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Paranormed I-convergent Double Sequence Spaces Associated with Multiplier Sequences

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Abstract. In this article we introduce different types of multiplier I-convergent double sequence spaces. We study their different algebraic and topological properties like solidity, symmetricity, completeness etc. The decomposition theorem is established and some inclusion results are proved.

1. Introduction

The notion of paranormed sequence space was introduced by Nakano [12] and Simons [16]. It was further investigated from sequence space point of view and linked with summability theory by Maddox ([10], [11]), Lascarides [9], Tripathy [18], Tripathy and Hazarika [23], Tripathy and Sen [29] and many others.

The notion of I-convergence was studied at the initial stage by Kostyrko, Šalát and Wilczynski [8]. It generalizes and unifies different notions of convergence of sequences. The notion was further investigated by Šalát, Tripathy and Ziman ([14], [15]), Tripathy and Dutta [21], Tripathy and Hazarika ([22], [23], [24]), Tripathy, Hazarika and Choudhary [25], Tripathy and Tripathy [30] and many others.

The scope for the studies on sequence spaces was extended on introducing the notion of associated multiplier sequences. Goes and Goes [5] defined the differentiated sequence space dE and the integrated sequence space $\int E$ for a given sequence spacee E, with the help of multiplier sequences (k^{-1}) and (k) respectively. Kamthan

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[7] used the multiplier sequence $(k!)$. It was further investigated from sequence space point of view by Tripathy and Hazarika [22], Tripathy and Mahanta [26] and others. In this article we shall consider a general multiplier sequence $\Lambda = (\lambda_k)$ of non-zero scalars.

2. Definitions and Background

Throughout N, R and C denote the sets of natural, real and complex numbers respectively.

Let $X \neq \emptyset$, then a non-void class $I \subseteq 2^X$ is called an ideal if I is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e. $A \in I$ and $B \subseteq A \Rightarrow B \in I$). An ideal is said to be nontrivial if $I \neq 2^X$. A non-trivial ideal I is said to be admissible if I contains every finite subset of N. A non-trivial ideal I is said to be maximal if there does not exist any non-trivial ideal $J \neq I$ containing I as a subset.

For any ideal I, there is a filter $\Psi(I)$ corresponding to I, given by $\Psi(I) = \{K \subseteq N : N \setminus K \in I\}$.

A sequence (x_n) is said to be I-convergent to $L \in C$, if for a given $\varepsilon > 0$, the set $\{n \in N : |x_n - L| \geq \varepsilon\} \in I$.

Throughout the article a double sequence A is denoted by (a_{nk}) i.e. a double infinite array of complex numbers, a_{nk} for all $n, k \in N$.

A double sequence $A = (a_{nk})$ a is said to converge in Pringsheims sense if there exists a number L such that (a_{nk}) converges to L as both n and k tend to ∞ independent of one another. It can be easily seen that convergence of (a_{nk}) in Pringsheims sense does not guarantee the boundedness of (a_{nk}).

Hardy [6] introduced the notion of regular convergence for double sequences. A double sequence (a_{nk}) is said to converge regularly if it convergences in Pringsheims sense and in addition, the following limits exist

$$\lim_{n\to\infty} a_{nk} = x_k (k = 1, 2, 3, ...)$$

and

$$\lim_{k\to\infty} a_{nk} = y_n (n = 1, 2, 3, ...).$$

Throughout w_2 , $(\ell_\infty)_2$, c, $(c_0)_2$, $(c)_2^R$, $(c_0)_2^R$ denote the spaces of all, bounded, convergent in Pringsheims sense, null in Pringsheims sense, regularly convergent and regularly null double sequences respectively. The space $(\ell_\infty)_2$ is a normed linear space with respect to the norm $||A|| = \sup_{nk} |a_{nk}|$.

A double sequence space E is said to be solid if $(\alpha_{nk} a_{nk}) \in E$, whenever $(a_{nk}) \in E$ and for all sequences (α_{nk}) of scalars with $|\alpha_{nk}| \leq 1$, for all $n, k \in N$.

A double sequence space E is said to be symmetric if $(a_{\pi(nk)}) \in E$, whenever $(a_{nk}) \in E$, where π is a permutation of $N \times N$. A double sequence space E is said to be monotone if it contains the canonical pre-images of its step spaces.

