GLOBAL WEAK MORREY ESTIMATES FOR SOME ULTRAPARABOLIC OPERATORS OF KOLMOGOROV-FOKKER-PLANCK TYPE

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Abstract. We consider a class of hypoelliptic operators of the following type
\[ L = \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t, \]
where \((a_{ij}), (b_{ij})\) are constant matrices and \((a_{ij})\) is symmetric positive definite on \(\mathbb{R}^{p_0}\) \((p_0 \leq N)\). By establishing global Morrey estimates of singular integral on the homogenous space and the relation between Morrey space and weak Morrey space, we obtain the global weak Morrey estimates of the operator \(L\) on the whole space \(\mathbb{R}^{N+1}\).

1. Introduction and main results

Let us concern a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type in \(\mathbb{R}^{N+1}\):
\[ L_0 = \text{div}(A\nabla) + \langle x, B\nabla \rangle - \partial_t = \sum_{i,j=1}^{N} a_{ij} \partial^2_{x_i x_j} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t, \]
where \(1 \leq p_0 \leq N\), \(A = (a_{ij})\) and \(B = (b_{ij})\) are \(N \times N\) matrices with constant real entries, \(\nabla = (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_N})\), \(\text{div}\) and \(\langle \cdot, \cdot \rangle\) denote the gradient, the divergence and the inner product in \(\mathbb{R}^N\), separately. The matrix \(A\) is supposed to be symmetric and positive semidefinite. We also assume that the following condition holds:
\((H_0)\) Ker\((A)\) does not contain nontrivial subspaces which are invariant for \(B\).

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Hörmander in [12] pointed out that (H\(_0\)) implies (actually, is equivalent to) the hypoellipticity of (1.1). By introducing the matrix
\[
C(t) = \int_0^t E(s)AE^T(s)ds,
\]
where \(E(s) = \exp(-sB^T)\), the authors in [14] showed that (H\(_0\)) is equivalent to the condition
\[
C(t) > 0 \quad \text{for every } t > 0.
\]
It is interesting to remark that the condition (1.3) can also be expressed in geometric-differential terms. In fact, setting
\[
X_i = \sum_{j=1}^N a_{ij} \partial x_j, \quad i = 1, \ldots, N, \quad Y = \langle x, B\nabla \rangle,
\]
then (1.3) is equivalent to the following Hörmander’s condition
\[
\text{rank } L(X_1, X_2, \ldots, X_N, Y)(x) = N, \quad x \in \mathbb{R}^N,
\]
where \(L(X_1, X_2, \ldots, X_N, Y)\) denotes the Lie algebra generated by \(X_1, X_2, \ldots, X_N, Y\). The proof of the equivalence between (H\(_0\)) and (1.4) is implicitly contained in the introduction of [12], and Kuptsov in [13] gave an explicit proof of the equivalence between (1.3) and (1.4).

The authors in [14] also proved that (1.4) implies that, for some basis on \(\mathbb{R}^N\), the matrices \(A\) and \(B\) take the form:
\[
A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}
\]
and
\[
B = \begin{pmatrix} * & B_1 & 0 & \cdots & 0 \\ * & * & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & B_r \\ * & * & * & \cdots & * \end{pmatrix},
\]
respectively, where \(A_0 = (a_{ij})_{i,j=1}^{p_0}\) is a \(p_0 \times p_0\) constant matrix \((p_0 \leq N)\) with rank \(p_0\); \(B_j\) is a \(p_{j-1} \times p_j\) block with rank \(p_j\), \(j = 1, 2, \ldots, r\). Moreover \(p_0 \geq p_1 \geq \cdots \geq p_r \geq 1\) and \(p_0 + p_1 + \cdots + p_r = N\).

Specially, if we denote by \(B_0\) the matrix obtained by annihilating all the * blocks of the matrix written as (1.6), then the operator \(L_0\) becomes
\[
L = \text{div}(A\nabla) + \langle x, B_0\nabla \rangle - \partial_t = \sum_{i,j=1}^{p_0} a_{ij} \partial_{x_i x_j}^{2} + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} - \partial_t,
\]
which is the principal part of \(L_0\). In this paper, we will consider the operator \(L\) and make the following assumption:
(H₁) $A₀ = (aᵢⱼ)ᵢⱼ=1^{p₀}$ is symmetric and positive definite, and there exists positive constant $ν$ such that

$$ν|ξ|^2 \leq \sum_{ᵢ,ⱼ=1}^{p₀} aᵢⱼξᵢξⱼ \leq \frac{1}{ν}|ξ|^2$$

for all $ξ \in \mathbb{R}^{p₀}$.

