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On Strongly Extending Modules

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Abstract. The purpose of this paper is to introduce the concept of strongly extending modules which are particular subclass of the class of extending modules, and study some basic properties of this new class of modules. A module M is called strongly extending if each submodule of M is essential in a fully invariant direct summand of M. In this paper we examine the behavior of the class of strongly extending modules with respect to the preservation of this property in direct summands and direct sums and give some properties of these modules, for instance, strongly summand intersection property and weakly co-Hopfian property. Also such modules are characterized over commutative Dedekind domains.

1. Introduction

The theory of extending modules has come to play an important role and major contributions to this theory have been made in recent years, providing extensively interesting results on extending properties in the module-theoretical setting. An R-module M is called (strongly FI-) extending if each (fully invariant) submodule is essential in a (fully invariant) direct summand. Now it is natural to ask: *When does a module have the property that every submodule is essential in a fully invariant direct summand?* The main purpose of this paper is to answer this question and investigate these modules. Here is a brief summary of our paper. In fact, we will show that direct summands of a strongly extending module are strongly extending, and some conditions are given to show direct sum of two strongly extending modules is strongly extending. Also we prove that an R-module M is strongly extending if and only if $M = Z_2(M) \oplus N$ for some submodule N of M, where $Z_2(M)$ and N are both strongly extending and $\mathrm{Hom}(K, Z_2(M)) = 0$ for each submodule K of N. We introduce the notion of strongly Rickart modules and use this to show that

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endomorphism ring of each strongly extending module has a ring direct summand which is nonsingular and semiprime. Moreover, a number of results concerning strongly extending modules and examples of such modules are given. In the end of this paper, we investigate strongly extending modules over commutative Dedekind domains.

Throughout all rings (not necessarily commutative rings) have identity and all modules are unital right modules. For the sake of completeness, we state some definitions and notations used throughout this paper. Let M be a module over a ring R. For submodules N and K of M, $N \leq K$ denotes N is a submodule of K and $\mathrm{End}(M)$ denotes the ring of right R-module endomorphisms of M. $r_M(.)$ denotes the right annihilator of a subset of $\mathrm{End}(M)$ with elements from M. In what follows, by $\leq^{\oplus}$, $\leq^{ess}$ and $E(M)$ we denote, respectively, a module direct summand, an essential submodule and the injective hull of M. The symbols $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ stand for the ring of integers, the ring of residues modulo n and ring of rational numbers, respectively.

Definition 1.2. (a) An R-module M is called *extending* (or *CS*), if each submodule of M is essential in a direct summand. Equivalently, each closed submodule (i.e. has no proper essential extensions in M, for example direct summands) is a direct summand of M (see [5]). The second statement is clear because each submodule of M is contained in a closed submodule of M in which is essential due to Zorn's lemma.

(b) An R-module M is called *FI-extending*, provided that each fully invariant submodule of M is essential in a direct summand of M (see [1],[3]).

(c) An R-module M is called *strongly FI-extending* if each fully invariant submodule of M is essential in a fully invariant direct summand of M (see [1],[2],[4]).

(d) An R-module M is said to be *Baer* (resp., *Rickart*), if for any left ideal I of $\mathrm{End}(M)$ (resp., $\phi \in \mathrm{End}(M)$), $r_M(I)$(resp., $r_M(\phi)$) is a direct summand of M (see [11],[14]).

(e) An R-module M is called *duo* (resp., *weak duo*), provided that each submodule (resp., direct summand) is fully invariant in M (see [13]).

(f) An R-module M is said to have *SIP*, if the intersection of any two direct summands is a direct summand of M. A module M is said to have the *strongly summand intersection property (SSIP)* if the intersection of any family of direct summands is a direct summand of M (see [11]).

(g) An R- module M is called *weakly co-Hopfian* if every injective endomorphism has an essential image (see [6]).

(h) An idempotent $e \in R$ is called left semicentral if $re = ere$ for each $r \in R$. Equivalently, eR is an ideal of R. The set of all semicentral idempotents of R will be denoted by $S_l(R)$ (see [1]).

