Korean J. Math. **22** (2014), No. 3, pp. 583–590
http://dx.doi.org/10.11568/kjm.2014.22.3.583

QUASI-ELLIS GROUPS AND SOME SUBGROUPS OF THE AUTOMORPHISM GROUP

Hyungsoo Song

Abstract. In this paper we give some relationships between quasi-Ellis groups and some subgroups of the automorphism group. In particular, we investigate several characterizations on some subgroups of the automorphism group.

1. Introduction

Universal minimal sets were studied by R. Ellis in [3]. S. Glasner introduced the Ellis group which is a certain group of the universal minimal set In [5]. Given a homomorphism of pointed minimal sets $\pi : (X, x_0) \to (Y, y_0)$, we can define quasi-Ellis groups $\mathcal{S}(X, x_0)$ and $\mathcal{S}(Y, y_0)$ which are generalizations of the Ellis groups and give some relationships between the homomorphism and quasi-Ellis groups.

Let G be the automorphism group of universal minimal set M. Given a minimal set X and a homomorphism $\gamma : M \to X$, we may define subgroups $G(X, \gamma)$ and $S(X, \gamma)$ of G, and study some characterizations on $G(X, \gamma)$ and $S(X, \gamma)$. In particular, we investigate G and the subgroup of M are isomorphic.

Received June 13, 2014. Revised September 16, 2014. Accepted September 16, 2014.

2010 Mathematics Subject Classification: 37B05.

Key words and phrases: proximal homomorphism, distal homomorphism, regular homomorphism, minimal set, Ellis group, quasi-Ellis group, automorphism group.

The present research has been conducted by the Research Grant of Kwangwoon University in 2013.

2. Preliminaries

A *transformation group*, or *flow*, (X, T), will consist of a jointly continuous action of the topological group T on the compact Hausdorff space X. The group T, with identity e, is assumed to be topologically discrete and remain fixed throughout this paper, so we may write X instead of (X, T).

A *point transitive flow*, (X, x_0) consists of a flow X with a distinguished point x_0 which has dense orbit.

A *homomorphism* of flows is a continuous, equivariant map. A homomorphism whose domain is point transitive is determined by its value at a single point. A one-one homomorphism of X onto X is called an automorphism of X. We denote the group of automorphisms of X by $A(X)$.

A flow is said to be *minimal* if every point has dense orbit. Minimal flows are also referred to as minimal sets. M is said to be a *universal minimal set* if it is a minimal set such that every minimal set is a homomorphic image of M. A homomorphism whose range is minimal is always surjective.

Given a flow (X, T), we may regard T as a set of self-homeomorphisms of X. We define $E(X)$, the *enveloping semigroup* of X to be the closure of T in X^X, taken with the product topology. $E(X)$ is at once a flow and a sub-semigroup of X^X. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$.

The points $x, x' \in X$ are said to be *proximal* if there exists a net (t_i) in T such that $\lim xt_i = \lim x't_i$. The points $x, x' \in X$ are said to be *distal* if either $x = x'$ or x and x' are not proximal. Thus if x and x' are both proximal and distal, they must be equal. The set of all proximal pairs in X will be denoted $P(X, T)$ or simply $P(X)$. X is said to be *distal* if $P(X) = \triangle_X$, the diagonal of $X \times X$ and is said to be *proximal* if $P(X) = X \times X$. Given any point $x \in X$, we define $P(x) = \{x' \in X \mid (x, x') \in P(X)\}$.

A homomorphism $\pi : X \to Y$ is said to be *proximal* (resp. *distal*) if whenever $x, x' \in \pi^{-1}(y)$ then x and x' are proximal (resp. distal).

A homomorphism $\pi : X \to Y$ is said to be *regular* if whenever $x, x' \in X$ with $\pi(x) = \pi(x')$, then $(\phi(x), x') \in P(X)$ for some $\phi \in A(X)$.

If E is some enveloping semigroup, and there exists a homomorphism $\theta : (E, e) \to (E(X), e)$ we say that E is an *enveloping semigroup for X*.

If such a homomorphism exists, it must be unique, and, given $x \in X$ and $p \in E$ we may write xp to mean $x\theta(p)$ unambiguously.