A double sequence space E is said to be sequence algebra if $A \circ B = (a_{nk} b_{nk}) \in E$, whenever $A = (a_{nk}) \in E$ and $B = (b_{nk}) \in E$.

A double sequence space E is said to be convergence free if $B = (b_{nk}) \in E$, whenever $A = (a_{nk}) \in E$ and $a_{nk} = 0 \Rightarrow b_{nk} = 0$.

Tripathy [18] introduced the notion of density for the subsets of $N \times N$.

A subset E of $N \times N$ is said to have density $\rho(E)$ if

$$\lim_{p,q \to \infty} \frac{1}{pq} \sum_{n \leq p} \sum_{k \leq q} \chi_E(n,k) \quad \text{exists.}$$

Tripathy and Tripathy [30] introduced the notions of Logarithmic density and uniform density of a subset E of $N \times N$ as follows : let $s_n = \sum_{k=1}^{n} \frac{1}{k}$.

Then a subset E of $N \times N$ is said to have logarithmic density $\rho^*(E)$ if

$$\rho^*(E) = \lim_{p,q \to \infty} \frac{1}{s_p s_q} \sum_{n=1}^{p} \sum_{k=1}^{q} \frac{\chi_E(n,k)}{nk} \quad \text{exists.}$$

The above expression if exists is equivalent to the following :

$$\rho^*(E) = \lim_{p,q \to \infty} \frac{1}{\log p \log q} \sum_{n=1}^{p} \sum_{k=1}^{q} \frac{\chi_E(n,k)}{nk} \quad \text{exists.}$$

Let $p, q > 1$ and $s, t > 1$, be integers. Let $D \subseteq N \times N$ and $D(p+1, p+t; q+1, q+s) = \text{card } \{(n,k) \in D : p+1 \leq n \leq p+t \text{ and } q+1 \leq k \leq q+s\}$.

Put $\beta_{t,s} = \liminf_{p,q \to \infty} D(p+1, p+t; q+1, q+s)$ and $\beta^{t,s} = \limsup_{p,q \to \infty} D(p+1, p+t; q+1, q+s)$. Let $u_-(D) = \lim_{t,s \to \infty} \frac{\beta_{t,s}}{ts}$ exists and $u^-(D) = \lim_{t,s \to \infty} \frac{\beta^{t,s}}{ts}$ exists.

If $u_-(D) = u^-(D)$ then $u_-(D) = u^-(D)$ is called the uniform density of D.

3. Paranormed I-convergent Double Sequences

Throughout the ideals of 2^N will be denoted by I and the ideals of $2^{N\times N}$ will be denoted by I_2. A double sequence (a_{nk}) is said to be I-convergent to L in Pringsheims sense if for every $\varepsilon > 0$, the set $\{(n,k) \in N \times N : |a_{nk} - L| \geq \varepsilon\} \in I_2$. Let $(c^I)_2$ and $(c_0^I)_2$ denote the spaces of I-convergent and I-null double sequences respectively.

Throughout $p = (p_{nk})$ is a double sequence of positive numbers. The notion of paranormed double sequences was investigated by Turkmenoglu [31]. We define the following multiplier double sequence spaces.

$$(c^I)_2(\Lambda, p) = \{(a_{nk}) \in w_2 : I - \lim |\lambda_{nk}a_{nk} - L|^{p_{nk}} = 0 \text{ for some } L \in C\}$$

$$(c_0^I)_2(\Lambda, p) = \{(a_{nk}) \in w_2 : I - \lim |\lambda_{nk}a_{nk}|^{p_{nk}} = 0\}$$

$(a_{nk}) \in (c^I)_2^R(\Lambda, p)$ if and only if $(a_{nk}) \in (c^I)_2(\Lambda, p)$ and the following limits exist:

$I - \lim |\lambda_{nk}a_{nk} - L_k|^{p_{nk}} = 0$ for some L_k, for each $k \in N$

and $I - \lim |\lambda_{nk}a_{nk} - M_n|^{p_{nk}} = 0$ for some M_n, for each $n \in N$.

$(a_{nk}) \in (c_0^I)_2^R(\Lambda, p)$ if and only if $(a_{nk}) \in (c_0^I)_2(\Lambda, p)$ and

$I - \lim |\lambda_{nk}a_{nk}|^{p_{nk}} = 0$ for each $k \in N$

$I - \lim |\lambda_{nk}a_{nk}|^{p_{nk}} = 0$ for each $n \in N$.