The following propositions are used in the sequel.

Proposition 1.2. (i) $[7, Lemma 1]$ *Let M be any module and $L \subseteq K$ be two submodules of M. If L is closed in K and K is closed in M, then L is closed in M.*

(ii) $[2, Lemma 1.2(i)]$ *Let* $e^2 = e \in \mathrm{End}(M)$. *Then* $e \in S_l(End(M))$ *if and only if* eM *is a fully invariant direct summand.*

(iii) $[13, Lemma 2.1]$ *If* N *is a fully invariant submodule of* M *and the module* $M = \oplus_{i\in I} M_i$ *is a direct sum of submodules* M_i $(i \in I)$, *then* $N = \oplus_{i\in I}(N \cap M_i)$.

Proposition 1.3. $[13, Theorem 3.10]$ *Let* R *be a Dedekind domain. Then the following statements are equivalent for a non-zero torsion* R*-module* M.

(i) M *is a duo module;*

(ii) M *is a weak duo module;*

(iii) *There exist distinct maximal ideals* P_i $(i \in I)$ *of* R *and submodules* M_i $(i \in I)$ *of* M *such that* $M = \oplus_{i\in I} M_i$ *and for each* $i \in I$, *either* $M_i \cong E(R/P_i)$ *or* $M_i \cong R/P_i^{n_i}$ *for some positive integer* n_i.

2. Strongly Extending Modules

We begin with the key definition of this paper. Motivated by the definition of a strongly FI-extending module, we define the strongly extending notion.

Definition 2.1. An R-module M is called *strongly extending*, if for every submodule N of M, there exists a fully invariant direct summand K of M such that N is an essential submodule of K. A ring R is called *strongly right extending*, if the module R_R has the corresponding property.

Remark 2.2. The diagram below offers a summary of the results mentioned above.

$$\begin{array}{ccccc} Duo & \Rightarrow & Weak\ duo & & \\ \Downarrow & & \Uparrow & & \\ Weakly\ co-Hopfian & \Leftarrow & Strongly\ extending & \Rightarrow & Strongly\ FI-extending \\ & & \Downarrow & & \Downarrow \\ & & Extending & \Rightarrow & FI-extending \end{array}$$

It is known that an R-module M is extending if and only if every closed submodule of M is a direct summand of M. The following states similar property for closed submodules of strongly extending modules.

Proposition 2.3. *An* R*-module* M *is strongly extending if and only if every closed submodule of* M *is a fully invariant direct summand.*

Proof. Is straightforward. □

We next give two other characterizations of strongly extending modules.

Theorem 2.4. *The following are equivalent:*

(i) M *is a strongly extending module;*

(ii) M *is extending and each idempotent of* $End(M)$ *is left semicentral;*

(iii) M *is extending and* End(M) *is an abelian ring.*

Proof. (i)⇒ (ii) If M is a strongly extending module, then it is clear that M

is extending. Let e be an idempotent element of $\mathrm{End}(M)$. Since M is strongly extending, eM is fully invariant by Proposition 2.3. Hence $e \in S_l(\mathrm{End}(M))$ by Proposition 1.2(ii).

(ii)$\Rightarrow$ (iii) It suffices to show that each idempotent of $\mathrm{End}(M)$ is central. If $e = e^2 \in \mathrm{End}(\mathrm{M})$, then e and $(1 - e)$ are left semicentral by (ii), which implies that e is central.

(iii)$\Rightarrow$ (i) By (iii), every direct summand of M is fully invariant. Thus every closed submodule is a fully invariant direct summand. Therefore by Proposition 2.3, M is strongly extending. $\square$

We study some examples for motivation.

