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Poset Properties Determined by the Ideal - Based Zero-divisor Graph

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Abstract. In this paper, we study some properties of finite or infinite poset P determined by properties of the ideal based zero-divisor graph properties $G_J(P)$, for an ideal J of P.

1. Introduction

Throughout this paper, $(P, \leq)$ denotes a poset and the graph G denotes the ideal based zero-divisor graph of a poset P with respect to ideal I of P. For $M \subseteq P$, let $L(M) := \{x \in P : x \leq m$ for all $m \in M\}$ denotes the lower cone of M in P, and dually let $U(M) := \{x \in P : m \leq x$ for all $m \in M\}$ be the upper cone of M in P. For $A, B \subseteq P$ we shall write $L(A, B)$ instead of $L(A \cup B)$ and dually for upper cones. If $M = \{x_1, ..., x_n\}$ is finite, then we use the notation $L(x_1, ..., x_n)$ instead of $L(\{x_1, ..., x_n\})$ (and dually). By an ideal we mean a non-empty subset $I \subseteq P$ such that if $b \in I$ and $a \leq b$, then $a \in I$. A proper order-ideal I of P is called prime if for any $a, b \in P$, $L(a, b) \subseteq I$ implies $a \in I$ or $b \in I$. In [2], Beck introduced the concept of a zero-divisor graph of a commutative ring with identity, but this work was mostly concerned with coloring of rings. Later D. F. Anderson and Livingston in [1] studied the subgraph $\Gamma(R)$ of $G(R)$ whose vertices are the nonzero zero-divisors of R. In [10], Redmond has generalized the notion of the zero-divisor graph. For a given ideal I of a commutative ring R, he defined an undirected graph $\Gamma_I(R)$ with vertices $\{x \in R \backslash I : xy \in I \; for \; some \; y \in R \backslash I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. The zero-divisor graph of various algebraic structures has been studied by several authors [[4],[5],[7] and[11]].

In [8], Radomr Halas and Marek Jukl have introduced the concept of a graph structure of a posets, let $(P, \leq)$ be a poset with 0. Then the zero-divisor graph of P, denoted by $\Gamma(P)$, is an undirected graph whose vertices are just the elements of P with two distinct vertices x and y are joined by an edge if and only if $L(x, y) = \{0\}$,

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and proved some interesting results related with clique and chromatic number of this graph structure. In [6], we generalized the notion of zero-divisor graph of P. Let P be a poset and J be an ideal of P. Then the graph of P with respect to the ideal J, denoted by $G_J(P)$, is the graph whose vertices are the set $\{x \in P\backslash J \ : \ L(x,y) \subseteq J$ for some $y \in P\backslash J\}$ with distinct vertices x and y are adjacent if and only if $L(x,y) \subseteq J$. If $J = \{0\}$, then $G_J(P) = G(P)$, and J is a prime ideal of P if and only if $G_J(P) = \phi$. And investigated the interplay between the poset properties of P and the graph-theoretics properties of $G_J(P)$. Following [9], let I be an ideal of P. Then the extension of I by $x \in P$ is meant the set $< x,\ I >= \{a \in P \ : \ L(a,x) \subseteq I\}$. For any $s \in V(G)$, $N(s)$ denotes the set of all vertices adjacent to s and $K(G)$ denotes the core of G. In this paper the notations of graph theory are from [3], the notations of posets from [8].

2. Poset Properties Related to a Single Vertex

Theorem 2.1. Let G be the graph of a poset P. If there exist $s,t \in V(G)$ such that $N(s) \neq \phi,\ N(t) \neq \phi$, then $L(x,y) \subseteq (N(s) \cap N(t)) \cup I$ for $x \in N(s), y \in N(t)$. In addition, if x is an end vertex, then $I \cup \{s\}$ is an ideal of P.

Proof. Let $t_1 \in L(x,y)\backslash I$ for $x \in N(s)$ and $y \in N(t)$. Then $L(x,s) \subseteq I$ and $L(y,t) \subseteq I$. If $t_1 \in \{x,y\}$, then it is easy to see that $t_1 \in N(s) \cap N(t)$. If $t_1 = s$, then $s \in L(x,s) \subseteq I$, a contradiction. So $t_1 \neq s$. Similar way, we can get $t_1 \neq t$. Now, $L(t_1,s) \subseteq L(x,s) \subseteq I$ and $L(t_1,t) \subseteq L(y,t) \subseteq I$ which imply $t_1 \in N(s) \cap N(t)$. If $x \in N(s)$ is an end vertex of G, then $< x, I >= I \cup \{s\}$ is an ideal of P. □

Corollary 2.2. Let G be a graph of a poset P and $y - s - t - x$ be a path in G. Then

(i) $K(G)$ is non - empty and it contains atleast $|L(x,y)\backslash I|$ triangles.

(ii) If x and y are end vertices, then P has at least two ideals of the form $I \cup \{s\}$.