Example 3.1. Let $I_2(P)$ be the class of all subsets of $N \times N$ such that $D \in I_2(P)$ implies that there exist $n_0, k_0 \in N$, such that $D \subseteq N \times N \setminus \{(n,k) \in N \times N : n \geq n_0, k \geq k_0\}$. Then $I_2(P)$ is an ideal of $2^{N\times N}$ and we will get definitions of paranormed convergent multiplier double sequence spaces. With $I_2(P)$, if we consider the ideal $I(f)$, the class of all finite subsets of N, then we get the paranormed regular convergent multiplier double sequence spaces.

Example 3.2. Let us consider $I_2(\rho) \subset 2^{N\times N}$ i.e. the class of all subsets of $N \times N$ of zero density. Then $I_2(\rho)$ is an ideal of $2^{N\times N}$. If $I(d)$ is the class of all subsets $A \subset N$ such that $d(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_A(k) = 0$, then $I(d)$ is an ideal of 2^N. Considering $I(d)$ along with $I_2(\rho)$ one will get different types of paranormed statistically convergent multiplier double sequence spaces.

Example 3.3. Let us consider $I_2(\rho^*) \subset 2^{N\times N}$ i.e. the class of all subsets of $N \times N$

of zero logarithmic density. Then $I_2(\rho^*)$ is an ideal of $2^{N\times N}$. If $I(\delta)$ is the class of all subsets A of N with $\delta(A) = \lim_{n\to\infty} \frac{1}{s_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k} = 0$, then $I(\delta)$ is an ideal of 2^N. With $I_2(\rho^*)$ if we consider $I(\delta)$, then we will get different types of paranormed logarithmic convergent multiplier double sequence spaces.

Example 3.4. Let us consider $I_2(u^*) \subset 2^{N\times N}$ i.e. the class of all subsets of $N \times N$ of zero uniform density. Then $I_2(u^*)$ is an ideal of $2^{N\times N}$. Along with $I_2(u^*)$, if we consider $I(u)$, the class of all subsets of N of uniform density zero, then we will get the definitions of different types of paranormed uniformly convergent multiplier double sequence spaces.

Let $(\ell_\infty)_2(\Lambda, p) = \{(a_{nk}) \in w_2 : \sup_{n,k} |\lambda_{nk}a_{nk}|^{p_{nk}} < \infty\}$

Also we define the following sequence spaces:

$$(c^I)_2^{BP}(\Lambda, p) = (c^I)_2(\Lambda, p) \cap (\ell_\infty)_2(\Lambda, p);$$
$$(c_0^I)_2^{BP}(\Lambda, p) = (c_0^I)_2(\Lambda, p) \cap (\ell_\infty)_2(\Lambda, p);$$
$$(c^I)_2^{BR}(\Lambda, p) = (c^I)_2^{R}(\Lambda, p) \cap (\ell_\infty)_2(\Lambda, p);$$
$$(c_0^I)_2^{BR}(\Lambda, p) = (c_0^I)_2^{R}(\Lambda, p) \cap (\ell_\infty)_2(\Lambda, p);$$

Note 3.1. If $p = (p_{nk})$ is a double sequence of positive real numbers and $H = \sup_{n,k} p_{nk} < \infty$, then for sequences (a_{nk}) and (b_{nk}) of complex numbers, we have the following inequality:

$$|a_{nk} + b_{nk}|^{p_{nk}} \le J(|a_{nk}|^{p_{nk}} + |b_{nk}|^{p_{nk}})$$

where $J = \max(1, 2^{H-1})$.

If $\sup_{n,k} p_{nk} < \infty$, then using the above inequality we can easily check that all the spaces defined above are linear spaces.

Let $A = (a_{nk})$ and $B = (b_{nk})$ be two double sequences. Then we say that $a_{nk} = b_{nk}$ for almost all n and k relative to I_2 (in short *a.a.n&kr.* I_2) if $\{(n, k) \in N \times N : a_{nk} \ne b_{nk}\} \in I_2$.

Let $A = (a_{nk})$ be a double sequence and I_2 be an ideal of $2^{N\times N}$. A subset D of C, the field of complex numbers is said to contain a_{nk} for *a.a.n&k r.* I_2 if $\{(n, k) \in N \times N : a_{nk} \notin D\} \in I_2$.

