thus for every $\varepsilon > 0$, there is $N \in N$ such that for every $n \geq N$,

$$\| f(x_n) - f(x) \| < 2 \| f(x) \|^{1/2}\, \varepsilon.$$

Now for every $n \geq N$ we have

$$\begin{aligned} | \varphi(x_n) - \varphi(x) | &= |\| f(x_n) \|^{1/2} - \| f(x) \|^{1/2}| \\ &= \left| \frac{\| f(x_n) \| - \| f(x) \|}{\| f(x_n) \|^{1/2} + \| f(x) \|^{1/2}} \right| \\ &\leq \frac{\| f(x_n) - f(x) \|}{\| f(x_n) \|^{1/2} + \| f(x) \|^{1/2}} \\ &\leq \frac{2 \| f(x) \|^{1/2}\, \varepsilon}{2 \| f(x) \|^{1/2}} \\ &= \varepsilon, \qquad \big(f(x) \neq 0\big). \end{aligned}$$

Also this holds for $f(x) = 0$.

This implies that $\varphi(x_n) \longrightarrow \varphi(x)$, so $\varphi \in C(X)$. Now, we show that $p_\alpha(\varphi) < \infty$. For every $x, y \in X$ such that $x \neq y$, we have

$$p_\alpha(\varphi) = \sup_{x\neq y} \frac{| \varphi(x) - \varphi(y) |}{d^\alpha(x,y)}\ .$$

Since $f \in\mathrm{Lip}_\alpha(X,B)$, $p_\alpha(f) < \infty$. So

$$\sup_{x\neq y} \frac{\| f(x) - f(y) \|}{d^\alpha(x,y)} < \infty\ ,$$

and then

$$\sup_{x\neq y} \frac{|\, \| f(x) \| - \| f(y) \|\, |}{d^\alpha(x,y)} < \infty\ .$$

So

$$\sup_{x\neq y} \frac{\Big|\Big(\| f(x) \|^{1/2} + \| f(y) \|^{1/2} \Big)\Big(\| f(x) \|^{1/2} - \| f(y) \|^{1/2} \Big)\Big|}{d^\alpha(x,y)} < \infty\ ,$$

Since B is bounded, $\|f\| < \infty$, $(x \in X)$. Thus

$$\sup_{x\neq y} \frac{|\| f(x) \|^{1/2} - \| f(y) \|^{1/2}|}{d^{\alpha}(x,y)} < \infty,$$

and so

$$\sup_{x\neq y} \frac{| \varphi(x) - \varphi(y) |}{d^{\alpha}(x,y)} < \infty.$$

Therefore $p_{\alpha}(\varphi) < \infty$. Hence $\varphi \in \text{Lip}_{\alpha}(X)$. □

Remark 2.3. Note that, in lemma 2.2., we suppose that $0 < \alpha < 1$. Because for $\alpha = 1$, the function $f(x) = x^{1/2}$ on [0,1] is not Lipschitz, where $B = C$ and $d(x,y) = |x - y|$, $(x, y \in X)$.

Lemma 2.4. Let $f \in \text{Lip}_{\alpha}(X,B)$ and

$$g(x) := \begin{cases} \| f(x) \|^{-\frac{1}{2}} f(x), & f(x) \neq 0\ ; \\ 0, & f(x) = 0\ . \end{cases} \qquad (x \in X).$$

Then $g \in \text{Lip}_{\alpha}(X,B)$.

Proof. Case 1: $f(x) \neq 0$, $(x \in X)$. Let

$$\varphi(x) := \| f(x) \|^{1/2}, \quad (x \in X).$$

Then by Lemma 2.2, $\varphi \in \text{Lip}_{\alpha}(X)$. Let $x \in X$ and $\{x_n\} \subset X$ be a sequence such that $x_n \longrightarrow x$ in X. Since $f \in C(X,B)$, $f(x_n) \longrightarrow f(x)$ with $\| \,.\, \|$. So

$$\| f(x_n) \|^{-1/2} \longrightarrow \| f(x) \|^{-1/2}\ .$$

For every $\varepsilon > 0$, we have

$$\begin{aligned} \| g(x_n) - g(x) \| &= \left\| \|f(x_n)\|^{-1/2} f(x_n) - \|f(x)\|^{-1/2} f(x) \right\| \\ &\leq \|f(x_n)\|^{-1/2} \|f(x_n) - f(x)\| \\ &+ \|f(x)\| \left| \|f(x_n)\|^{-1/2} - \|f(x)\|^{-1/2} \right| \\ &< \varepsilon. \end{aligned}$$

So $g \in C(X,B)$. Now we have

$$f(x) = \| f(x) \|^{1/2} g(x), \quad (x \in X).$$

Since $f \in \text{Lip}_{\alpha}(X,B)$, $\| f \|_{\alpha} < \infty$. So $\| f \|_{\infty} < \infty$. Then $\| g \|_{\infty} < \infty$. Also $p_{\alpha}(f) < \infty$, thus

$$\sup_{x\neq y} \frac{\| f(x) - f(y) \|}{d^{\alpha}(x,y)} < \infty,$$

$$\sup_{x\neq y} \frac{\|\| f(x) \|^{1/2} g(x) - \| f(y) \|^{1/2} g(y) \|}{d^{\alpha}(x,y)} < \infty.$$

Then

$$\sup_{x\neq y}\frac{\|\| f(x)\|^{1/2} g(x)-\| f(x)\|^{1/2} g(y)+\| f(x)\|^{1/2} g(y)-\| f(y)\|^{1/2} g(y)\ \|}{d^{\alpha}(x,y)}<\infty.$$

