KYUNGPOOK Math. J. 54(2014), 249-262
http://dx.doi.org/10.5666/KMJ.2014.54.2.249

A Note on Certain Properties of Mock Theta Functions of Order Eight

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Abstract. In this paper, we have developed a non-homogeneous q- difference equation of first order for the generalized Mock theta function of order eight and besides these established limiting case of Mock theta functions of order eight. We have also established identities for Partial Mock theta function and Mock theta function of order eight and provided a number of cases of the identities.

1. Introduction

"The Mock Theta Functions", a last gift of Ramanujan to the Mathematical world, a substantial portion of which lay buried in the debris in the attire of Watson's house for almost half a century. Later on Andrews (1976) rediscovered Ramanujan's lost note book found in the library of Trinity college Cambridge, which contains many further results on Mock theta functions, certain unresolved problems, and their possible future interest and relationship in the general context of the basic hypergeometric theory. In the first phase of development G. N. Watson [22], R. P. Agrawal [3, 4, 5, 6] and G. E. Andrews [1, 2] had contributed significantly. The emergence of Ramanujan's Lost Notebook offered a new platform for further development in the literature of Mock theta functions and consequently Choi [7], Gordon and McIntosh [12] provide valuable information regarding the structure and possible construction of new Mock theta functions other than ones given by Ramanujan. The Mathematical world came across about the new class of Mock theta function that enhanced domain of generalized form of Mock theta functions developed by G. E. Andrews [1, 2], R. P. Agarwal [3, 4, 5, 6], Sneh D. Prasad [16], Bhaskar Srivastava [18], Anju Gupta [14]. The cultivation of these new classes provides a new platform

* Corresponding Author.
Received February 29, 2012; accepted July 18, 2013.
2010 Mathematics Subject Classification: 33D15, 33D90.
Key words and phrases: Mock Theta Functions, Basic Hypergeometric series, q-difference equation.

to develop and formalize a new shape of Mock theta function of various orders.

A. K. Srivastava [17] in his finding, defined partial Mock theta functions of order three and five by taking partial sum of the series defining these functions and also the relationship between partial Mock theta function of these orders. R. Y. Denis and S. N. Singh [9] developed the representation of Mock theta functions of order six and order ten, R. Y. Denis, S. N. Singh and S. P. Singh [10] established relationship connecting two Mock theta functions of arbitrary orders where each Mock theta function is on a different base and R. Y. Denis [8] has also given the certain expansion of basic hypergeometric function. In this context recently, authors [19, 20, 21] have also identified the basic hypergeometric structure of Hikami's [15] Mock theta functions with some of its properties. Perusal of literature shows that basic hypergeometric structure of Mock Theta function of order 3, 5 and 7 have been defined and shown by R. P. Agarwal [3] that they are limiting cases of ${}_2\phi_1$'s, ${}_3\phi_2$'s and ${}_4\phi_3$'s respectively and further got supported by the findings of Anju Gupta [13] in her work of fifth, seventh and the seven sixth order Mock theta functions mentioned in the 'Lost' Notebook.

In this paper, we have given a first order non homogeneous q- difference equation for the generalized form of Mock theta functions of order eight. It has also been shown that eighth order Mock theta functions are the limiting cases of basic hypergeometric series ${}_3\phi_2$'s. In addition, the double series representation of Mock theta function of order eight along with the relations among partial Mock theta functions and Mock theta functions of order eight and the special cases of such relationship are also discussed.

2. Definitions and Notations

In this section, we will adopt the following definitions and notations.
For $|q| < 1$ and $|q^r| < 1$, q- shifted factorial is defined by,

$$(a;q)_n = \prod_{s=0}^{n-1}(1-aq^s),\ n \geq 1,$$

$$(a;q^r)_n = \prod_{s=0}^{n-1}(1-aq^{rs}),\ n \geq 1,$$

$$(a;q)_0 = 1,\ (a;q^r)_0 = 1,$$

$$(a;q^r)_\infty = \prod_{s=0}^{\infty}(1-aq^{rs}).$$

Following [11] a generalized basic hypergeometric function ${}_r\phi_s$ is defined for $|q^\mu| < 1$ as,

$$(2.1)\quad {}_r\phi_s\left[\begin{array}{c} a_1, a_2, ...a_r; q^\mu; z \\ b_1, b_2, ...b_s; q^i \end{array}\right] = \sum_{n=0}^{\infty} q^{in(n-1)/2}\frac{[a_1, a_2, ...a_r; q^\mu]_n z^n}{[b_1, b_2, ...b_s; q^\mu]_n [q; q^\mu]_n}.$$