It is known that $L$ is hypoelliptic (see [14]). On the other hand, $L$ is a heat operator when $p₀ = N$, $B = 0$ and the degenerate operators (i.e., with $p₀ < N$) appear in many research fields. For instance, the Kolmogorov equation

$$\partial^2_{x₁} u + x₁ \partial_{x₂} u = \partial_t u, \ (x, t) \in \mathbb{R}^3$$

occurs in the financial problem (see [1, 10]), in the kinetic theory (see [6, 16]) as well as in the visual perception problem (see [18]).

We know that $L$ is a class of Kolmogorov-Fokker-Planck ultraparabolic operator. Owing to its importance in physics and in mathematical finance, it has been extensively studied (see [3, 4, 11, 14, 19, 20]). The authors in [11, 14, 19, 20] proved an invariant Harnack inequality for the non-negative solutions of the equation $Lu = 0$. The local $L^p$ estimates have been studied by the authors in [3] and [4].

There are also some authors studied the Morrey estimates for some operators (see [7, 15, 22]). The local Morrey estimates for second-order nondivergence elliptic operators in Euclidean spaces were established by G. Fazio and M. Ragusa in [7]. G. Lieberman in [15] derived directly the local Morrey estimates for some second-order nondivergence elliptic and parabolic operators. For parabolic nondivergence operators of Hörmander type, S. Tang and P. Niu in [22] checked the local Sobolev-Morrey estimates. In this paper, we investigate the global weak Morrey estimates for the operator $L$. Moreover, the Hölder estimates of the operator $L$ under some certain conditions are given.

To state our main results, we introduce some notations and function spaces.

**Definition 1.1** (Morrey space). We say that a measurable function $f \in L^p_{loc}(\mathbb{R}^{N+1})$ belongs to the Morrey space $L^{p,λ}(\mathbb{R}^{N+1})$ with $p \in (1, +\infty)$ and $λ \in [0, Q+2]$, if the norm

$$\|f\|_{L^{p,λ}(\mathbb{R}^{N+1})} = \left(\sup_{r>0} \frac{1}{r^p} \int_{B_r} |f(x)|^p \, dx \right)^{1/p}$$

is finite, where $Q$ and $B_r$ are described in (2.2) and (2.5), separately.

Letting $Y_i = \partial_{x_i}$ ($i = 1, 2, \ldots, p₀$), and noting $Y₀ = \langle x, B∇ \rangle - \partial_t$, we use the simplified notations:

$$\|Du\|_{L^{p,λ}(\mathbb{R}^{N+1})} = \sum_{j=1}^{p₀} \|Y_j u\|_{L^{p,λ}(\mathbb{R}^{N+1})};$$
Theorem 1.1. For every measurable function \( f \) is finite.

Definition 1.2. Let \( p \in (1, +\infty), \lambda \in [0, Q + 2] \) and \( k \) be a nonnegative integer. We define Sobolev-Morrey spaces \( S^{k, p, \lambda}(\mathbb{R}^{N+1}) \) which consists of all \( L^{p, \lambda}(\mathbb{R}^{N+1}) \) functions with \( k \)-th derivatives with respect to vector fields \( Y_i \)’s (\( i = 0, 1, \ldots, p_0 \)) of degree \( k \). The Sobolev-Morrey norm is defined by

\[
\|u\|_{S^{k, p, \lambda}(\mathbb{R}^{N+1})} = \sum_{h=0}^{k} \|D^h u\|_{L^{p, \lambda}(\mathbb{R}^{N+1})}.
\]

Definition 1.3. Let \( f \in L^p(\mathbb{R}^{N+1}), z_0 \in \mathbb{R}^{N+1}, \rho > 0, \tau \geq 0 \), and set

\[
A_{\tau, \rho}(f) = \{ z \in B_\rho(z_0) | |f(z)| > \tau \},
\]

where \( B_\rho(z_0) \) is given in (2.5). The function \( \lambda_f(\tau, \rho) = |A_{\tau, \rho}(f)| \) is called a distribution function of \( f \).