LEMMA 2.1. ([6]) *Let E be an enveloping semigroup for X and let I be a minimal right ideal in E. The following are true :*

(1) *The set $J(I)$ of idempotent elements in I is non-empty.*
(2) *$pI = I$ for all $p \in I$.*
(3) *$up = p$ whenever $p \in I$ and $u \in J(I)$.*
(4) *If $u \in J(I)$ then Iu is a group with identity u.*
(5) *If $p \in I$ then there exists a unique $u \in J(I)$ with $pu = p$.*
(6) *Given $x \in X$, the following are equivalent :*
  (a) *x is an almost periodic point.*
  (b) *$\overline{xT} = xI$.*
  (c) *$x = xu$ for some $u \in J(I)$.*

LEMMA 2.2. ([6]) *Let E be an enveloping semigroup for X, Then for any points $x, x' \in X$, (a) and (b) are equivalent :*

(a) *$(x, x') \in P(X, T)$.*

(b) *There exists a minimal right ideal I in E such that $xp = x'p$ for every $p \in I$.*

Moreover, if X is minimal, (a) and (b) are equivalent to :

(c) *There exists $u \in J(I)$ such that $x' = xu$.*

LEMMA 2.3. ([6]) *If (X, x) and (Y, y) are point transitive flows, and E is an enveloping semigroup for both X and Y, there exists a unique homomorphism $\psi : (X, x) \to (Y, y)$ if and only if $xp = xq$ for $p, q \in E$ implies $yp = yq$.*

LEMMA 2.4. ([3]) *The following are equivalent :*

(a) *(X, T) is distal.*

(b) *(X^I, T) is pointwise almost periodic where I is a set with at least two elements.*

3. Some results on homomorphisms of pointed minimal sets

Let βT be the Stone-Čech compactification of T. Then $(\beta T, e)$ is a universal point transitive flow. It is also clear that βT is an enveloping semigroup for X, whenever X is a flow with acting group T. Now let M be a fixed minimal right ideal in βT. Then (M, T) is a universal minimal set. We choose a distinguished idempotent u in $J(M) = J$

and let $\mathcal{G}$ denote the group Mu. Given a minimal set X, we choose a distinguished onto homomorphism $\gamma : M \to X$ and let $\gamma(u) = x_0$. Then $x_0 u = x_0$. Thus $x_0 \in Xu$.

Now we define the Ellis group $\mathcal{G}(X, x_0)$ and the quasi-Ellis group $\mathcal{S}(X, x_0)$ as follows ;

$$\mathcal{G}(X, x_0) = \{\alpha \in \mathcal{G} \mid x_0\alpha = x_0\} \text{ ([5])}$$

$$\mathcal{S}(X, x_0) = \{\alpha \in \mathcal{G} \mid h(x_0)\alpha = x_0 \text{ for some h} \in \text{A(X)}\}.$$

Clearly $\mathcal{G}(X, x_0) \subset \mathcal{S}(X, x_0)$, and $\mathcal{G}(X, x_0)$ and $\mathcal{S}(X, x_0)$ are subgroups of $\mathcal{G}$.

Let $G = A(M)$. Given a homomorphism $\gamma : M \to X$ we define the subgroups $G(X, \gamma)$ and $S(X, \gamma)$ of G as follows(see [1], [8]);

$$G(X, \gamma) = \{\theta \in G \mid \gamma \circ \theta = \gamma\}$$

$$S(X, \gamma) = \{\theta \in G \mid h \circ \gamma \circ \theta = \gamma \text{ for some h} \in \text{A(X)}\}.$$

Lemma 3.1. *The following are true :*

(1) Let $\alpha \in \mathcal{G}(X, x_0)$ and let $\theta : M \to M$ be the map with $\theta(u) = \alpha$. Then $\theta \in G(X, \gamma)$.

(2) Let $\theta \in G(X, \gamma)$ and let $\theta(u) = \alpha$. Then $\gamma(\alpha) = x_0$ and $\alpha \in \mathcal{G}(X, x_0)$.

Proof. (1) Let $\alpha \in \mathcal{G}(X, x_0)$ and define the map $\theta : M \to M$ by $\theta(u) = \alpha$. Given elements p and q in M with $up = uq$, we also have $\alpha p = \alpha q$. This imples that θ is a unique homomorphism by Lemma 2.3. Hence it follows from [3, Proposition 14] that $\theta \in A(M)$.

Moreover, $\gamma \circ \theta(u) = \gamma(\theta(u)) = \gamma(\alpha) = \gamma(u\alpha) = \gamma(u)\alpha = x_0\alpha = x_0 = \gamma(u)$. Thus $\theta \in G(X, \gamma)$.