Example 2.5. (1) *Every commutative domain R as a right R-module is strongly extending.*

(2) *Every uniform module is strongly extending. The ring $\mathbb{Z}_6$ as $\mathbb{Z}_6$-module is a strongly extending module which is not uniform.*

(3) *It can be seen that strongly extending modules are strongly FI-extending. The converse is not true: consider the ring $R = \begin{pmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$ of upper triangular matrices over the ring $\mathbb{Z}_4$. If $M = R_R$, then by [1, Theorem 2.8], M is a strongly FI-extending module that is not strongly extending.*

(4) *Let F be a field and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then R_R is not strongly extending, but R_R is extending.*

Theorem 2.6. *Let M be a strongly extending module. Then every direct summand of M is strongly extending.*

Proof. If N is a direct summand of M, then $N = eM$ for some $e = e^2 \in \mathrm{End}(M)$. Since M is extending, N is extending. It is known that $\mathrm{End}(N) = e(\mathrm{End}(M))e$. By usual argument, each idempotent of $\mathrm{End}(N)$ is central. Thus by Theorem 2.4, N is strongly extending. $\square$

In the following example, it is shown that an arbitrary direct sum of strongly extending modules is not necessarily strongly extending.

Example 2.7. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. It is clear that M_R is not strongly extending, because $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is an idempotent of* $\mathrm{End}(M) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ *which is not central. But $\mathbb{Z}_\mathbb{Z}$ is strongly extending.*

By Example 2.7, we obtain that strongly extending property is not a morita equivalent property.

Proposition 2.8. *Let $M = \oplus_{i \in I} M_i$. If M is a strongly extending module, then $Hom(M_i, M_j) = 0$ for each $i \neq j$ of I.*

Proof. As M is strongly extending, every direct summand of M is fully invariant in M. Hence the result is clear. □

The following example shows that if for each $i \in I$, M_i is strongly extending and $Hom(M_i, M_j) = 0$ for each $i \neq j$ of I, then $M = \oplus_{i\in I} M_i$ may be not strongly extending.

Example 2.9. *Consider $M = \mathbb{Z}_p \oplus \mathbb{Q}$ as $\mathbb{Z}$-module where p is prime. It easily can be seen* $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_p) = 0$ *and* $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Q}) = 0$. *Also $\mathbb{Q}$ and $\mathbb{Z}_p$ are strongly extending $\mathbb{Z}$-module. If $N = (1 + p\mathbb{Z}, 1)\mathbb{Z}$ and $K = ((p-1) + p\mathbb{Z}, 0)\mathbb{Z}$, then $N \cap K = 0$. Hence N is not essential in M. If M is strongly extending, then $N \leq^{ess} T$ for some fully invariant direct summand T of M. By Proposition 1.2(iii), $T = T \cap \mathbb{Z}_p \oplus T \cap \mathbb{Q}$. Since $N \leq T$, $0 \neq T \cap \mathbb{Z}_p$ and $0 \neq T \cap \mathbb{Q}$. Since $T \leq^{\oplus} M$, $T \cap \mathbb{Z}_p \leq^{\oplus} \mathbb{Z}_p$ and $T \cap \mathbb{Q} \leq^{\oplus} \mathbb{Q}$. Since $\mathbb{Z}_p$ and $\mathbb{Q}$ are indecomposable, $T = M$. Hence $N \leq^{ess} M$, a contradiction. Therefore M is not strongly extending.*

When direct sums of extending modules are extending are considered in [7] and [12].

Theorem 2.10. *Let $M = M_1 \oplus M_2$. Then M is strongly extending if and only if each closed submodule K of M with $K \cap M_1 = 0$ or $K \cap M_2 = 0$ is a fully invariant direct summand of M.*

Proof. ($\Rightarrow$) is clear.

