

# Stationary analysis of the surplus process in a risk model with investments<sup>†</sup>

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## Abstract

We consider a continuous time surplus process with investments the sizes of which are independent and identically distributed. It is assumed that an investment of the surplus to other business is made, if and only if the surplus reaches a given sufficient level. We establish an integro-differential equation for the distribution function of the surplus and solve the equation to obtain the moment generating function for the stationary distribution of the surplus. As a consequence, we obtain the first and second moments of the level of the surplus in an infinite horizon.

*Keywords:* Integro-differential equation, moment generating function, risk model, stationary distribution, surplus process.

## 1. Introduction

In this paper, we consider a continuous time surplus process in a risk model with investments. The surplus, with initial level of  $u > 0$ , linearly increases at a constant rate  $c > 0$  due to the incoming premium. Meanwhile, claims arrive according to a Poisson process of rate  $\lambda > 0$  and decrease the level of the surplus jump-wise by random amounts which are independent and identically distributed with distribution function  $G$  of mean  $\mu > 0$ . The premium rate  $c$  is usually assumed to be larger than  $\lambda\mu$  which is the expected amount of claims per unit time. Hence, the surplus process goes eventually to infinity with probability 1 in the classical risk model.

In the present risk model, it is assumed that an investment of the surplus to other business is made whenever the surplus reaches a sufficient level  $V > u$ . The amounts of investments are independent and identically distributed with distribution function  $H$ . To analyze stochastically the level of the surplus in an infinite horizon, we also assume that the surplus process continues to move even though the level of the surplus becomes negative. In the classical risk model, we stop the surplus process and say that a ruin occurs if the level of the surplus goes below 0. However, in practice, the insurance company never lets the surplus of a policy get

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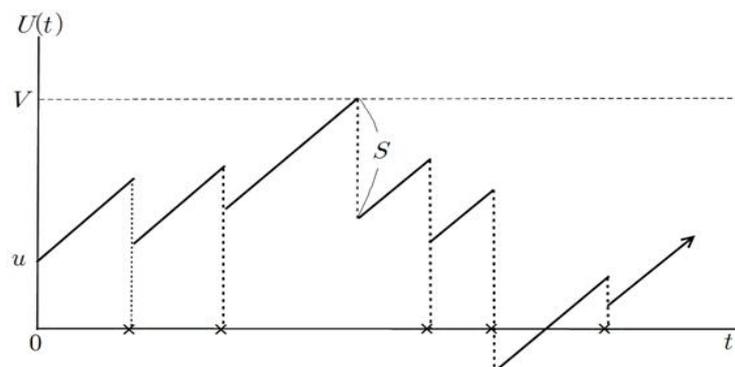
infinitely large and/or still manages to operate the policy though the surplus of the policy is exhausted. Hence, our assumptions are very practical.

The concept of the investment of the surplus was introduced by Jeong *et al.* (2009) and Jeong and Lee (2010). They studied some optimal policies related to managing the surplus in the risk model. Cho *et al.* (2013) also studied some transient and stationary behaviors of the surplus in the risk model with investments. However, in all previous models, the amount of the investment was a constant.

The classical risk model has been studied by many authors by assuming that a ruin occurs when the surplus becomes negative. They have obtained the ruin probability of the surplus and some interesting characteristics of the risk model. The core result on the ruin probability is well summarized in Klugman *et al.* (2004). The first passage time of the surplus to a certain level was introduced by Gerber (1990), thereafter, Gerber and Shiu (1997) obtained the joint distribution of the time of ruin, the surplus before ruin and the deficit at ruin. Dickson and Willmot (2005) calculated the density of the time to ruin by an inversion of its Laplace transform. Kwon *et al.* (2014) studied some approximations of the ruin probability of the surplus in the classical risk model.

Dufresne and Gerber (1991) generalized the classical risk model by assuming that the surplus is perturbed by diffusion between the time points of occurrence of claim and studied the ruin probabilities of the surplus. Won *et al.* (2013) generalized the risk model of Dufresne and Gerber (1991) by assuming that there are two types of claim, where one occurs more frequently but the size is stochastically less than the other, and studied the ruin probabilities of the surplus. Song *et al.* (2012) generalized the classical risk model by assuming that the premium rate depends on the current level of the surplus and obtained the joint distribution of the time of ruin, the surplus before ruin and the deficit at ruin.

However, until now, most works have been concentrated on the ruin probability of the surplus and its related characteristics. In this paper, by assuming that the surplus process continues though the level of it goes below 0 and an investment to other business is made when the level of it reaches a sufficient level  $V$ , we study the long-term behavior of the surplus in our modified risk model. In Section 2, we establish an integro-differential equation for the distribution function of the surplus and solve the equation to obtain the moment generating function for the stationary distribution of the surplus. In Section 3, by differentiating the moment generating function, we obtain the first and second moments of the level of the surplus in the long-run (in an infinite horizon).