Theorem 2.3. Let P be a poset with corresponding graph G such that $P = V(G) \cup \{I\}$. For an element $x \in P\backslash I$, assume that $V(G) = C_x \cup \{x\} \cup T(x)$ is a disjoint union of three subsets satisfying the following conditions:

(i) $T(x)$ contains all end vertices adjacent to x.

(ii) There is no edge linking a vertex in $T(x)$ with a vertex in C_x, whenever $T(x) \neq \phi$ and $C_x \neq \phi$.

(iii) Either $C_x \neq \phi$ or $|V(G)| \geq 3$ and x is adjacent to at least one end vertex. Then $L(a,b) \subseteq C_x \cup \{x\} \cup I$ for all $a,b \in C_x \cup \{x\} \cup I$.

Proof. Let us assume that $T(x) \neq \phi$ and let $a,b \in C_x \cup \{x\} \cup I$. If $C_x = \phi$, by assumption (iii), there exists an end vertex y adjacent with x which gives $I \cup \{x\}$ is an ideal of P. So $L(a,b) \subseteq I \cup \{x\}$. If $C_x \neq \phi$, then there is at least one element $z \in C_x$ such that $z - x$. Suppose $L(x) \cap T(x) \neq \phi$. Then there exists $y \in L(x) \cap T(x)$ with such that $L(z,y) \subseteq I$, contradicting condition (ii). So $L(x) \cap T(x) = \phi$, i.e., $L(x) \subseteq C_x \cup \{x\} \cup I$. It remains to consider the case $a,b \in C_x\backslash\{x\} \cup I$. Assume to the contrary that there is an element $t \in L(a,b)$ such that $t \notin C_x \cup \{x\} \cup I$. If a

is not adjacent to x, then there exists $c \in C_x$ such that $L(a, c) \subseteq I$ which implies there is an edge $c - t$, where $c \in C_x, t \in T(x)$, contradicting condition (ii). If a is adjacent to x, then by condition (i), a is not an end vertex, then by condition (ii), there is an element $c(\neq a) \in C_x$ such that $a - c$. In this case also we have an edge $c - t$, contradicting condition (ii). So $L(a, b) \subseteq C_x \cup \{x\} \cup I$. □

For any vertex $x \in V(G)$, T_x denotes the set of all end vertices adjacent to x in G.

Corollary 2.4. Let P be a poset with corresponding graph G such that $V(G) = P \backslash I$. If G is not a star graph, then for any $x \in V(G)$, we have $L(a, b) \subseteq P \backslash T_x$ for all $a, b \in P \backslash T_x$.

Proof. In Theorem 2.3, let $T(x) = T_x$. If G is not a star graph, then $C_x \neq \phi$ and $P \backslash T(x) = C_x \cup \{x\} \cup I$. The result then follows from Theorem 2.3. □

Theorem 2.5. Let G be the graph of a poset P and assume that G has a cycle. For any vertex x in G that is not an end vertex. If any two vertices in $L(u)$ are comparable ((i.e) $a \leq b$, for $a, b \in L(u)$), then $L(u, v) \subseteq T_x \cup I$ for all $u, v \in T_x \cup I$.

Proof. Suppose $L(u, v) \nsubseteq I$ for some $u, v \in T_x$. Then there exists $c \in L(u, v) \backslash I$ such that $c \neq x$. If c is not an end vertex of G, by Theorem 3.4 of [6], it is in the core of G. Then there exists a vertex d in the core such that $d \notin \{x, c\}$ and $x - c - d$. Since $L(d, u) \nsubseteq I$, there exists $e \in L(d, u) \backslash I$ such that $e \in I$ or $c \in I$ as $L(e, c) \subseteq I$, a contradiction. So c is an end vertex of G. □

Note that if we consider $x = \{a\}$ and $u = \{b, c\}$ in Example 2.8, then {b} and {c} are not comparable, but $L(\{b, c\}) \nsubseteq T_x \cup I$. Therefore, Theorem 2.5 is not valid in general. Hence, the condition comparable on the set $L(u)$ is not superficial in Theorem 2.5.

Theorem 2.6. Let G be a graph of a poset P. If G does not contain an infinite clique, then P satisfies the a.c.c on $< x, I >$ for $x \in P$.

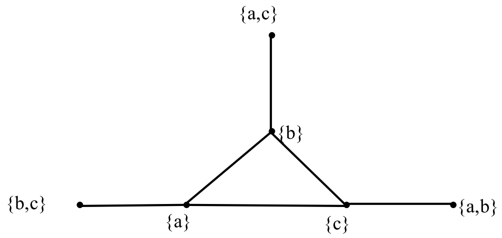
Proof. Suppose that $< x_1, I > \subseteq < x_2, I > \subseteq \ldots < x_i, I > \subseteq \ldots$ be an increasing chain of ideals, for $x_i \in P$. If some $x_i \in I$, then the proof is trivial. So assume that $x_i \in P \backslash I$ for all i. For each $i \geq 2$, let $a_i \in < x_i, I > \backslash < x_{i-1}, I >$. Then $L(x_{n-1}, a_n) \nsubseteq I$ for $n = 2, 3, \ldots$. So there exists $y_n \in L(x_{n-1}, a_n) \backslash I$ such that $L(y_i, y_j) \subseteq I$ for any $i \neq j$. i.e., we have an infinite clique in G, a contradiction. So P satisfies the a.c.c on $< x, I >$ for $x \in P$. □

Example 2.7. Let G be a graph of a poset P. For any $x, y \in V(G)$ with $U(x, y) \cap V(G) \neq \phi$, then the edge $x - y$ is contained in a triangle.