Definitions and Notations of Mock theta function and Partial Mock theta functions that shall be used in our analysis are as:

Mock theta functions of order eight:
B. Gordon and R.J. McIntosh [12] found eight Mock theta functions of order 8:-

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} = -1 + 2\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.$$

If F(q)=$\sum_{n=0}^{\infty} f(q,n)$ is a mock theta function, then the corresponding partial mock theta function is denoted by the truncated series $F_p(q) = \sum_{n=0}^{p} f(q,n)$.

The following identities have been used to develop our main results are as:
The identity due to Srivastava [17] is as,

(2.2)
$$\frac{(aq-e)(e-bq)}{(q-e)(e-abq)} \sum_{m=0}^{\infty} \frac{(a,b)_m q^m}{(e,abq^2/e)_m} \sum_{r=0}^{m} \alpha_r$$

$$= -\frac{(a,b)_\infty}{(e/q,abq/e)_\infty} \sum_{m=0}^{\infty} \alpha_m + \sum_{m=0}^{\infty} \frac{(a,b)_m}{(e/q,abq/e)} \alpha_m.$$

The identity due to Denis [8] is as

$$\sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \frac{x^m[(a)]_m}{[(b)]_m} {}_{A+1}\phi_B \left[\begin{array}{c} q^{-n+m}, (a)q^m; xq^{n-m} \\ (b)q^m \end{array} \right] \tag{2.3}$$

$$ {}_{C+1}\phi_D \left[\begin{array}{c} q^{-m}, (c); yq^m \\ (d) \end{array} \right] = {}_{A+1}\phi_B \left[\begin{array}{c} q^{-n}, (a), (c); xyq^n \\ (b), (d) \end{array} \right],$$

where $\left(\begin{array}{c} n \\ m \end{array} \right)$ stands for $\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}$.

3. Main Results

We have classified main results in five sections.

3.1 Generalized function for Mock theta function of order eight:

In this section, we have given the generalized form of mock theta functions of order eight and these generalized form of Mock theta functions are satisfying a first order non homogeneous q-difference equation,

$$S_0(q,z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n (-q;q^2)_n}{(-zq;q^2)_n},$$

$$S_1(q,z) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)} z^n (-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_0(q,z) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n+2} z^n (-zq;q^2)_n}{(-z;q^2)_{n+1}},$$

$$T_1(q,z) = \sum_{n=0}^{\infty} \frac{q^{n^2} z^n (-zq;q^2)_n}{(-z;q^2)_{n+1}},$$

$$U_0(q,z) = 2\sum_{n=0}^{\infty} \frac{q^{n^2-n} z^n (-z;q^2)_n}{(-z^2/q^2;q^4)_{n+1}},$$

$$U_1(q,z) = q\sum_{n=0}^{\infty} \frac{q^{n^2+n} z^n (-z;q^2)_n}{(-z^2;q^4)_{n+1}},$$

$$V_0(q,z) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n(n-1)} z^n (-q;q^2)_n}{(z;q^2)_n},$$

$$V_1(q,z) = q\sum_{n=0}^{\infty} \frac{q^{n(n+1)} z^n (-z;q^2)_n}{(z;q^2)_{n+1}}.$$

For $z = q$, we get mock theta functions of order eight $S_0(q), S_1(q), T_0(q), T_1(q)$, $U_0(q), U_1(q), V_0(q)$ and $V_1(q)$.

Further, We shall show that these generalized functions satisfy a first order non homogeneous q- difference equation.