**Figure 1.1** A sample path of the surplus process with investment

The surplus process in the risk model with investments is illustrated in Figure 1.1, where  $U(t)$  is the level of the surplus at time  $t \geq 0$ ,  $\times$  denotes the time epoch where the claim occurs, and  $S$  is the amount of the investment to other business following distribution function  $H$ .

### 2. Moment generating function of the surplus

Let  $F(x, t)$  be the distribution function of  $U(t)$ , the surplus at time  $t > 0$ , that is,

$$F(x, t) = P\{U(t) \leq x\}, \text{ for } -\infty < x \leq V.$$

Let  $Y$  be a random variable denoting the amount of a claim which follows distribution function  $G$  with mean  $\mu$ . Conditioning on whether a claim arrives in a small interval  $[t, t + \Delta t]$ , we can have the following relations between  $U(t)$  and  $U(t + \Delta t)$ :

(i) If no claims arrive, then

$$U(t + \Delta t) = \begin{cases} U(t) + c\Delta t, & \text{when } U(t) \leq V - c\Delta t \\ V - S + c\Delta t', & \text{when } V - c\Delta t < U(t) \leq V, \end{cases}$$

where  $0 < \Delta t' \leq \Delta t$ .

(ii) If a claim arrives, then

$$U(t + \Delta t) = U(t) + c\Delta t - Y.$$

Note that two or more claims arrive in a small interval  $[t, t + \Delta t]$  is  $o(\Delta t)$ . Hence, we can obtain the following equation for  $-\infty < x \leq V$ ,

$$\begin{aligned} P\{U(t + \Delta t) \leq x\} &= [1 - \lambda\Delta t + o(\Delta t)]P\{U(t) \leq x - c\Delta t, U(t) \leq V - c\Delta t\} \\ &\quad + [1 - \lambda\Delta t + o(\Delta t)]P\{V - S \leq x - c\Delta t', V - c\Delta t < U(t) \leq V\} \\ &\quad + [\lambda\Delta t + o(\Delta t)]P\{U(t) - Y \leq x - c\Delta t\} + o(\Delta t). \end{aligned}$$

Representing the above equation for  $F(x, t)$ , we have

$$\begin{aligned} F(x, t + \Delta t) &= [1 - \lambda\Delta t + o(\Delta t)][F(x - c\Delta t, t) + P\{S \geq V - x + c\Delta t'\}\{F(V, t) - F(V - c\Delta t, t)\}] \\ &\quad + [\lambda\Delta t + o(\Delta t)]P\{U(t) - Y \leq x - c\Delta t\} + o(\Delta t). \end{aligned}$$

Here, applying Taylor series expansion on  $F(x - c\Delta t, t)$  gives

$$F(x - c\Delta t, t) = F(x, t) - c\Delta t \frac{\partial}{\partial x} F(x, t) + o(\Delta t).$$

Observe also that conditioning on  $Y$  yields

$$\begin{aligned} P\{U(t) - Y \leq x - c\Delta t\} &= \int_0^\infty F(x - c\Delta t + y, t) dG(y) \\ &= \int_0^{V-x} F(x + y, t) dG(y) + \bar{G}(V - x) \end{aligned}$$

where  $\bar{G}(y) = 1 - G(y), y > 0$ , since  $F(x, t) = 1$  for  $x \geq V$ . Inserting these two into the above equation, we have

$$\begin{aligned}
 F(x, t + \Delta t) &= F(x, t) - c\Delta t \frac{\partial}{\partial x} F(x, t) - \lambda\Delta t F(x, t) \\
 &\quad + \bar{H}(V - x + c\Delta t) \{F(V, t) - F(V - c\Delta t, t)\} \\
 &\quad + \lambda\Delta t \left[ \int_0^{V-x} F(x + y, t) dG(y) + \bar{G}(V - x) \right] + o(\Delta t).
 \end{aligned}$$

Subtracting  $F(x, t)$  from both sides of the equation, dividing by  $\Delta t$ , and letting  $\Delta t \rightarrow 0$ , we have the following integro-differential equation for  $F(x, t)$ :

$$\begin{aligned}
 \frac{\partial}{\partial t} F(x, t) &= -c \frac{\partial}{\partial x} F(x, t) - \lambda F(x, t) + cf(V, t) \bar{H}(V - x) \\
 &\quad + \lambda \int_0^{V-x} F(x + y, t) dG(y) + \lambda \bar{G}(V - x),
 \end{aligned} \tag{2.1}$$

where  $f(V, t) = \frac{\partial}{\partial x} F(x, t)|_{x=V}$ . Though the integro-differential equation (2.1) looks complicate, we can solve it, numerically, to obtain  $F(x, t)$ , the distribution function of the surplus at finite time  $t$ . However, in this paper, we concentrate on obtaining, theoretically, the stationary distribution of the surplus in the long-run of the process.

