NORM OF THE COMPOSITION OPERATOR FROM BLOCH SPACE TO BERGMAN SPACE

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Abstract. In this paper, we study some quantity equivalent to the norm of Bloch to $A^p_\alpha$ composition operator where $A^p_\alpha$ is the weighted Bergman space on the unit ball of $\mathbb{C}^n$ ($0 < p < \infty$ and $-1 < \alpha < \infty$).

1. Introduction

Let $n \geq 1$ be an integer. Let $H(B)$ denote the space of all holomorphic functions in the unit ball $B \equiv B_n$ of the complex $n$-dimensional Euclidean space $\mathbb{C}^n$. Let $U$ stand for $B^1$. The Bloch space $B = B(U)$ is defined by $B = \{f \in H(U) : \sup_{z \in U}|f'(z)|(1 - |z|^2) < \infty\}$. With the norm $\|f\|_B = |f(0)| + \sup_{z \in U}|f'(z)|(1 - |z|^2)$, $B$ is a Banach space. Let $\nu$ denote the Lebesgue measure on $\mathbb{C}^n$, so normalized that $\nu(B) = 1$. For $-1 < \alpha < \infty$, we set $c_\alpha = \Gamma(n + \alpha + 1)/\Gamma(n + 1)\Gamma(\alpha + 1)$ and $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$, $z \in B$. Note that $\nu_\alpha(B) = 1$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha \equiv A^p_\alpha(B)$ is defined by $A^p_\alpha = \{f \in H(B) : \int_B|f(z)|^pd\nu_\alpha(z) < \infty\}$.

For $f \in A^p_\alpha$, we write $\|f\|_{A^p_\alpha} = \left(\int_B|f(z)|^pd\nu_\alpha(z)\right)^{\frac{1}{p}}$.

In 2003, Kwon and Lee proved the following theorem. This result relates to the pull-back property (see [1], [2], [5], [7] for example).

Theorem 1 ([6]). Let $f : U \to U$ be a holomorphic function with $f(0) = 0$. For $1 \leq p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C^0_0 : B \to A^p_\alpha(U)$ defined by $C^0_0 h = h \circ f - h(0)$ has its operator norm equivalent to the quantity

$$\left\{\int_U (1 - |z|)^{\alpha+p} \left(\frac{|f'(z)|}{1 - |f(z)|^2}\right)^p dxdy\right\}^{\frac{1}{p}}.$$ (1.1)

In the present paper we study their analogues in “$B$ and $0 < p < \infty$” in place of “$U$ and $1 \leq p < \infty$”. The following is the main result of this paper.

Theorem 2. Let $f : B \to U$ be a holomorphic function with $f(0) = 0$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the bounded operator $C^0_f : B \to A^p_\alpha(B)$ defined...
by $C^0 h = h \circ f - h(0)$ has its operator norm equivalent to the quantity

$$\left\{ \int_B \left(1 - |z|^2 \right)^{\alpha + p} \left( \frac{|
abla f(z)|}{1 - |f(z)|^2} \right)^p \, d\nu(z) \right\}^{\frac{1}{p}},$$

where $\nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right)$ is the holomorphic gradient of $f$ at $z$.

That is, there are positive constants $C_1$ and $C_2$ independent of $f$ such that $C_1 ||C^0|| \leq (1.2) \leq C_2 ||C^0||$, where $||C^0||$ is the operator norm of $C^0$.

**Proposition 3** ([3]). If $f : B \to U$ is a holomorphic function, then

$$|\nabla f(z)| \leq \sqrt{n} \cdot \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in B.$$ 

**Proof.** By ([3] page 532, Remark 4),

$$\left| \frac{\partial f(z)}{\partial z_j} \right| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in B, \quad j = 1, \ldots, n.$$ 

By (1.3), the result holds. 

By Proposition 3, (1.2) stays bounded for any holomorphic function $f : B \to U$. In fact, (1.2) $\leq c_{\alpha}^{\frac{1}{p}} \sqrt{n} < \infty$.