(2) Let $\theta \in G(X, \gamma)$ and let $\theta(u) = \alpha$. Then $\gamma(\alpha) = \gamma(\theta(u)) = \gamma \circ \theta(u) = \gamma(u) = x_0$ and hence $x_0\alpha = \gamma(u)\alpha = \gamma(u\alpha) = \gamma(\alpha) = x_0$. But since $\alpha = \theta(u) = \theta(u)u = \alpha u \in \mathcal{G}$, it follows that $\theta \in A(M)$. □

Theorem 3.2. *The groups $\mathcal{G}$ and G are isomorphic.*

Proof. Define $\Phi : G \to \mathcal{G}$ by $\Phi(\theta) = \theta(u)$ for all $\theta \in G$. Since $\theta(u) = \theta(u)u \in \mathcal{G}$, Φ is well defined. Let $\Phi(\theta_1) = \Phi(\theta_2)$. Since $\theta_1(u) = \theta_2(u)$, it follows from the minimality of M that $\theta_1 = \theta_2$. This means that Φ is injective. Now let $\alpha \in \mathcal{G}$. Then there exists $p \in M$ with $\alpha = pu$ and we can choose $\theta \in G$ with $\theta(u) = p$ by Lemma 3.1 (1). Thus $\Phi(\theta) = \theta(u) = \theta(u)u = pu = \alpha$ whence Φ is surjective. Finally let $\theta, \eta \in G$. Then $\Phi(\theta\eta) = \theta\eta(u) = \theta(u\eta(u)) = \theta(u)\eta(u) = \Phi(\theta)\Phi(\eta)$, which means that Φ is a homomorphism. Therefore Φ is a isomorphism whence $\mathcal{G}$ and G are isomorphic. □

THEOREM 3.3. *The following are true :*
(1) $\mathcal{G}(X, x_0)$ *and* $G(X, \gamma)$ *are isomorphic.*
(2) $\mathcal{S}(X, x_0)$ *and* $S(X, \gamma)$ *are isomorphic.*

Proof. (1) By Lemma 3.1 (2), $\Phi|_{G(X,\gamma)} : G(X, \gamma) \to \mathcal{G}(X, x_0)$ is well defined. Also $\Phi|_{G(X,\gamma)}$ is surjective by Lemma 3.1 (1). Therefore it is trivial the fact that $\mathcal{G}(X, x_0)$ and $G(X, \gamma)$ are isomorphic.
(2) This follows from Definition and 3.3 (1). □

The next theorems follow from Theorem 3.3.

THEOREM 3.4. ([8]) *The following are true :*
(1) $\mathcal{G}(X, x_0)$ *is a normal subgroup of* $\mathcal{S}(X, x_0)$.
(2) $G(X, \gamma)$ *is a normal subgroup of* $S(X, \gamma)$.

THEOREM 3.5. ([5]) *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets. Then the following are true :*
(1) $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$.
(2) $G(X, \gamma) \subset G(Y, \pi \circ \gamma)$.

REMARK 3.6. Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets and let $\gamma : M \to X$ be a fixed homomorphism. Then the following are true :
(1) The groups $\mathcal{G}$, $\mathcal{G}(X, x_0)$, and $\mathcal{S}(X, x_0)$ can be identified with the groups G, $G(X, \gamma)$, and $S(X, \gamma)$, respectively.
(2) The group $\mathcal{G}(Y, y_0)$ can be identified with the group $G(Y, \pi \circ \gamma)$.

THEOREM 3.7. *Let* (X, T) *be a minimal set and let* $x_0 \in Xu$. *Then* $\beta \in \mathcal{G} - \mathcal{G}(X, x_0)$ *if and only if* $(x_0, x_0\beta) \notin P(X)$.

Proof. Let $\beta \in \mathcal{G} - \mathcal{G}(X, x_0)$ and suppose $(x_0, x_0\beta) \in P(X)$. Then there exists $q \in M$ with $x_0q = x_0\beta q$. Since $qM = M$, it follows that there exists $r \in M$ with $qr = u$. Hence $x_0 = x_0u = x_0qr = x_0\beta qr = x_0\beta u = x_0\beta$. This is a contradiction because $\beta \notin \mathcal{G}(X, x_0)$.