Proof. In order to prove that the generalized functions satisfy a first order non homogeneous q-difference equation.
First of all we take the case of $S_0(q)$ and proceed as follows.
Let

$$F(z)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}z^n(-q;q^2)_n}{(-zq;q^2)_n}, \tag{3.1.1}$$

For $z=q$ it becomes $S_0(q)$.
The auxiliary function is defined as

$$L(z)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}z^n(-q;q^2)_n}{(-zq;q^2)_{n+1}} \tag{3.1.2}$$

and observe that

$$L(z)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)}z^n(-q;q^2)_n}{(-zq;q^2)_{n+1}}(1+zq^{2n+1}-zq^{2n+1})= \tag{3.1.3}$$

$$F(z)-\frac{zq}{(1+zq)}F(zq^2)$$

we also have

$$L(z)=\frac{1}{(1+zq)}+\sum_{n=0}^{\infty}\frac{q^{n(n+1)}z^{n+1}(-q;q^2)_{n+1}}{(-zq;q^2)_{n+2}}=\frac{1}{(1+zq)}+$$

$$\frac{z}{(1+zq)}\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(zq^2)^n(-q;q^2)_n(1+q^{2n+1})}{(-zq^3;q^2)_{n+1}}$$

$$=\frac{1}{(1+zq)}+\frac{z}{(1+zq)}\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(zq^2)^n(-q;q^2)_n}{(-zq^3;q^2)_{n+1}}$$
$$+\frac{zq}{(1+zq)(1+zq^3)}\sum_{n=0}^{\infty}\frac{q^{n(n-1)}(zq^4)^n(-q;q^2)_n}{(-zq^5;q^2)_n}$$

$$L(z)=\frac{1}{(1+zq)}+\frac{z}{(1+zq)}L(zq^2)+\frac{zq}{(1+zq)(1+zq^3)}F(zq^4) \tag{3.1.4}$$

Using (3.1.3) to eliminate $L(z)$ and $L(zq^2)$ from (3.1.4), we obtain

$$F(z) = \frac{1}{(1+zq)} + \frac{z(1+q)}{(1+zq)}F(zq^2) + \frac{zq(1-zq)}{(1+zq^3)}F(zq^4). \tag{3.1.5}$$

Which is the first order non homogeneous q-difference equation. □

Similarly, one can establish and satisfy a first order non homogeneous q- difference equation for $S_1(q), T_0(q), T_1(q), U_0(q), U_1(q), V_0(q)$ and $V_1(q)$.

3.2 Mock theta function of order eight as a limiting case:

In this section, we express all the eighth order mock theta functions as a limiting case of basic hypergeometric series ${}_3\phi_2$,

$$S_0(q) = \lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q, q^2; -q/\alpha \\ -q^2, 0\end{matrix}\right],$$

$$S_1(q) = \lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q, q^2; -q^3/\alpha \\ -q^2, 0\end{matrix}\right],$$

$$T_0(q) = \frac{q^2}{(1+q)}\lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q^2, q^2; -q^4/\alpha \\ -q^3, 0\end{matrix}\right],$$

$$T_1(q) = \frac{1}{(1+q)}\lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q^2, q^2; -q^2/\alpha \\ -q^3, 0\end{matrix}\right],$$

$$U_0(q) = \lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q, q^2; -q/\alpha \\ iq^2, -iq^2\end{matrix}\right],$$

$$U_1(q) = \frac{q}{(1+q)}\lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q, q^2; -q^3/\alpha \\ iq^3, -iq^3\end{matrix}\right],$$

$$V_0(q) = -1 + 2\lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q, q^2; -q/\alpha \\ q, 0\end{matrix}\right],$$

$$V_1(q) = \frac{q}{(1-q)}\lim_{\alpha\to\infty} {}_3\phi_2^{(q^2)}\left[\begin{matrix}\alpha, -q, q^2; -q^3/\alpha \\ q^3, 0\end{matrix}\right].$$

Proof. Following the basic hypergeometric functions properties, representations of

$S_0(q)$, $S_1(q)$, $T_0(q)$, $T_1(q)$, $U_0(q)$, $U_1(q)$, $V_0(q)$ and $V_1(q)$ can be established. □

3.3 Transformations of Mock theta functions of order eight:

In this section, transformations of Mock theta function of order eight in double series is given.