($\Leftarrow$) We will show that, if K is a closed submodule of M, then K is a fully invariant direct summand of M. Consider $K \cap M_1$ as a submodule of K. By Zorn's Lemma, there exists a closed submodule L of K such that $K \cap M_1 \leq^{ess} L$. Since L is closed in K and K is closed in M, L is closed in M by Proposition 1.2(i). Since $(K \cap M_1) \cap (M_2 \cap L) = 0$ and $K \cap M_1 \leq^{ess} L$, $M_2 \cap L = 0$. By assumption L is a fully invariant direct summand of M. Let $M = L \oplus L'$ for some submodule L' of M. By modular law, $K = L \oplus (K \cap L')$. Since $K \cap L'$ is a direct summand of K (so it is closed) and K is closed in M, $K \cap L'$ is closed in M by Proposition 1.2(i). As $K \cap M_1 \leq L$, $(K \cap M_1) \cap L' = 0$. Thus $(K \cap L') \cap M_1 = 0$. By hypothesis, $K \cap L'$ is a fully invariant direct summand of M. Therefore $M = (K \cap L') \oplus Q$ for some submodule Q of M. Hence by modular law, $L' = (K \cap L') \oplus (Q \cap L')$. Thus we have $M = L \oplus L' = L \oplus (K \cap L') \oplus (Q \cap L')$, where $L \oplus (K \cap L') = K$. Therefore $M = K \oplus (Q \cap L')$. As L and $K \cap L'$ are fully invariant in M, K is fully invariant by [3, Lemma 1.1]. Hence M is strongly extending. □

In the following theorem the necessary condition are given for that the direct sum of two strongly extending modules is strongly extending.

Theorem 2.11. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are strongly extending and for each $N_1 \leq M_1$ and $N_2 \leq M_2$,* $\mathrm{Hom}(N_1, M_2) = 0$ *and* $\mathrm{Hom}(N_2, M_1) = 0$, *then M is strongly extending.*

Proof. Let K be a closed submodule of M such that $K \cap M_1 = 0$. Suppose that $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ denote the canonical projections. Since $Ker(\pi_2|_K) = Ker(\pi_2) \cap K = M_1 \cap K = 0$, we have monomorphism $\pi_2|_K : K \to M_2$.

As $\pi_1|_K = \pi_1((\pi_2|_K)^{-1}) : \pi_2(K) \to M_1 \in \mathrm{Hom}(\pi_2(K), M_1)$ and by assumption $\mathrm{Hom}(\pi_2(K), M_1) = 0$, we have $\pi_1(K) = 0$. Thus $K \subseteq Ker(\pi_1) = M_2$. Since K is closed in M, K is closed in M_2 and so by strongly extending property of M_2, K is a fully invariant direct summand of M_2. Hence K is a direct summand of M. To complete the proof of theorem, it suffices to show that K is fully invariant in M. Since $\mathrm{Hom}(M_1, M_2) = 0$ and $\mathrm{Hom}(M_2, M_1) = 0$, we have $\mathrm{End}(M) = \begin{pmatrix} \mathrm{End}(M_1) & 0 \\ 0 & \mathrm{End}(M_2) \end{pmatrix}$. Thus M_1 and M_2 are fully invariant direct summand of M. Since $K \leq^{\oplus} M_2 \leq^{\oplus} M$ and K is fully invariant in M_2 and M_2 is fully invariant in M, K is fully invariant in M by [3, Lemma 1.1]. Thus K is a fully invariant direct summand of M. Similarly, if K is a closed submodule of M with $K \cap M_2 = 0$, then K is a fully invariant direct summand of M. Therefore by Proposition 2.10, M is strongly extending. □

It is well-known from [5] that a free Z-module F is extending if and only if F has finite rank. In the following theorem, we extend this fact to the general setting of strongly extending free modules.

Theorem 2.12. *A free R- module F is strongly extending if and only if* $\mathrm{rank}(F) = 1$ *and R is strongly extending.*

Proof. Assume that F is a strongly extending and free R-module. If $\mathrm{rank}(F) \geq 2$, then Proposition 2.8 gives $\mathrm{Hom}(R, R) = 0$, a contradiction. Thus $\mathrm{rank}(F) = 1$, and R is strongly extending. The converse is clear. □

Theorem 2.13. *If M is a strongly extending module, then M has SSIP.*

Proof. Assume that M is a strongly extending module and let $\{M_i\}_{i \in I}$ be a family of direct summands of M and $M_i = e_i M$ for some idempotent e_i of End(M). If $\cap_{i \in I} M_i = 0$, then there is nothing to prove. Suppose that $\cap_i M_i \neq 0$ and so $\cap_i M_i \leq^{ess} eM$ for some $e \in S_l(End(M))$. Therefore for each $i \in I$, $(1-e_i)M \cap eM = 0$, whence $eM \subseteq e_i M$. Thus $\cap_i M_i = eM$ and M has SSIP. □

Motivated by the definition of Rickart modules and strongly extending modules, we define the following notion. There is a subclass of Baer modules say abelian Baer [15], which contains the class of modules defined next.