Lemma 3.1. *If a sequence space is solid, then it is monotone.*

Lemma 3.2. *The following are equivalent:*

(i) $(a_{nk}) \in (c^I)_2$ *and* $I - \lim a_{nk} = L$.

(ii) $(a_{nk} - L) \in (c^I)_2$.

(iii) *there exists a sequence* $(b_{nk}) \in c_2$ *such that* $a_{nk} = b_{nk}$ *for a.a.n&kr.*I_2.

(iv) *there exists a subset* $M = \{(n_i, k_j) \in N \times N : i, j \in N\}$ *of* $N \times N$ *such that* $M \in \Psi(I_2)$ *and* $(a_{n_i k_j} - L) \in (c_0)_2$.

(v) *there exists sequences* (x_{nk}), (y_{nk}) *such that* $a_{nk} = x_{nk} + y_{nk}$, *for all* $n, k \in N$, *where* $\lim x_{nk} = L$ *and* $(y_{nk}) \in (c_0^I)_2$.

Proof. The equivalence of (i) and (ii) is clear from definitions.

(i) $\Rightarrow$ (iii) Since $I - \lim |a_{nk} - L| = 0$, we have, for any $\varepsilon > 0$, the set $\{(n,k) \in N \times N : |a_{nk} - L| \geq \varepsilon\} \in I_2$. We select the increasing sequence (T_j) and (M_j) of natural numbers such that if $p > T_j$ and $q > M_j$, then the set

$$\{(n,k) \in N \times N : n \leq p,\ k \leq q \text{ and } |a_{nk} - L| \geq \frac{1}{j}\} \in I_2.$$

We define the sequence (b_{nk}) as follows:

$b_{nk} = a_{nk}$ if $n \leq T_1$ or $k \leq M_1$. Also for all (n,k) with $T_j < n \leq T_{j+1}$ or $M_j < k \leq M_{j+1}$, let $b_{nk} = a_{nk}$ if $|a_{nk} - L| < \frac{1}{j}$, otherwise $b_{nk} = L$.

We show that (b_{nk}) converges to L. Let $\varepsilon > 0$. We choose ε such that $\varepsilon > \frac{1}{j}$. We see that for $n > T_j$ and $k > M_j$, $|b_{nk} - L| < \varepsilon$. Hence $\lim b_{nk} = L$.

Next we assume that $T_j < n \leq T_{j+1}$ or $M_j < k \leq M_{j+1}$, then the set $A = \{(n,k) \in N \times N : a_{nk} \neq b_{nk}\} \subseteq \{(n,k) \in N \times N : |a_{nk} - L| \geq \frac{1}{j}\} \in I_2$.

Hence $A \in I_2$ and $a_{nk} = b_{nk}$ for *a.a.n&kr.* I_2.

(iii) $\Rightarrow$ (iv) Suppose there exist a sequence $(b_{nk}) \in c_2$ such that $a_{nk} = b_{nk}$ for *a.a.n&kr.* I_2. Let $M = \{(n,k) \in N \times N : a_{nk} = b_{nk}\}$. Then $M \in \Psi(I_2)$. We can enumerate M as $M = \{(n_i, k_j) \in N \times N : i, j \in N\}$, on neglecting the rows and columns those contain finite number of elements. Then $|a_{n_i k_j} - L| \to 0$ as $i, j \to \infty$.

(iv) $\Rightarrow$ (v) we define the sequences (x_{nk}), (y_{nk}) as follows:

$$\begin{aligned} x_{nk} &= a_{nk}, \text{ if } (n,k) \in M \\ &= L, \text{ otherwise} \end{aligned}$$

$$\begin{aligned} y_{nk} &= 0, \text{ if } (n,k) \in M \\ &= a_{nk} - L, \text{ otherwise.} \end{aligned}$$

Then we can easily check that the conditions of (v) hold.
(v) $\Rightarrow$ (i) Suppose the conditions of (v) hold. Then for any $\varepsilon > 0$, the sets

$A = \{(n,k) \in N \times N : |x_{nk} - L| < \frac{\varepsilon}{2}\} \in \Psi(I_2)$ and $B = \{(n,k) \in N \times N : |y_{nk}| < \frac{\varepsilon}{2}\} \in \Psi(I_2)$.
Now for each $(n,k) \in A \cup B$ we have $|a_{nk} - L| < \varepsilon$ and $A \cap B \in \Psi(I_2)$.
Hence $I - \lim a_{nk} = L$. This completes the proof of the lemma. $\square$