Definition 1.4 (Weak Morrey space). For \( p \in (1, \infty), \lambda \in [0, Q + 2] \), a measurable function \( f \) is said to belong to a weak Morrey space (denoted by \( L_{w, \lambda}^{p}(\mathbb{R}^{N+1}) \)), if

\[
\|f\|_{L_{w, \lambda}^{p}(\mathbb{R}^{N+1})} = \sup_{\rho > 0} \inf_{\tau > 0} \{ A | \lambda_f(\tau, \rho) \leq \rho^{\lambda} \tau^{-p} A^p \}
\]

is finite.

The main results in this paper are as follows.

Theorem 1.1. For every \( p \in (1, \infty), \lambda \in [0, Q + 2] \), there exists a positive constant \( C \), depending on \( p, p_0, \) the matrix \( B \) and the number \( \nu \) in (1.7) such that

\[
\|D^{k+2} u\|_{L_{w, \lambda}^{p}(\mathbb{R}^{N+1})} \leq C \|Lu\|_{S^{k, p, \lambda}(\mathbb{R}^{N+1})}
\]

for every \( u \in C_0^\infty(\mathbb{R}^{N+1}) \) and nonnegative integer \( k \).

Theorem 1.2. If \( 2p + \lambda > Q + 2, p + \lambda < Q + 2 \) and \( \theta = \frac{2p + \lambda - (Q + 2)}{p} \), then there exists a positive constant \( C \), depending only on \( p, \lambda \) and the operator \( L \), such that for every \( u \in C_0^\infty(\mathbb{R}^{N+1}) \),

\[
\frac{|u(z) - u(w)|}{\|z^{-1} \circ w\|^\theta} \leq C \|Lu\|_{L_{w, \lambda}^{p}(\mathbb{R}^{N+1})},
\]
for every \( z, w \in \mathbb{R}^{N+1}, z \neq w \), where \( \circ \) is the group law given in Section 2.

If \( p + \lambda > Q + 2 \) and \( \delta = \frac{p + \lambda \cdot (Q + 2)}{p} \), then there exists a positive constant \( C \), depending only on \( p, \lambda \) and the operator \( L \), such that for every \( u \in C_0^\infty (\mathbb{R}^{N+1}) \),

\[
\frac{|\partial_x u(z) - \partial_x u(w)|}{\|z^{-1} \circ w\|^\delta} \leq C\|Lu\|_{L^p, \lambda(\mathbb{R}^{N+1})}
\]

for every \( z, w \in \mathbb{R}^{N+1}, z \neq w \) and \( j = 1, 2, \ldots, p_0 \).

The paper is organized as follows: In Section 2, we introduce some preliminary and known results which will be used later. Section 3 is first devoted to obtaining global second order Morrey estimates and the higher order Morrey estimates by using global \( L^p \) estimate. And then, we check the relation between the global Morrey space and the global weak Morrey space. Using these conclusions, Theorem 1.1 is proved. The proof of Theorem 1.2 is given in Section 4.

2. Preliminary

It is proved in [14] that the operator \( L \) is left-invariant with respect to the Lie group \( K \) whose underlying manifold is \( \mathbb{R}^{N+1} \), endowed with the composition law

\[
(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau),
\]

where \( E(\tau) = \exp(-\tau B^T) \) and \( B^T \) denotes the transpose of \( B \). Note that

\[
(\xi, \tau)^{-1} = (-E(-\tau)\xi, -\tau).
\]

There exists a group of dilations on \( \mathbb{R}^{N+1} \), which we denote by \( (D(\lambda))_{\lambda > 0} \).

More precisely, \( D(\lambda) \) is defined by

\[
D(\lambda) = \text{diag}(\lambda^{\alpha_1}, \lambda^{\alpha_2}, \ldots, \lambda^{\alpha_N}, \lambda^2),
\]

where

\[
\alpha_1 = \cdots = \alpha_{p_0} = 1, \quad \alpha_{p_0+1} = \cdots = \alpha_{p_0+p_1} = 3, \ldots,
\]

\[
\alpha_{p_0+p_1+\cdots+p_r+1} = \cdots = \alpha_N = 2r + 1.
\]