Let  $F(x) = \lim_{t \rightarrow \infty} F(x, t)$  be the stationary (limiting) distribution function of  $U(t)$ . Note that in our process the stationary distribution is equal to the limiting distribution, since  $U(t)$  is a regenerative process. Now, in the stationary case,  $\frac{\partial}{\partial t} F(x, t) = 0$ , and hence,  $F(x)$  satisfies

$$0 = -c \frac{\partial}{\partial x} F(x) - \lambda F(x) + cf(V) \bar{H}(V - x) + \lambda \int_0^{V-x} F(x + y) dG(y) + \lambda \bar{G}(V - x). \tag{2.2}$$

Let  $f(x) = \frac{d}{dx} F(x)$  be the stationary density function of  $U(t)$ .  $f(x)$  is the long-run density that the level of the surplus is near  $x$  in an infinite horizon. By differentiating (2.2) with respect to  $x$ , we can see that  $f(x)$  satisfies

$$0 = -cf'(x) - \lambda f(x) + cf(V)h(V - x) + \lambda \int_0^{V-x} f(x + y)g(y)dy, \tag{2.3}$$

where  $g(y) = \frac{d}{dy} G(y)$  and  $h(x) = \frac{d}{dx} H(x)$ .

Let  $f^*(r) = \int_{-\infty}^V e^{rx} f(x) dx$  be the moment generating function of  $f(x)$ . Observe that

$$\begin{aligned}
 \int_{-\infty}^V e^{rx} f'(x) dx &= e^{rV} f(V) - rf^*(r), \\
 \int_{-\infty}^V e^{rx} h(V - x) dx &= e^{rV} h^*(-r) \text{ and} \\
 \int_{-\infty}^V e^{rx} \int_0^{V-x} f(x + y)g(y) dy dx &= f^*(r)g^*(-r),
 \end{aligned}$$

where  $g^*(r) = \int_{-\infty}^V e^{ry}g(y)dy$  and  $h^*(r) = \int_{-\infty}^V e^{rx}h(x)dx$ . Hence, taking the moment generating function on (2.3) gives

$$0 = -c[e^{rV}f(V) - rf^*(r)] - \lambda f^*(r) + cf(V)e^{rV}h^*(-r) + \lambda f^*(r)g^*(-r). \tag{2.4}$$

Solving (2.4) for  $f^*(r)$ , we have

$$f^*(r) = \frac{ce^{rV}f(V)[1 - h^*(-r)]}{cr - \lambda + \lambda g^*(-r)}. \tag{2.5}$$

Since  $f^*(0) = 1$ , we can obtain  $f(V)$  by applying l'Hôpital's rule to (2.5), which is given by  $f(V) = \frac{c-\lambda\mu}{cE(S)}$ .

Finally,  $f^*(r)$  is given by

$$f^*(r) = \frac{e^{rV}(c - \lambda\mu)[1 - h^*(-r)]}{E(S)[cr - \lambda + \lambda g^*(-r)]}. \tag{2.6}$$

### 3. Moments of the surplus

Differentiating  $f^*(r)$  with respect to  $r$ , we can obtain the moments of  $U$ , the level of the surplus in the long-run (in an infinite horizon). To do that, let

$$A(r) = cr - \lambda + \lambda g^*(-r),$$

then  $A(0) = 0, A'(0) = c - \lambda\mu, A''(0) = \lambda\mu_2$  and  $A'''(0) = -\lambda\mu_3$ , where  $\mu_k = E(Y^k)$  for  $k = 2, 3, \dots$ . Let  $B(r) = 1 - h^*(-r)$ , then  $B(0) = 0, B'(0) = E(S), B''(0) = -E(S^2)$  and  $B'''(0) = E(S^3)$ .

Now,  $f^*(r)$  can be written as

$$f^*(r) = \frac{e^{rV}(c - \lambda\mu)B(r)}{E(S)A(r)}.$$

Differentiating  $f^*(r)$  with respect to  $r$  gives

$$f^{*'}(r) = \frac{(c - \lambda\mu)\{[Ve^{rV}B(r) + e^{rV}B'(r)]A(r) - e^{rV}B(r)A'(r)\}}{E(S)[A(r)]^2}.$$

Applying L'Hopital's rule twice gives the first moment of  $U$ , given by

$$E(U) = f^{*'}(0) = \frac{(c - \lambda\mu)[2VE(S) - E(S^2)] - \lambda\mu_2E(S)}{2(c - \lambda\mu)E(S)}.$$

Similarly, after tedious calculations, we can obtain the second moment of  $U$ , which is given by

$$\begin{aligned} E(U^2) &= f^{*''}(0) \\ &= \frac{(\lambda\mu_2)^2}{2(c-\lambda\mu)^2} + \frac{2\lambda\mu_3E(S) - 3\lambda\mu_2\{2VE(S) - E(S^2)\}}{6(c-\lambda\mu)E(S)} + \frac{3V^2E(S) - 3VE(S^2) + E(S^3)}{3E(S)}. \end{aligned}$$

Finally, the variance of  $U$  is  $Var(U) = E(U^2) - [E(U)]^2$ .

$E(U)$  can be interpreted as the average amount of the surplus in the long-run of the process and  $Var(U)$  may represent the variability of the level of the surplus in the long-run of the process.

The result of the paper can be a useful information to help the insurance company how to manage the surplus of an insurance policy, efficiently, in the long-run.

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