Proof. Let $x, y \in V(G)$ with $x - y$ and $U(x, y) \cap V(G) \neq \phi$. Then there exists $t \in U(x, y) \cap V(G)$ such that $t \notin \{x, y\}$. Since $diam(G) \leq 3$, we have either $x - a - t$ or $x - a - b - t$ for some $a, b \in V(G)$. If $x - a - t$, then $x - a - y - x$. If $x - a - b - t$, then $x - b - t$ which implies $x - b - y - x$. □

We now show by an example that the Theorem 2.7 will fail if $U(x, y) \cap V(G) = \phi$ for any edge $x - y$ in G.

Theorem 2.8. Let $P(X)$ be the power set of a set $X = \{a, b, c\}$. Then $P(X)$ is a poset under the set inclusion $\subseteq$. If $I = \{\phi\}$, then the graph G is:



Here $U(\{a, b\},\ \{c\}) \cap V(G) = \phi$ but the edge $\{a, b\} - \{c\}$ is not contained in a triangle.

The distance $d(v)$ of a vertex v in a connected finite graph G is the sum of the distances v to each vertex of G. The median $M(G)$ of a graph G is the subgraph induced by the set of vertices having minimum distance. Let G be a connected graph, and $T \subseteq V(G)$. We say T is a cut vertex set if $G\backslash T$ is disconnected. Also the cut vertex set T is called a minimal cut vertex set for G if no proper subset of T is a cut vertex set. In addition, if $T = \{x\}$, then x is called a cut vertex. □

Theorem 2.9. Let G be a graph of a poset P. Then $V(M(G)) \cup I$ is an ideal of P. In addition, if T is a minimal cut vertex set of G, then $T \cup I$ is an ideal of P.

Proof. Let $x \in V(M(G))\backslash I$ and $y \in P$ with $y \leq x$. Suppose $y \notin I$. Then $y \in V(G)$ and $d(y, z) \leq d(x, z)$ for any $z \in V(G)$ which implies $d(y) = \sum_{z\in V(G)} d(y, z) \leq \sum_{z\in V(G)} d(x, z) = d(x)$. Since $x \in V(M(G))$, we have $d(y) = d(x)$, and hence $y \in V(M(G))$. Let T be a minimal cut vertex set of G and $x \in T, p \in P$ such that $p \leq x$. Then there exist two vertices z, y of the graph G such that $y - x - z$ is a path in G and y, z belong to two distinct connected components of $G\backslash T$ as $T\backslash\{x\}$ is not a cut vertex. Suppose $p \notin T \cup I$. Then there exists a path $y - p - z$ in $G\backslash T$, a contradiction. □

Corollary 2.10. Let G be a graph of a poset P. If x is cut-vertex of G, then $I \cup \{x\}$ is an ideal of P. For $y \in G\backslash\{x\}$, x is adjacent to y or $x \leq y$.

Corollary 2.11. Let G be a graph of G, and let $x - y$ be a bridge e of G such that G_1 and G_2 are the two connected components of $G\backslash\{e\}$. Then the following conclusions hold:

(i) If G_1 and G_2 have at least two vertices, then $I \cup \{x\}$ and $I \cup \{y\}$ are ideals of P. Also, if G_1 or G_2 has only one vertex, then $I \cup \{x\}$ or $I \cup \{y\}$ is an ideal of P.

(ii) If G_1 and G_2 have exactly one vertex, then $I \cup \{x\}$ and $I \cup \{y\}$ are ideals of P, and hence $I \cup \{x, y\}$ is an ideal of P.

Proof. It follows from Corollary 2.10 and Theorem 2.1. □

The center $C(G)$ of a connected finite graph G is the subgraph induced by the vertices of G with eccentricity equal the radius of G.

Theorem 2.12. Let G be a graph of a poset P. For a finite poset, $V(C(G)) \cup I$ is an ideal of P.

Proof. Let $x \in V(C(G)) \cup I$ and $p \in P$ such that $p \leq x$. Suppose $p \notin I$. Then $p \in V(G)$ and $e(p) = max\{d(u,p) \ : \ u \in V(G)\} \leq max\{d(u,x) \ : \ u \in V(G)\} = e(x)$. Since $x \in V(C(G))$, we have $e(p) = e(x)$, hence $p \in V(C(G))$. □

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