Now let $\beta \in \mathcal{G}$ and suppose $(x_0, x_0\beta) \notin P(X)$. Then $x_0p \neq x_0\beta p$ for all $p \in \beta T$. Hence $x_0 = x_0u \neq x_0\beta u = x_0\beta$. This implies $\beta \notin \mathcal{G} - \mathcal{G}(X, x_0)$. □

LEMMA 3.8. *Let* (X, T) *be a flow. Then the following are true :*
(1) *If* $(x, x') \notin P(X)$, *then* $xu \neq x'u$.

(2) *Let* (X,T) *be a minimal set and let* $x, x' \in Xu$. *Then there exists* $\beta \in \mathcal{G}$ *such that* $x = x'\beta$.

Proof. (1) Suppose $xu = x'u$. Then $xp = xup = x'up = x'p$ for all $p \in \mathcal{G}$ and hence $(x,x') \in P(X)$.

(2) Let $x, x' \in Xu$. Since $x'M = X$, it follows that there exists $q \in M$ such that $x'q = x$. Set $qu = \beta$. Then $\beta \in \mathcal{G}$ and $x'\beta = x'qu = xu = x$. □

Remark 3.9. (1) Note that if $p \in M$, p has a unique decomposition as $p = \alpha v$ where $\alpha \in \mathcal{G}$ and $v \in J$ by Lemma 2.1 (5).

(2) If $x, x' \in Xu$, then there exists $\beta \in \mathcal{G}$ such that $\mathcal{G}(X,x') = \beta\mathcal{G}(X,x)\beta^{-1}$. In fact, if $\beta \in \mathcal{G}$ with $x'\beta = x$ and $\alpha \in \mathcal{G}(X,x)$, then $\beta\alpha\beta^{-1} \in \mathcal{G}(X,x')$ by Lemma 3.8 (2). Also, if $\delta \in \mathcal{G}(X,x')$, then it is immediate that $\delta \in \beta\mathcal{G}(X,x)\beta^{-1}$.

Theorem 3.10. *Let* $\pi : (X,x_0) \to (Y,y_0)$ *be a homomorphism of pointed minimal sets. Then the following are true :*

(1) If $P(x_0) = X$, *then* $P(X) = X \times X$.

(2) If $P(x_0) = X$, *then* $P(y_0) = Y$.

(3) If $\mathcal{G}(X,x_0) = \mathcal{G}$, *then* $P(X) = X \times X$ *and* $P(Y) = Y \times Y$.

Proof. (1) Let $x, x' \in X$. Then we have from Lemma 2.2 and $P(x_0) = X$ that there exist $v, w \in J$ such that $x = x_0 v$ and $x' = x_0 w$. Then $xw = (x_0 v)w = x_0 w = x'$. Therefore $(x,x') \in P(X)$ and hence $P(X) = X \times X$.

(2) Let $y \in Y$. Since π is surjective, it follows that there exists $x \in X$ with $\pi(x) = y$. Since $P(x_0) = X$, we have that there exists $v \in J$ such that $x = x_0 v$. Then $y = \pi(x) = \pi(x_0 v) = y_0 v$. Thus $y \in P(y_0)$. This means that $P(y_0) = Y$.

(3) Let $\mathcal{G}(X,x_0) = \mathcal{G}$, $x \in X$, and $y \in Y$. Since X is minimal, it follows that there exists $p \in M$ with $x = x_0 p$. Also we have from Remark 3.9 (1) that there exist $\alpha \in \mathcal{G}$, $v \in J$ such that $p = \alpha v$. Since $\mathcal{G}(X,x_0) = \mathcal{G}$, it follows that $x = x_0 p = x_0 \alpha v = x_0 v$. Thus $P(x_0) = X$ and hence $P(y_0) = Y$ by (2). It follows from (1) that $P(X) = X \times X$ and $P(Y) = Y \times Y$. □

Theorem 3.11. ([5]) *Let* $\pi : (X,x_0) \to (Y,y_0)$ *be a homomorphism of pointed minimal sets. Then the following are equivalent :*

(a) π *is proximal.*

(b) $\mathcal{G}(X,x_0) = \mathcal{G}(Y,y_0)$

(c) *Given* $y \in Y$, $\pi^{-1}(y) \subset xJ(M)$ *for all* $x \in \pi^{-1}(y)$.

The following theorem will show that in the case of distality, the group determines the flow homomorphism. We prove J. Auslander's result as a corollary.