$$S_{0n}(q) = \sum_{r=0}^{n} \frac{(-q;q^2)_r q^{r^2}}{(-q^2;q^2)_r} \tag{3.3.1}$$

$$= \sum_{m=0}^{n} \frac{(-q;q^2)_m^2}{(-q^2;q^2)_m} \;\phi \left[\begin{matrix} -q^{2m+1}, q^{2m+2}; q^2; -1 \\ -q^{2m+2}; q^2 \end{matrix} \right]_{n-m}$$

Here $S_{on}(q)$ denotes the partial mock theta function of $S_0(q)$.
Taking $n \to \infty$ in the above, we get

$$S_0(q) = \sum_{m=0}^{\infty} \frac{(-q;q^2)_m^2}{(-q^2;q^2)_m} \;\phi \left[\begin{matrix} -q^{2m+1}, q^{2m+2}; q^2; -1 \\ -q^{2m+2}; q^2 \end{matrix} \right]. \tag{3.3.2}$$

This is the double series representation of $S_0(q)$.

$$S_{1n}(q) = \sum_{r=0}^{n} \frac{(-q;q^2)_r q^{r(r+2)}}{(-q^2;q^2)_r} \tag{3.3.3}$$

$$= \sum_{m=0}^{n} \frac{(-q;q^2)_m (-q^3;q^2)_m}{(-q^2;q^2)_m} \;\phi \left[\begin{matrix} -q^{2m+1}, q^{2m+2}; q^2; -1 \\ -q^{2m+2}; q^2 \end{matrix} \right]_{n-m}$$

Here $S_{1n}(q)$ denotes the partial mock theta function of $S_1(q)$.
Taking $n \to \infty$ in the above, we get

$$S_1(q) = \sum_{m=0}^{\infty} \frac{(-q;q^2)_m (-q^3;q^2)_m}{(-q^2;q^2)_m} \;\phi \left[\begin{matrix} -q^{2m+1}, q^{2m+2}; q^2; -1 \\ -q^{2m+2}; q^2 \end{matrix} \right]. \tag{3.3.4}$$

This is the double series representation of $S_1(q)$.

$$T_{0n}(q) = \sum_{r=0}^{n} \frac{(-q^2;q^2)_r q^{(r+1)(r+2)}}{(-q;q^2)_{r+1}} \tag{3.3.5}$$

$$= q^2 \sum_{m=0}^{n} \frac{(-q^2;q^2)_m (-q^4;q^2)_m}{(-q;q^2)_{m+1}} \;\phi \left[\begin{matrix} -q^{2m+2}, q^{2m+2}; q^2; -1 \\ -q^{2m+3}; q^2 \end{matrix} \right]_{n-m}$$

Here $T_{0n}(q)$ denotes the partial mock theta function of $T_0(q)$.
Taking $n \to \infty$ in the above, we get

$$(3.3.6) \qquad T_0(q) = q^2 \sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m(-q^4;q^2)_m}{(-q;q^2)_{m+1}}$$

$$\phi\left[\begin{array}{c} -q^{2m+2}, q^{2m+2}; q^2; -1 \\ -q^{2m+3}; q^2 \end{array}\right].$$

This is the double series representation of $T_0(q)$.

$$(3.3.7) \qquad T_{1n}(q) = \sum_{r=0}^{n} \frac{(-q^2;q^2)_r q^{r(r+1)}}{(-q;q^2)_{r+1}}$$

$$= \sum_{m=0}^{n} \frac{(-q^2;q^2)_m^2}{(-q;q^2)_{m+1}} \ \phi\left[\begin{array}{c} -q^{2m+2}, q^{2m+2}; q^2; -1 \\ -q^{2m+3}; q^2 \end{array}\right]_{n-m}$$

Here $T_{1n}(q)$ denotes the partial mock theta function of $T_1(q)$.
Taking $n \to \infty$ in the above, we get

$$(3.3.8) \qquad T_1(q) = \sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m^2}{(-q;q^2)_{m+1}} \ \phi\left[\begin{array}{c} -q^{2m+2}, q^{2m+2}; q^2; -1 \\ -q^{2m+3}; q^2 \end{array}\right].$$