Definition 2.14. An R-module M is called *strongly Rickart*, if for each $\phi \in \mathrm{End}(M)$, $r_M(\phi)$ is a fully invariant direct summand of M.

Proposition 2.15. *For an R-module M, the following are equivalent:*

(i) *M is strongly Rickart;*

(ii) *M is Rickart and every direct summand of M is fully invariant;*

(iii) *M is Rickart and* $\mathrm{End}(M)$ *is an abelian ring.*

Proof. (i) $\Rightarrow$(ii) We only show that every direct summand of M is fully invariant. Let $e^2 = e \in \mathrm{End}(M)$ and $eM \leq^{\oplus} M$. Since $r_M(1-e) = eM$, eM is a fully invariant direct summand.

(ii)⇒(iii) and (iii)⇒(i) are similar to Theorem 2.4. □

Example 2.16. *Every strongly Rickart module is Rickart, however the converse is not true, for example, let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, *where* F *is a field. Then* R_R *is a Rickart module which is not strongly Rickart.*

Theorem 2.17. *Every direct summand of a strongly Rickart module is a strongly Rickart module.*

Proof. Let M be a strongly Rickart module and $N = eM$ be a direct summand of M. By Proposition 2.15, $\mathrm{End}(M)$ is an abelian ring and so $\mathrm{End}(N) = e\mathrm{End}(M)e$ is abelian. Since M is strongly Rickart, M is Rickart, whence $N = eM$ is Rickart by [11, Theorem 2.7]. Thus N is strongly Rickart by Proposition 2.15. □

Theorem 2.18. *Let* $\{M_i\}_{i\in I}$ *be a class of* R*-modules, for an index set* I*. The following are equivalent.*

(1) $M = \oplus_{i\in I} M_i$ *is strongly Rickart.*

(2) (i) *For each distinct* $i, j \in I$, $\mathrm{Hom}(M_i, M_j) = 0$.

(ii) *For each* $i \in I$, M_i *is strongly Rickart.*

Proof. (1) ⇒ (2) (i) Since M is strongly Rickart, each idempotent of $\mathrm{End}(M)$ is central by Proposition 2.15. As proof of Proposition 2.8, we can prove $\mathrm{Hom}(M_i, M_j) = 0$ for each distinct $i, j \in I$.

(ii) is clear from Theorem 2.17.

(2) ⇒ (1) The endomorphism ring of M is a ring of matrices, with elements of $End(M_i)$ in the ii-position and elements of $\mathrm{Hom}(M_i, M_j)$ in ij-position, for each $i, j \in I,\ i \neq j$. By $2(i)$, for each $i, j \in I$ with $i \neq j$, $\mathrm{Hom}(M_i, M_j) = 0$. Therefore every element of $\mathrm{End}(M)$ is a matrix where for each $i, j \in I$ with $i \neq j$, the ij-position is zero.