THEOREM 3.12. *Let X, Y be minimal sets and let $x_0 \in Xu,\ y_0 \in Yu$, and Y distal. If $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$, then there exists a homomorphism $\pi : (X, x_0) \to (Y, y_0)$.*

Proof. Let $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$, $(x_0, y_0) = z_0$, and $\overline{z_0 T} = Z$. Given $\alpha \in \mathcal{G}$ with $z_0\alpha = z_0$, we have that $(x_0, y_0)\alpha = (x_0, y_0)$ whence $\alpha \in \mathcal{G}(X, x_0)$ and $\alpha \in \mathcal{G}(Y, y_0)$. Thus $\alpha \in \mathcal{G}(X, x_0)$. Now let $\alpha \in \mathcal{G}(X, x_0)$. Since $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$, it follows from the Definition that $z_0\alpha = z_0$. This implies that $\mathcal{G}(X, x_0) = \mathcal{G}(Z, z_0)$. Therefore, by Theorem 3.11, there exists a proximal homomorphism $\psi : (Z, z_0) \to (X, x_0)$. Define $\phi : (Z, z_0) \to (Y, y_0)$ by $\phi(z_0) = y_0$. Then ϕ is a unique homomorphism by Lemma 2.3. Now define $\pi : (X, x_0) \to (Y, y_0)$ by $\pi(x_0) = \phi(z_0)$. If $\psi(z_1) = x_0$, then $(z_0, z_1) \in P(Z)$ whence $(\phi(z_0), \phi(z_1)) \in P(Y)$. Since Y is distal, it follows Lemma 2.4 that $\phi(z_0) = \phi(z_1)$. Thus π is a well defined homomorphism such that $\pi \circ \psi = \phi$. □

COROLLARY 3.13. ([2]) *Let X, Y be minimal sets and let $x_0 \in Xu,\ y_0 \in Yu$, and Y distal. Then $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$ if and only if there exists a homomorphism $\pi : (X, x_0) \to (Y, y_0)$.*

Proof. This follows from Theorem 3.5 and Theorem 3.12. □

In [7], Song proved the following theorems :

THEOREM 3.14. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then π is regular if and only if $\mathcal{G}(Y, y_0) \subset \mathcal{S}(X, x_0)$.*

THEOREM 3.15. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then the following are true :*

(1) *If π is regular, then $\mathcal{G}(X, x_0)$ is a normal subgroup of $\mathcal{G}(Y, y_0)$.*

(2) *Let π be regular. For each $y \in Y$ and $x \in \pi^{-1}(y)$, there exists $\phi \in A(X)$ such that $\phi(x) \in x'J$ for all $x' \in \pi^{-1}(y)$.*

J. Auslander and S. Glasner proved the following theorem :

THEOREM 3.16. ([2], [5]) *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then the following are equivalent :*

(a) *π is distal.*

(b) *If $y \in Y$ and $v \in J$ such that $yv = y$, then $\pi^{-1}(y)v = \pi^{-1}(y)$.*

(c) *If* $y \in Y$, *then* $\pi^{-1}(yp) = \pi^{-1}(y)p$ *for all* $p \in M$.

(d) *Given* $y \in Y$ *and* $p \in M$ *with* $y_0 p = y$, *we have that* $\pi^{-1}(y) = x_0 \mathcal{G}(Y, y_0)p$.

REMARK 3.17. (1) If X is proximal, then $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0) = \mathcal{S}(X, x_0)$. Note that if X is proximal and minimal, then the only homomorphism of X into X is the identity (see 1 in [4]).

(2) Note that a homomorphism is both proximal and distal if and only if it is an isomorphism.

References

[1] J. Auslander, *Homomorphisms of minimal transformation group*, Topology **9** (1970), 195–203.

[2] J. Auslander, *Minimal flows and their extensions*, Elsevier Science Publishers B.V., 1988.

[3] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[4] S. Glasner, *Compressibility properties in topological dynamics*, Amer. J. Math. **97** (1975), 148–171.

[5] S. Glasner, *Proximal flows*, Springer-Verlag, New York, 1975.

[6] P. S. Shoenfeld, *Regular homomorphisms of minimal sets*, Doctoral Dissertation, Univ. of Maryland (1974).

[7] H. S. Song, *A homomorphism of pointed minimal sets and quasi-Ellis groups*, Korean J. Math. **21** (2013), 103–111.

[8] J. O. Yu, *Regular relation with respect to a homomorphism*, J. Chungcheong Math. Soc.**11** (1998), 59–71.

Hyungsoo Song
Department of Mathematics
Kwangwoon University
Seoul 139–701, Korea
E-mail: songhs@kw.ac.kr