4. Main Results

Theorem 4.1. *If* $0 < \inf_{n,k} p_{nk} < \sup_{n,k} p_{nk} < \infty$, *then the spaces* $(\ell_\infty)_2(\Lambda, p)$, $(c^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$ *are paranormed spaces, paranormed by*

$$g((a_{nk})) = \sup_{n,k} |\lambda_{nk} a_{nk}|^{\frac{p_{nk}}{M}}, \quad \text{where} \quad M = \max(1, \sup_{n,k} p_{nk}).$$

Proof. We consider the space $(\ell_\infty)_2(\Lambda, p)$. Clearly for any $A = (a_{nk}) \in (\ell_\infty)_2(\Lambda, p)$, $g(\theta_2) = 0$, where θ_2 is the double sequence, whose all the terms are zero, $g(-A) = g(A)$ and $g(A) \geq 0$. Also $g(A + B) \leq g(A) + g(B)$.

Let $A \to \theta_2$, $\alpha \to 0$, then $g(A) \to 0$. We have for a given scalar α

$$g(\alpha A) = \sup_{n,k} |\alpha \lambda_{nk} a_{nk}|^{\frac{p_{nk}}{M}} < \max(1, |\alpha|) g(A).$$

Thus α fixed and $A \to \theta_2 \Rightarrow g(A) \to 0$. Next let $\alpha \to 0$ and A is fixed. Without loss of generality we can take $|\alpha| < 1$. Then for given $A = (a_{nk})$,

$$g(\alpha A) = \sup_{n,k} |\alpha \lambda_{nk} a_{nk}|^{\frac{p_{nk}}{M}} \leq |\alpha|^{\frac{h}{H}} g(A) \to 0 \text{ as } \alpha \to 0,$$

where $h = \inf_{n,k} p_{nk} > 0$.
Hence $(\ell_\infty)_2(\Lambda, p)$ is paranormed by g.
Similarly we can prove that the other spaces are paranormed by g. $\square$

The proof of the following result is easy, so omitted.

Theorem 4.2. *If* $0 < \inf_{n,k} p_{nk} < \sup_{n,k} p_{nk} < \infty$, *then the spaces* $(\ell_\infty)_2(\Lambda, p)$, $(c^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$ *are complete paranormed spaces, paranormed by* g.

Theorem 4.3. *The following are equivalent:*
(i) *The double sequence* $(a_{nk}) \in (c_0^I)_2(\Lambda, p)$.

(ii) *there exists a sequence* $(b_{nk}) \in (c_0)_2(\Lambda, p)$ *such that* $a_{nk} = b_{nk}$ *for a.a.n&kr.* I_2.

(iii) *there exists a subset* $M = \{(n_i, k_j) \in N \times N : i, j \in N\}$ *of* $N \times N$ *such that* $M \in \Psi(I_2)$ *and* $\lim_{i,j} |\lambda_{n_i k_j} a_{n_i k_j}|^{p_{n_i k_j}} = 0$.

(iv) *there exists sequences* (x_{nk}), (y_{nk}) *such that* $a_{nk} = x_{nk} + y_{nk}$, *for all* $n, k \in N$, *where* $(x_{nk} \in (c_0)_2(\Lambda, p))$ *and* $(y_{nk} \in (c_0^I)_2(\Lambda, p))$ *such that* $\{(n,k) \in N \times N : y_{nk} \neq 0\} \in I_2$.

Proof. Let $(a_{nk}) \in (c_0^I)_2(\Lambda, p)$. Then $I - \lim |\lambda_{nk} a_{nk}|^{p_{nk}} = 0$. Let us write $d_{nk} = |\lambda_{nk} a_{nk}|^{p_{nk}}$. Then $(d_{nk}) \in (c_0^I)_2$. The result follows from lemma 3.2. □

Theorem 4.4. *The spaces* $(c_0^I)_2(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^R(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$ *are solid as well as monotone.*

Proof. Let $(a_{nk}) \in (c_0^I)_2(\Lambda, p)$. Let (α_{nk}) be a double sequence of scalars such that $|\alpha_{nk}| \leq 1$, for all $n, k \in N$. Let $\varepsilon > 0$ be given. Then the solidness of the space follows from the following inclusion relation

$$\{(n,k) \in N \times N : |\lambda_{nk} a_{nk}|^{p_{nk}} \geq \varepsilon\} \supseteq \{(n,k) \in N \times N : |\alpha_{nk} \lambda_{nk} a_{nk}|^{p_{nk}} \geq \varepsilon\}.$$