Let $f \in \mathrm{End}(M)$. We will show that $r_M(f)$ is a fully invariant direct summand. By the structure of $\mathrm{End}(M)$, f is a matrix of the form $f_i \in \mathrm{End}(M_i)$ in the ii-position and elsewhere zero. We can show f by $\oplus_{i\in I} f_i$. Hence $r_M(f) = Ker(f) = \oplus_{i\in I} Ker(f_i) = \oplus_{i\in I} r_{M_i}(f_i)$. By (ii), for each $i \in I$, M_i is strongly Rickart, thus $r_{M_i}(f_i) = e_i M_i$ for some $e_i^2 = e_i \in \mathrm{End}(M_i)$. Since M_i is strongly Rickart, $\mathrm{End}(M_i)$ is an abelian ring by Proposition 2.15. Therefore e_i is central in $\mathrm{End}(M_i)$. Let e be a matrix that e_i in ii-position and zero elsewhere. Since for each $i \in I$, e_i is central in $\mathrm{End}(M_i)$, e is central in $\mathrm{End}(M)$. Thus we have $r_M(f) = \oplus_{i\in I} e_i M = eM$. Therefore M is strongly Rickart. □

If M_1 and M_2 are strongly Rickart modules, then the module $M_1 \oplus M_2$ need not be a strongly Rickart module, as the following example shows. Thus we need condition (2)(i) in Theorem 2.18.

Example 2.19. ($[11, Example 2.5]$) *The* $\mathbb{Z}$*-modules* $\mathbb{Z}$ *and* $\mathbb{Z}_2$ *are both strongly Rickart, however the* $\mathbb{Z}$*-module* $\mathbb{Z} \oplus \mathbb{Z}_2$ *is not strongly Rickart. Because the map* $(m, n) \to (0, \bar{m})$ *has the kernel* $2\mathbb{Z} \oplus \mathbb{Z}_2$ *which is not a direct summand of* $\mathbb{Z} \oplus \mathbb{Z}_2$.

In general a strongly Rickart module need not be strongly extending.

Example 2.20. (1) *Consider* $\mathbb{Q} \oplus \mathbb{Z}_2$ *as* $\mathbb{Z}$*-module. Clearly* $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_2) = 0$ *and* $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Q}) = 0$*. Since* $\mathbb{Q}$ *and* $\mathbb{Z}_2$ *are strongly Rickart* $\mathbb{Z}$*-modules,* $\mathbb{Q} \oplus \mathbb{Z}_2$ *is strongly Rickart by Theorem 2.18. Although* $\mathbb{Q} \oplus \mathbb{Z}_2$ *is not strongly extending by Example 2.9.*

(2) *In view of Example 2.19 and part (1), the strongly Rickart property does not always transfer to each of its submodules.*

Remark 2.21. Let M be a strongly Rickart module. By Proposition 2.15, $\mathrm{End}(M)$ is abelian. Since M is strongly Rickart, M is Rickart, whence $\mathrm{End}(M)$ is a Rickart ring by [11, Proposition 3.2]. Thus $\mathrm{End}(M)$ is strongly Rickart, by Proposition 2.15.

The following proposition is similar to the Proposition 3.15 of [11].

Proposition 2.22. *If* $S = \mathrm{End}(M)$ *is strongly regular (i.e. abelian von Neumann regular), then* M *is a strongly Rickart module.*

Proof. Since S is von Neumann regular, M is Rickart by [11, Proposition 3.15]. As S is abelian, M is a strongly Rickart module, by Proposition 2.15. □

Lemma 2.23. *Let* M *be a nonsingular* R*-module. If* M *is a strongly extending module, then* M *is a strongly Rickart module.*

Proof. Suppose that $\phi \in \mathrm{End}(M)$. Since M is strongly extending, $r_M(\phi) \leq^{ess} eM$ for some $e \in S_l(\mathrm{End}(M))$. By nonsingularity of M and $\frac{M}{r_M(\phi)} \cong Im(\phi) \leq M$, we have $r_M(\phi) = eM$. □

The second singular submodule $Z_2(M)$ is the submodule of M which is defined by $Z(\frac{M}{Z(M)}) = \frac{Z_2(M)}{Z(M)}$. The next theorem is similar to [9, Theorem 1] for extending modules. In our proof of the following theorem, we use the method described in the proof of Theorem 1 of [9].