This is the double series representation of $T_1(q)$.

$$(3.3.9) \qquad U_{0n}(q) = \sum_{r=0}^{n} \frac{(-q;q^2)_r q^{r^2}}{(-q^4;q^4)_r} =$$

$$\sum_{m=0}^{n} \frac{(-q;q^2)_m^2}{(-q^4;q^4)_m} \ \phi\left[\begin{array}{c} -q^{2m+1}, q^{2m+2}; q^2; -1 \\ iq^{2m+2}, -iq^{2m+2}; q^2 \end{array}\right]_{n-m}$$

Here $U_{0n}(q)$ denotes the partial mock theta function of $U_0(q)$.
Taking $n \to \infty$ in the above, we get

$$(3.3.10) \qquad U_0(q) = \sum_{m=0}^{\infty} \frac{(-q;q^2)_m^2}{(-q^4;q^4)_m} \ \phi\left[\begin{array}{c} -q^{2m+1}, q^{2m+2}; q^2; -1 \\ iq^{2m+2}, -iq^{2m+2}; q^2 \end{array}\right].$$

This is the double series representation of $U_0(q)$.

$$(3.3.11) \qquad U_{1n}(q) = \sum_{r=0}^{n} \frac{(-q;q^2)_r q^{(r+1)^2}}{(-q^2;q^4)_{r+1}}$$

$$= q\sum_{m=0}^{n}\frac{(-q;q^2)_m(-q^3;q^2)_m}{(-q^2;q^4)_{m+1}}\ \phi\left[\begin{matrix}-q^{2m+1},q^{2m+2};q^2;-1\\ iq^{2m+3},-iq^{2m+3};q^2\end{matrix}\right]_{n-m}$$

Here $U_{1n}(q)$ denotes the partial mock theta function of $U_1(q)$.
Taking $n\to\infty$ in the above, we get

(3.3.12)
$$U_1(q) = q\sum_{m=0}^{\infty}\frac{(-q;q^2)_m(-q^3;q^2)_m}{(-q^2;q^4)_{m+1}}$$
$$\phi\left[\begin{matrix}-q^{2m+1},q^{2m+2};q^2;-1\\ iq^{2m+3},-iq^{2m+3};q^2\end{matrix}\right].$$

This is the double series representation of $U_1(q)$.

(3.3.13)
$$1+V_{0n}(q) = 2\sum_{r=0}^{n}\frac{(-q;q^2)_r q^{r^2}}{(q;q^2)_r}$$

$$= 2\sum_{m=0}^{n}\frac{(-q;q^2)_m^2}{(q;q^2)_m}\ \phi\left[\begin{matrix}-q^{2m+1},q^{2m+2};q^2;-1\\ q^{2m+1};q^2\end{matrix}\right]_{n-m}$$

Here $V_{0n}(q)$ denotes the partial mock theta function of $V_0(q)$.
Taking $n\to\infty$ in the above, we get

(3.3.14)
$$1+V_0(q) = 2\sum_{m=0}^{\infty}\frac{(-q;q^2)_m^2}{(q;q^2)_m}\ \phi\left[\begin{matrix}-q^{2m+1},q^{2m+2};q^2;-1\\ q^{2m+1};q^2\end{matrix}\right].$$

This is the double series representation of $V_0(q)$.

(3.3.15)
$$V_{1n}(q) = \sum_{r=0}^{n}\frac{(-q;q^2)_r q^{(r+1)^2}}{(q;q^2)_{r+1}}$$

$$= q\sum_{m=0}^{n}\frac{(-q;q^2)_m(-q^3;q^2)_m}{(q;q^2)_{m+1}}\ \phi\left[\begin{matrix}-q^{2m+1},q^{2m+2};q^2;-1\\ q^{2m+3};q^2\end{matrix}\right]_{n-m}$$

Here $V_{1n}(q)$ denotes the partial mock theta function of $V_1(q)$.
Taking $n\to\infty$ in the above, we get

(3.3.16)
$$V_1(q) = q\sum_{m=0}^{\infty}\frac{(-q;q^2)_m(-q^3;q^2)_m}{(q;q^2)_{m+1}}$$
$$\phi\left[\begin{matrix}-q^{2m+1},q^{2m+2};q^2;-1\\ q^{2m+3};q^2\end{matrix}\right].$$

This is the double series representation of $V_1(q)$.