So

$$\sup_{x\neq y}\frac{\| \varphi(x)\Big(g(x)-g(y)\Big)+g(y)\big(\varphi(x)-\varphi(y)\big)\|}{d^{\alpha}(x,y)}<\infty,$$

$$\Big(\sup_{x\neq y}\varphi(x)\times\frac{\| g(x)-g(y)\|}{d^{\alpha}(x,y)}\Big)-\Big(\sup_{x\neq y}\| g(y)\|\times\frac{\| \varphi(x)-\varphi(y)\|}{d^{\alpha}(x,y)}\Big)<\infty.$$

Hence

$$\| \varphi\|_{\infty}\, p_{\alpha}(g)-\| g\|_{\infty}\, p_{\alpha}(\varphi)<\infty.$$

Since $\|g\|_\infty<\infty$, $\|\varphi\|_\infty<\infty$ and $p_\alpha(\varphi)<\infty$, $p_\alpha(g)<\infty$. So $g\in\text{Lip}_\alpha(X,B)$.

Case 2: $f(x)=0$, $(x\in X)$. Firstly, we show that g is continuous. Let $x\in X$ with $f(x)=0$ be fixed. Let $\varepsilon>0$ and $n\in N$ with $\frac{2}{n}<\varepsilon$. Then V defined by

$$V:=\{t\in X:\ \ \| f(t)\|<\frac{1}{n^2}\},$$

is a neighborhood of x satisfying $\| g(t)\|<\infty$ for each $t\in V$. Indeed, $f(t)=0$ implies that

$$\| g(t)\|=\| 0\|=0<\varepsilon.$$

If $t\in V$ satisfies $f(t)\neq 0$, then there is $k\geq n$ with

$$\frac{1}{(k+1)^2}<\| f(t)\|\leq\frac{1}{k^2}.$$

Since $\frac{1}{(k+1)^2}<\| f(t)\|$, $\frac{1}{k+1}<\| f(t)\|^{1/2}$. So

$$\frac{1}{k+1}\| f(t)\|^{-1/2}<1.$$

Thus we get

$$\begin{aligned}\| g(t)\| &= \|\| f(t)\|^{-1/2} f(t)\|\\ &= \|\frac{1}{k+1}\| f(t)\|^{-1/2}(k+1)f(t)\|\\ &< (k+1)\| f(t)\|\\ &\leq \frac{k+1}{k^2}\leq\frac{2k}{k^2}=\frac{2}{k}\leq\frac{2}{n}<\varepsilon.\end{aligned}$$

Which proves the continuity of g. Now for every $x, y \in X$, $x \neq y$ we have

$$\begin{aligned} p_\alpha(g) &= \sup_{x\neq y} \frac{\| g(x) - g(y) \|}{d^\alpha(x,y)} \\ &= \sup_{x\neq y} \frac{\| 0 - 0 \|}{d^\alpha(x,y)} = 0 < \varepsilon, \end{aligned}$$

so $g \in \text{Lip}_\alpha(X, B)$. □

Let A be a commutative Banach algebra with identity. An ideal J of A is maximal if $J \neq A$, while J is contained in no other proper ideal of A. The set of maximal ideals of A is called the maximal ideal space of A.

Theorem 2.5. Every character χ on $\text{Lip}_\alpha(X, B)$ is of form $\chi = \psi o \delta_z$ for some character ψ on B and some $z \in X$.

Proof. Let

$$j : \texttt{Lip}_\alpha(X) \to \texttt{Lip}_\alpha(X, B)$$
$$h \mapsto h \otimes \mathbf{e} \ ,$$

be the canonical embedding. Since $\big(\text{Lip}_\alpha(X, B), \| \ . \ \|_\alpha \ \big)$ is isometrically isomorphic to $\big(\text{Lip}_\alpha(X)\check{\otimes}B, \| \ . \ \|_\varepsilon \ \big)$ by theorem 2.3., j is a well define map. Then there is $z \in X$ such that $\chi o j$ is the evaluation in z. Consider the ideal

$$I := \big\{ f \in \texttt{Lip}_\alpha(X, B) : f(z) = 0 \big\}.$$

We will show that I is contained in the kernel of χ. Given $f \in I$ we define

$$\varphi(x) :=\| f(x) \|^{1/2} \ (x \in X).$$

By Lemma 2.4., $\varphi \in \text{Lip}_\alpha(X)$ and has the same zeros as f. The function $g : X \to B$ defined by

$$g(x) := \begin{cases} \| f(x) \|^{-1/2} f(x), & \text{if } f(x) \neq 0, \\ 0, & \text{if } f(x) = 0 \end{cases} ,$$

is in $\text{Lip}_\alpha(X, B)$, by Lemma 2.5. Now for every $x \in X$ with $f(x) \neq 0$ we have

$$\begin{aligned} f(x) &= \| f(x) \|^{1/2} g(x) = \varphi(x)g(x) \\ &= \varphi(x) \ \mathbf{e} \ g(x) = (\varphi \otimes \mathbf{e})(x)g(x) \\ &= \Big((\varphi \otimes \mathbf{e})g\Big)(x) = \Big(j(\varphi)g\Big)(x). \end{aligned}$$

So $f = j(\varphi)g$. Since φ has the same zeros as f, we conclude

$$\chi(f) = \chi\big(j(\varphi)g\big) = \big(\chi o j\big)(\varphi)\chi(g) = \delta_z(\varphi)\chi(g) = \varphi(z)\chi(g) = 0.$$

The evaluation δ_z is an epimorphism and since $\ker \delta_z = I \subset \ker \chi$, we obtain the desired factorization $\chi = \psi o \delta_z$ for some character ψ on B. □

Example 2.6. For $0 < \alpha \leq 1$, $X = [0,1]$ and $B = C$, the maximal ideal space of $\mathrm{Lip}_\alpha([0,1])$ is [0,1].

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