Note that we may see $A^p_{\alpha+1} = H^p$, where $H^p$ is the Hardy space on $B$. When $n = 1$, Kwon and Lee researched the norm of Bloch-$H^p$ pull-back operator ([6]). When $n \geq 1$, Kwon studied their result extensively ([5]).

## 2. Notation

For $a \in B$, let $\varphi_a$ be the standard automorphism of $B$ taking 0 to $a$ ([8], page 25). Let $\lambda$ be the measure on $B$ defined by $d\lambda(z) = \frac{1}{(1 - |z|^2)^{\alpha + 1}} \, d\nu(z)$, $z \in B$.

For $f \in C^2(B)$ and $a \in B$, define $\bar{\Delta} f(a) = \frac{1}{n+1} \Delta (f \circ \varphi_a)(0)$, where $\Delta \equiv 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ is the ordinary Laplacian. We say that a function $f \in C^2(B)$ is $\mathcal{M}$-harmonic in $B$, if $\bar{\Delta} f(z) = 0$ for all $z \in B$. Note that $f \in H(\Delta)$ then $f$ is $\mathcal{M}$-harmonic in $B$ ([8], page 59). Let $\nabla$ denote the gradient with respect to the Bergman metric on $B$ ([10], page 27). For $f \in H(B)$,

$$\left| \nabla f(a) \right|^2 = \frac{2}{n+1} (1 - |a|^2) \left[ \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(a) \right|^2 - \sum_{j=1}^{n} a_j \frac{\partial f}{\partial z_j}(a) \right]^2,$$

$a \in B$ ([10], page 30).

When $n = 1$, we note that $|\nabla f(a)|^2 = (1 - |a|^2) |f'(a)|^2$, $a \in U$ ([10], page 30).
3. Preliminaries

For the proof of Theorem 2, the following lemmas will be needed.

**Lemma 4** ([4], page 78). Let $f$ be $\mathcal{M}$-harmonic on $B$.

(i) For all $p$, $0 < p < \infty$, and $s \in \mathbb{R}$, there exists a constant $C$, independent of $f$, such that
\[
\int_B (1 - |z|^2)^s |\nabla f(z)|^p d\lambda(z) \leq C \int_B (1 - |z|^2)^s |f(z)|^p d\lambda(z).
\]

(ii) For all $p$, $0 < p < \infty$, and $s > n$, there exists a constant $C$, independent of $f$, such that
\[
\int_B (1 - |z|^2)^s |f(z)|^p d\lambda(z) \leq C \left( |f(0)|^p + \int_B (1 - |z|^2)^{s+p} |\nabla f(z)|^p d\lambda(z) \right).
\]

**Lemma 5.** If $f$ is holomorphic in $B$, then
\[
\sqrt{2} \sqrt{n + 1} (1 - |z|^2) |\nabla f(z)| \leq |\tilde{\nabla} f(z)|
\]
for all $z \in B$.

**Proof.** By (2.1) and the Schwarz inequality, the result holds. \[\square\]

**Lemma 6** ([7], page 601). There exist two functions $h_1, h_2 \in B(U)$ such that
\[
|h_1'(z)| + |h_2'(z)| \geq \frac{1}{1 - |z|}
\]
for all $z \in U$.

4. Proof of the main theorem

The essential idea of the proof is the same as the one in [6] except using Lemma 4. For the sake of completeness, we give the proof. Hereafter the letter $K$ stands for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

By Lemma 4(ii) and $|\nabla (h \circ f)(z)| = |h'(f(z))\nabla f(z)| \leq \|h\|_B \frac{\nabla f(z)}{|f(z)|}$,
\[
\|C_f^p\| = c_0^p \sup_{h \in B, \|h\|_B \leq 1} \left\{ \int_B (1 - |z|^2)^\alpha (h \circ f)(z) - h(0)^p d\nu(z) \right\} \frac{1}{p}
\]
\[
\leq K \sup_{h \in B, \|h\|_B \leq 1} \left\{ \int_B (1 - |z|^2)^{\alpha+p} |\nabla (h \circ f)(z)|^p d\nu(z) \right\} \frac{1}{p}
\]
\[
\leq K \left\{ \int_B (1 - |z|^2)^{\alpha+p} \left( \frac{\nabla f(z)}{1 - |f(z)|^2} \right)^p d\nu(z) \right\} \frac{1}{p}.
\]