Theorem 2.24. *An* R*-module* M *is strongly extending if and only if* $M = Z_2(M) \oplus N$ *for some submodule* N *of* M *where* $Z_2(M)$ *and* N *are both strongly extending and* $\mathrm{Hom}(K, Z_2(M)) = 0$ *for each submodule* K *of* N*.*

Proof. Let M be a strongly extending module. It is known that $Z_2(M)$ is a closed submodule of M and so it is a fully invariant direct summand. Thus $M = Z_2(M) \oplus N$ for some submodule N of M. By Theorem 2.6, $Z_2(M)$ and N are strongly extending. Now we will show $\mathrm{Hom}(K, Z_2(M)) = 0$ for each submodule K of N. Let $f \in \mathrm{Hom}(K, Z_2(M))$. Set $Y = \{k - f(k) | \ k \in K\}$. It is clear that Y is a submodule of M. Since M is strongly extending, $Y \leq^{ess} H$ for some fully invariant direct summand H of M. Let $M = H \oplus H'$. We claim that $Y \cap Z_2(M) = 0$. Let $y = k - f(k) \in Y \cap Z_2(M)$ for some $k \in K$. Since $f(k) \in Z_2(M)$, we have $k \in K \cap Z_2(M) = 0$, and so $y = 0$. From $Y \leq^{ess} H$ and $Y \cap Z_2(M) = 0$, we have $H \cap Z_2(M) = 0$. Hence $Z_2(M) = Z_2(H')$, because $Z_2(M) = Z_2(H) \oplus Z_2(H')$.

Since $Z_2(H')$ is closed in H' and by Theorem 2.6, H' is strongly extending, we have $H' = Z_2(H') \oplus H''$ for some submodule H'' of H'. Therefore $M = H \oplus H'' \oplus Z_2(M)$. Let $\pi : M \to Z_2(M)$ be the canonical projection, then $\pi|_N : N \to Z_2(M)$ is the extension of f. Since $\mathrm{Hom}(N, Z_2(M)) = 0$ (by Proposition 2.8), $\pi|_N = 0$ and so $f = 0$. Thus for each $K \leq N$, $\mathrm{Hom}(K, Z_2(M)) = 0$.

Conversely, assume that N and $Z_2(M)$ are both strongly extending and $\mathrm{Hom}(K, Z_2(M)) = 0$ for each $K \leq N$. We will show $\mathrm{Hom}(X, N) = 0$ for each $X \leq Z_2(M)$. Let $g \in \mathrm{Hom}(X, N)$. Since $X \leq Z_2(M)$, $Z_2(X) = X$. Thus $f(X) = f(Z_2(X)) \subseteq Z_2(N) = 0$. Thus $f = 0$. Therefore by Theorem 2.11, M is strongly extending. □

Theorem 2.25. *Let M be a strongly extending module. Then M has a direct summand N such that N is a strongly Rickart module and* $\mathrm{End}(N)$ *is a reduced strongly Rickart ring.*

Proof. By Theorem 2.24, $M = Z_2(M) \oplus N$ for some nonsingular submodule N of M. By Theorem 2.6, N is a strongly extending module, whence by Lemma 2.23, N is a strongly Rickart module. By Remark 2.21, $\mathrm{End}(N)$ is strongly Rickart. We will show $\mathrm{End}(N)$ is reduced. Let $f^2 = 0$ for some element f of $\mathrm{End}(N)$. Then $r_{\mathrm{End}(N)}(f) = eEnd(N)$ for some central idempotent e of $\mathrm{End}(N)$. Thus $f = ef = fe = 0$, as desired. □

Corollary 2.26. *Let M be a strongly extending module. Then* $\mathrm{End}(M)$ *has a ring direct summand which is a semiprime nonsingular ring.*