Similarly we can show that the other spaces are solid. Also by lemma 3.1, it follows that the spaces are monotone. □

Theorem 4.5. *If* I_2 *is not maximal, then the spaces* $(c^I)_2(\Lambda, p)$, $(c^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^R(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$ *are neither solid nor monotone.*

Proof. We first show that $(c^I)_2(\Lambda, p)$ is not monotone. Let $a_{nk} = 1$ for all $n, k \in N$, $\lambda_{nk} = 1$ for all $n, k \in N$. Then $(a_{nk}) \in (c^I)_2(\Lambda, p)$. Since I_2 is not maximal, there exists a subset K of $N \times N$ such that $K \notin I_2$ and $N \setminus K \notin I_2$. Let us define the sequence (b_{nk}) by

$$\begin{aligned} b_{nk} &= a_{nk}, \text{ if } (n,k) \in K; \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then (b_{nk}) belongs to the canonical preimage of K-step space of (a_{nk}). But $(b_{nk}) \notin (c^I)_2(\Lambda, p)$. Hence $(c^I)_2(\Lambda, p)$ is not monotone and so from lemma 3.1 it is not solid. Similarly we can prove that the other spaces are not solid by constructing suitable examples. □

Theorem 4.6. *If* I_2 *is neither maximal nor* $I_2 = I_2(f)$ (*ideal of all finite subsets of* $N \times N$), *then the spaces* $(c_0^I)_2(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$, $(c_0^I)_2^R(\Lambda, p)$, $(c^I)_2(\Lambda, p)$, $(c^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^R(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$ *are not symmetric.*

Proof. The proof follows from the following example:

Example 4.1. Let $\lambda_{nk} = 1$ for all $n, k \in N$. We first consider the space $(c_0^I)_2(\Lambda, p)$. Let us consider the sequence (a_{nk}) defined by

$$\begin{aligned} a_{nk} &= 1, \text{ if } (n,k) \in A; \\ &= 0, \text{ otherwise,} \end{aligned}$$

where $A \in I_2$ is infinite. Then $(a_{nk}) \in (c_0^I)_2(\Lambda, p)$

Let $K \subseteq N \times N$ be such that $K \notin I_2$ and $N \backslash K \notin I_2$. Consider the rearrangement (b_{nk}) of (a_{nk}) as follows:

$$\begin{aligned} b_{nk} &= 1, \text{ if } (n,k) \in K; \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then $(b_{nk}) \notin (c_0^I)_2(\Lambda, p)$. Hence $(c_0^I)_2(\Lambda, p)$ is not symmetric. □

Theorem 4.7. *The spaces* $(c_0^I)_2(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^R(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$, $(c^I)_2(\Lambda, p)$, $(c^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^R(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$ *are not sequence algebras.*

Proof. The proof follows from the following example:

Example 4.2. Consider the space $(c_0^I)_2(\Lambda, p)$. Let $I_2 = I_2(P)$. Also let $\lambda_{nk} = \frac{1}{n^2k^2}$ for all $n, k \in N$ and $p_{nk} = 1$ for all $n, k \in N$. Let us consider the sequences (a_{nk}) and (b_{nk}) defined as follows.

$a_{1k} = k^3, a_{n1} = n^3$ for all $n, k \in N$ and $a_{nk} = nk$ otherwise.

$b_{1k} = 2k^3, a_{n1} = 2n^3$ for all $n, k \in N$ and $a_{nk} = 2nk$ otherwise.

Then $(a_{nk}) \in (c_0^I)_2(\Lambda, p)$ and $(b_{nk}) \in (c_0^I)_2(\Lambda, p)$, but $(a_{nk}b_{nk}) \notin (c_0^I)_2(\Lambda, p)$. □

Similarly we can show that the other spaces are not sequence algebras.

Theorem 4.8. *The spaces* $(c_0^I)_2(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^R(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$, $(c^I)_2(\Lambda, p)$, $(c^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^R(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$ *are not convergence free.*

Proof. We consider the space $(c^I)_2(\Lambda, p)$. The proof follows from the following example:

Example 4.3. Let $I = I_2(P)$. Also let $\lambda_{nk} = (nk)^{-1}$, for all $n, k \in N$ and $p_{nk} = 1$ for all $n, k \in N$. Let us consider the sequence (a_{nk}) defined by $a_{1k} = 0 = a_{n1}$ for all $n, k \in N$ and $a_{nk} = nk$ otherwise.