We note that
\[
(a + b)^p \leq \gamma_p (a^p + b^p)
\]
for $0 < p < \infty$, $a \geq 0$ and $b \geq 0$, where $\gamma_p = \max(1, 2^{p-1})$ ([9], page 73).
Conversely, by Lemma 6, Minkowski’s inequality or the above, Lemma 4(i) and Lemma 5,

\[
\left\{ \int_B (1 - |z|^2)^{\alpha + p} \left( \frac{|\nabla f(z)|}{1 - |f(z)|^2} \right)^p d\nu(z) \right\}^{\frac{1}{p}} 
\leq \left\{ \int_B (1 - |z|^2)^{\alpha + p} \left( \sum_{j=1}^{2} |\nabla (h_j \circ f)(z)| \right)^p d\nu(z) \right\}^{\frac{1}{p}} 
\leq K \sum_{j=1}^{2} \left\{ \int_B (1 - |z|^2)^{\alpha + p} |(h_j \circ f)(z) - h_j(0)|^p d\nu(z) \right\}^{\frac{1}{p}}.
\]

Since

\[
\left\{ \int_B (1 - |z|^2)^{\alpha} |(h_j \circ f)(z) - h_j(0)|^p d\nu(z) \right\}^{\frac{1}{p}} 
\leq \|h_j\|_B \sup_{h \in B, \|h\|_B \leq 1} \left\{ \int_B (1 - |z|^2)^{\alpha} |(h \circ f)(z) - h(0)|^p d\nu(z) \right\}^{\frac{1}{p}}, \quad j = 1, 2,
\]

from (4.1),

\[
\left\{ \int_B (1 - |z|^2)^{\alpha + p} \left( \frac{|\nabla f(z)|}{1 - |f(z)|^2} \right)^p d\nu(z) \right\}^{\frac{1}{p}} 
\leq K \sup_{h \in B, \|h\|_B \leq 1} \left\{ \int_B (1 - |z|^2)^{\alpha} |(h \circ f)(z) - h(0)|^p d\nu(z) \right\}^{\frac{1}{p}}
\]

\[
= K \sup_{h \in B, \|h\|_B \leq 1} \| (h \circ f)(z) - h(0) \|_{A^p_B}
\]

\[
= K \| C_0^f \|_p.
\]

5. Corollary

Let \(\mathcal{M}\) be the group of automorphisms of \(U\). For \(a \in U\), define \(\phi_a(z) = \frac{a - z}{1 - \overline{a}z}\). Kwon and Lee proved the following corollary.

**Corollary 1** ([6]). Let \(f : U \to U\) be a holomorphic function. For \(1 \leq p < \infty\) and \(-1 < \alpha < \infty\), the bounded operator \(C_0^f : \mathcal{B} \to A^p_B(U)\) defined by \(C_0^f h = h \circ \phi_{f(0)} \circ f - h(0)\) has its operator norm equivalent to the quantity (1.1).

The following is an immediate consequence of Theorem 2.

**Corollary 2.** Let \(f : B \to U\) be a holomorphic function. For \(0 < p < \infty\) and \(-1 < \alpha < \infty\), the bounded operator \(C_0^f : \mathcal{B} \to A^p_B(B)\) defined by \(C_0^f h = h \circ \phi_{f(0)} \circ f - h(0)\) has its operator norm equivalent to the quantity (1.2).
Proof. By Theorem 2 and $M$-invariance of \( \frac{|\nabla f(z)|}{1-|f(z)|^2} \), the result holds. \[\square\]

References


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