Proof. By Theorem 2.24, $M = Z_2(M) \oplus K$ for some submodule K of M. Thus $\mathrm{End}(M) = \begin{pmatrix} \mathrm{End}(Z_2(M)) & \mathrm{Hom}(K, Z_2(M)) \\ \mathrm{Hom}(Z_2(M), K) & \mathrm{End}(K) \end{pmatrix}$. Since M is strongly extending, $\mathrm{Hom}(Z_2(M), K) = 0$, $\mathrm{Hom}(K, Z_2(M)) = 0$, by Proposition 2.8. Therefore $\mathrm{End}(M) = \mathrm{End}(Z_2(M)) \oplus \mathrm{End}(K)$, and by Theorem 2.25, $\mathrm{End}(K)$ is reduced, therefore it is semiprime. We will show End(K) is nonsingular. Let $f \in \mathrm{End(K)}$ such that $r_{\mathrm{End(K)}}(f) \leq^{ess} \mathrm{End(K)}$. By Theorem 2.25, End(K) is strongly Rickart, thus $r_{\mathrm{End(K)}}(f) = t\mathrm{End(K)}$ for some idempotent t of End(K). Thus $t = 1$ and $r_{\mathrm{End(K)}}(f) = \mathrm{End(K)}$. Hence $f = 0$ and so $End(K)$ is nonsingular. □

Proposition 2.27. *If M is a strongly extending module, then M is weakly co-Hopfian.*

Proof. Let M be a strongly extending module and $f \in \mathrm{End}(M)$ be an injective endomorphism of M. Then $f(M) \leq^{ess} eM$ for some idempotent $e \in \mathrm{End}(M)$. Thus $(1-e)f(M) = 0$. As $1-e$ is central, $(1-e)f(M) = f((1-e)M) = 0$. Since f is injective, $(1-e)M = 0$, whence $eM = M$ and $f(M) \leq^{ess} M$. □

In the following, we want to characterize torsion strongly extending module over Dedekind domains. Torsion extending modules were characterized over Dedekind domains in [8, Theorem 7(1)] and [9, Corollary 23]. At first we state the following lemma.

Lemma 2.28. *Let $M = \oplus_{i\in I} M_i$. If M is a duo module and for each $i \in I$, M_i is strongly extending, then M is strongly extending.*

Proof. Let N be a submodule of M. It can be seen that $N = \oplus_{i\in I} N \cap M_i$, by Proposition 1.2 (iii). Since M_i is strongly extending, for each $i \in I$, $N \cap M_i \leq^{ess} M_i'$ for some fully invariant direct summand M_i' of M_i. Thus $N \leq^{ess} \oplus_{i\in I} M_i'$. It is clear that $\oplus_{i\in I} M_i'$ is a direct summand of M. Since M is duo, $\oplus_{i\in I} M_i'$ is fully invariant. Thus M is strongly extending. □

Theorem 2.29. *Let R be a Dedekind domain. A torsion R-module M is strongly extending if and only if $M = \oplus_{i\in I} M_i$ and for each $i \in I$, either $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i^{n_i}$ for some distinct maximal ideals P_i and positive integer n_i*

Proof. Let M be a torsion strongly extending module over a Dedekind domain R. Since M is a weak duo module, $M = \oplus_{i\in I} M_i$ and for each $i \in I$, either $M_i \cong E(R/P_i)$ or $M_i \cong R/P_i^{n_i}$ for some distinct maximal ideals P_i and positive integer n_i, by Proposition 1.3. Conversely, let $M \cong (\oplus_{i\in J} E(\frac{R}{P_i})) \oplus (\oplus_{i\in I} \frac{R}{P_i^{n_i}})$, where $P_i^{,}s$ $(i \in I \cup J)$ are distinct maximal ideals R and $n_i^{,}s$ are positive integers. It can be easily seen that $E(\frac{R}{P_i})$ and $\frac{R}{P_i^{n_i}}$ are uniform R-modules, so they are strongly extending. By Proposition 1.3, M is a duo module, thus Lemma 2.28 gives that M is strongly extending. □

Proposition 2.30. *Let R be a principle ideal domain. If M is a finitely generated torsion-free R-module, then M is strongly extending if and only if $M \cong R$.*

Proof. It is known that, if M is a finitely generated torsion-free module over a principle ideal domain, then M is a free R-module. Since M is strongly extending, $M \cong R$ by Theorem 2.12. The converse is clear. □

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