Proof. As an illustration, we shall prove the result (3.3.1).
We take $C = D = 0$, $B \to B+1$, $A \to A+1$, $b_{B+1} = q^{-n}$, $y = \beta$ and x is replaced by xq^{-n} in identity (2.3), we get

$$(3.3.17) \sum_{m=0}^{n} \sum_{t=0}^{n-m} \frac{[(a_{A+1})]_{m+t}[\beta]_m(-1)^m x^{m+t} q^{-mt}}{[q]_m[q]_t[(b)]_{m+t} q^{m(m-1)/2}} = \sum_{r=0}^{n} \frac{[(a_{A+1})]_r (x\beta)^r}{[q]_r[(b)]_r}.$$

Now replace q by q^2 in (3.3.17) and take $A = 2, B = 1, a_1 = 1/x, a_2 = -q, a_3 = q^2, b_1 = -q^2, \beta = -q$ and let $x \to 0$ and after simplifications, we get result (3.3.1). □

3.4 Relationship among partial Mock theta function and Mock theta function of order eight:

In order to develop relationships, we make use of identity (2.2) for some particular values of α_r that leads to a number of relations.

$$(3.4.1) \quad \frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)} \sum_{m=0}^{\infty} \frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m} S_{om}(q) =$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty} S_0(q) + \sum_{m=0}^{\infty} \frac{(a,b,-q;q^2)_m q^{m^2}}{(e/q^2,abq^2/e,-q^2;q^2)_m}.$$

$$(3.4.2) \quad \frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)} \sum_{m=0}^{\infty} \frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m} S_{1m}(q) =$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty} S_1(q) + \sum_{m=0}^{\infty} \frac{(a,b,-q;q^2)_m q^{m(m+2)}}{(e/q^2,abq^2/e,-q^2;q^2)_m}.$$

$$(3.4.3) \quad \frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)} \sum_{m=0}^{\infty} \frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m} T_{om}(q) =$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty} T_0(q) + \frac{1}{(1+q)} \sum_{m=0}^{\infty} \frac{(a,b,-q^2;q^2)_m q^{(m+1)(m+2)}}{(e/q^2,abq^2/e,-q^3;q^2)_m}.$$

$$(3.4.4) \quad \frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)} \sum_{m=0}^{\infty} \frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m} T_{1m}(q) =$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty}T_1(q)+\frac{1}{(1+q)}\sum_{m=0}^{\infty}\frac{(a,b,-q^2;q^2)_m q^{m(m+1)}}{(e/q^2,abq^2/e,-q^3;q^2)_m}.$$

(3.4.5)
$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m}U_{om}(q)=$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty}U_0(q)+\sum_{m=0}^{\infty}\frac{(a,b,-q;q^2)_m q^{(m+1)(m+2)}}{(e/q^2,abq^2/e,iq^2,-iq^2;q^2)_m}.$$

(3.4.6)
$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m}U_{1m}(q)=$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty}U_1(q)+\frac{1}{(1+q^2)}\sum_{m=0}^{\infty}\frac{(a,b,-q;q^2)_m q^{(m+1)^2}}{(e/q^2,abq^2/e,iq^3,-iq^3;q^2)_m}.$$

(3.4.7)
$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m}[1+V_{om}(q)]=$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty}[1+V_0(q)]+\sum_{m=0}^{\infty}\frac{(a,b,-q;q^2)_m q^{m^2}}{(e/q^2,abq^2/e,q;q^2)_m}.$$