Then $(a_{nk}) \in (c^I)_2(\Lambda, p)$.

Consider the sequence (b_{nk}) defined by $b_{1k} = 0 = a_{n1}$ for all $n, k \in N$ and $b_{nk} = n^2k^2$ otherwise.

Then $(b_{nk}) \notin (c^I)_2(\Lambda, p)$. Hence $(c^I)_2(\Lambda, p)$ is convergence free. □

Theorem 4.9. *Let $p = (p_{nk})$, $q = (q_{nk})$ be two sequences of positive real numbers. Then $(c_0^I)_2^{BP}(\Lambda, p) \supseteq (c_0^I)_2^{BP}(\Lambda, q)$ if and only if*

$\liminf_{(n,k)\in K} (\frac{p_{nk}}{q_{nk}}) > 0$ *where $K \subseteq N \times N$ such that $K \in \Psi(I_2)$.*

Proof. Let

$$\liminf_{(n,k)\in K} (\frac{p_{nk}}{q_{nk}}) > 0 \text{ for some } K \in \Psi(I_2). \tag{4.1}$$

Then there exists $\alpha > 0$ such that $p_{nk} > \alpha q_{nk}$ for sufficiently large pair $(n, k) \in K$. Let $(a_{nk}) \in (c_0^I)_2^{BP}(\Lambda, p)$. Then for given $\varepsilon > 0$ we have

$$A = \{(n, k) \in N \times N : |\lambda_{nk}a_{nk}|^{q_{nk}} < \varepsilon\} \in \Psi(I_2).$$

Let $B = A \cap K$. Then $B \in \Psi(I_2)$. Now for all $(n, k) \in B$, $|\lambda_{nk}a_{nk}|^{p_{nk}} \leq (|\lambda_{nk}a_{nk}|^{q_{nk}})^{\alpha}$.
Thus $(a_{nk}) \in (c_0^I)_2^{BP}(\Lambda, p)$.
Conversely let $(c_0^I)_2^{BP}(\Lambda, p) \supseteq (c_0^I)_2^{BP}(\Lambda, q)$, but there exists no $K \in \Psi(I_2)$ such that (4.1) holds. Then there exists $C = \{(n_i, k_j) : i, j \in N\} \subset N \times N$ with $C \notin I_2$ such that $ip_{n_ik_j} < q_{n_ik_j}$. We define the sequence (a_{nk}) by

$$\begin{aligned} a_{nk} &= \frac{(i^{-1})^{\frac{1}{q_{n_ik_j}}}}{|\lambda_{n_ik_j}|}, \text{ if } n = n_i; k = k_j; \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then $(a_{nk}) \in (c_0^I)_2^{BP}(\Lambda, q)$. But $|\lambda_{n_ik_j}a_{n_ik_j}|^{p_{n_ik_j}} > exp(-i^{-1}logi)$.
Hence $(a_{nk}) \notin (c_0^I)_2^{BP}(\Lambda, p)$, a contradiction.
Hence the theorem. □

Theorem 4.10. *The spaces $(c^I)_2^{BP}(\Lambda, p)$, $(c_0^I)_2^{BP}(\Lambda, p)$, $(c^I)_2^{BR}(\Lambda, p)$, $(c_0^I)_2^{BR}(\Lambda, p)$ are not separable. Proof.* Consider the space $(c^I)_2^{BP}(\Lambda, p)$. Let A be a countable dense subset of $(c^I)_2^{BP}(\Lambda, p)$. Then any point of $(c^I)_2^{BP}(\Lambda, p)$ is either a point of A or a limit point of A. Let $M \in I_2$ be such that M is infinite. Also let $L = \{(a_{nk}) : a_{nk} = 0 \text{ if } (n, k) \notin M \text{ and } a_{nk} = 0 \text{ or } 1 \text{ otherwise.}\}$ Then L is uncountable. Now we consider open balls of radius $\delta < \frac{1}{2}$ with centers at the points of L. Then these balls are disjoint and uncountable in number. Also each ball must contain at least one point of A. Hence we arrive at a contradiction. Thus $(c^I)_2^{BP}(\Lambda, p)$ is not separable. □

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