(3.4.8)
$$\frac{(aq^2-e)(e-bq^2)}{(q^2-e)(e-abq^2)}\sum_{m=0}^{\infty}\frac{(a,b;q^2)_m q^{2m}}{(e,abq^4/e;q^2)_m}V_{1m}(q)=$$

$$-\frac{(a,b;q^2)_\infty}{(e/q^2,abq^2/e;q^2)_\infty}V_1(q)+\frac{1}{(1-q)}\sum_{m=0}^{\infty}\frac{(a,b,-q;q^2)_m q^{(m+1)^2}}{(e/q^2,abq^2/e,q^3;q^2)_m}.$$

Proof:
As an illustration, we shall prove result (3.4.1).
To prove the result (3.4.1), changing the base $q\to q^2$ and taking $\alpha_r=\frac{(-q;q^2)_r q^{r^2}}{(-q^2;q^2)_r}$ in identity (2.2), simplifying further we obtain result (3.4.1). □

Similarly, appropriate selection of α_r, one can obtain results (3.4.2) to (3.4.8).

3.5 Special cases:

In this section, we have discussed special cases of the results in section (3.4).
(i)Putting $a = -q^2, b = 0, e = q^3$ in eq. (3.4.1), and after simplification we get.

$$\frac{-2q(1+q)}{(1-q)} \sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m q^{2m}}{(q^3;q^2)_m} S_{0m}(q) + \tag{3.5.1}$$

$$\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} S_0(q) = 1 + V_0(q).$$

(ii)Putting $a = -q^4, b = -q^3, e = -q^4$ in eq. (3.4.3), and after simplification we get

$$q^2(1-q)^2 \sum_{m=0}^{\infty} \frac{(-q;q^2)_{m+1} q^{2m}}{(-q^3;q^2)_{m+2}} T_{0m}(q) = T_1(q) - T_0(q). \tag{3.5.2}$$

(iii)Putting $a = -q^2, b = 0, e = q^5$ in eq. (3.4.2), and after simplification we get

$$\frac{-q^2(1+q)}{(1-q^3)} \sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m q^{2m}}{(q^5;q^2)_m} S_{1m}(q) + \tag{3.5.3}$$

$$\frac{(-q^2;q^2)_\infty}{(q^3;q^2)_\infty} S_1(q) = \frac{(1-q)}{q} V_1(q).$$

(iv)Putting $a = -q^4, b = 0, e = -q^4$ in eq. (3.4.4), and after simplification we get

$$q^2(1-q^2) \sum_{m=0}^{\infty} q^{2m} T_{1m}(q) = T_0(q). \tag{3.5.4}$$

(v)Putting $a = q, b = 0, e = -q^3$ in eq. (3.4.5), and after simplification we get

$$\frac{2q}{(1+q)} \sum_{m=0}^{\infty} \frac{(q;q^2)_m q^{2m}}{(-q^3;q^2)_m} U_{0m}(q) + \tag{3.5.5}$$

$$\frac{(q;q^2)_\infty}{(-q;q^2)_\infty} U_0(q) = S_0(q^2) - qS_1(q^2).$$

(vi)Putting $a = q, b = 0, e = -q^2$ in eq. (3.4.7), and after simplification we get

$$(1+q) \sum_{m=0}^{\infty} \frac{(q;q^2)_m q^{2m}}{-(q^2;q^2)_m} [1 + V_{0m}(q)] + \tag{3.5.6}$$

$$\frac{2(q;q^2)_\infty}{(-1;q^2)_\infty} [1 + V_0(q)] = S_0(q) + S_1(q).$$

(vii)Putting $a = -q^3, b = 0, e = q^3$ in eq. (3.4.8), and after simplification we get

$$2q(1-q^2)\sum_{m=0}^{\infty} q^{2m} V_{1m}(q) + 1 = 2V_1(q) + V_0(q). \tag{3.5.7}$$

(viii)Putting $a = q^3, b = 0, e = -q^4$ in eq. (3.4.8), and after simplification we get

$$\frac{q^2(1+q)}{(1+q^2)}\sum_{m=0}^{\infty} \frac{(q^3;q^2)_m q^{2m}}{(-q^4;q^2)_m} V_{1m}(q) + \tag{3.5.8}$$

$$\frac{(q^3;q^2)_\infty}{(-q^2;q^2)_\infty} V_1(q) = \frac{q}{(1-q)} S_1(q).$$

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