Therefore, we can write

\[
D(\lambda) = \text{diag}(\lambda I_{p_0}, \lambda^3 I_{p_1}, \ldots, \lambda^{2r+1} I_{p_r}, \lambda^2),
\]

where \( I_{p_j}, D(\lambda) \) denote the \( p_j \times p_j \) identity matrix and the matrix of dilations on \( \mathbb{R}^{N+1} \), respectively. Note that

\[
\det(D(\lambda)) = \lambda^{Q+2},
\]

where

\[
Q + 2 = p_0 + 3p_1 + \cdots + (2r + 1)p_r + 2
\]

is called the homogeneous dimension of \( \mathbb{R}^{N+1} \) with respect to \( (D(\lambda))_{\lambda > 0} \).
Definition 2.1. We say that a differential operator $Y$ on $\mathbb{R}^{N+1}$ is homogeneous of degree $\beta > 0$, if
\[ Y(\varphi(D(\lambda)z)) = \lambda^\beta (Y\varphi)(D(\lambda)z), \quad z \in \mathbb{R}^{N+1}, \lambda > 0 \]
for every test function $\varphi$. Also, we say that a function $f$ is homogeneous of degree $\alpha$ if
\[ f((D(\lambda)z)) = \lambda^\alpha f(z), \quad \lambda > 0, \quad z \in \mathbb{R}^{N+1}. \]

Clearly, if $Y$ is a homogeneous differential operator of degree $\beta$ and $f$ is a homogeneous function of degree $\alpha$, then $Yf$ is homogeneous of degree $\alpha - \beta$.

By Definition 2.1, it is easy to show that the operator $L$ is homogeneous of degree two with respect to the dilations $D(\lambda)$, i.e.,
\[ L(u(D(\lambda)z)) = \lambda^2 (Lu)(D(\lambda)z), \quad z \in \mathbb{R}^{N+1}, \lambda > 0 \]
for every $u \in C_0^\infty(\mathbb{R}^{N+1})$.

We introduce a norm and a quasidistance in $\mathbb{R}^{N+1}$, related to the groups of translations and dilations defined above.

Definition 2.2. Let $z = (x_1, x_2, \ldots, x_N, t) \in \mathbb{R}^{N+1}$, if $z = 0$ we set $\|z\| = 0$, while if $z \in \mathbb{R}^{N+1}\{0\}$ we define $\|z\| = \varrho$, where $\varrho$ is the unique positive solution to the equation
\[ \frac{x_1^2}{\varrho^{2\alpha_1}} + \frac{x_2^2}{\varrho^{2\alpha_2}} + \cdots + \frac{x_N^2}{\varrho^{2\alpha_N}} + \frac{t^2}{\varrho^2} = 1, \]
where $\alpha_1, \alpha_2, \ldots, \alpha_N$ are the positive integers in (2.1).

Bramanti and Cerutti in [4] showed that the norm $\| \cdot \|$ satisfies
\begin{equation}
\|z\|^{-1} \leq c_1 \|z\|, \quad z \in \mathbb{R}^{N+1}
\end{equation}
and
\begin{equation}
\|z \circ \zeta\| \leq c_2(<\|z\| + \|\zeta\|), \quad z, \zeta \in \mathbb{R}^{N+1},
\end{equation}
where the positive constants $c_1$ and $c_2$ depend only on the matrix $B$.

Remark 2.1. There is a natural homogeneous norm in $\mathbb{R}^{N+1}$, induced by dilation $D(\lambda)$:
\[ \|(x, t)\| = \sum_{j=1}^{N} |x_j|^{1/\alpha_j} + |t|^{1/2}. \]

Clearly, we have
\[ \|D(\lambda)z\| = \lambda \|z\|, \lambda > 0, \quad z \in \mathbb{R}^{N+1}. \]

Definition 2.3. For every $z, w \in \mathbb{R}^{N+1}$, define a quasidistance by
\[ d(z, w) = \|w^{-1} \circ z\|. \]
The ball with respect to $d$ is denoted by
\begin{equation}
B(z, r) = B_r(z) = \{w \in \mathbb{R}^{N+1} : d(z, w) < r\}.
\end{equation}
Since $B(0, r) = D(r)B(0, 1)$ and $\det (D(\lambda)) = \lambda^{Q+2}$, we also have
\[ |B_r(0)| = r^{Q+2} |B_1(0)|, \]
where $|B_1(0)| = \omega_{N+1}$ is the Lebesgue measure of the Euclidean unit ball of $\mathbb{R}^{N+1}$. This implies that the Lebesgue measure $dz$ is a doubling measure with respect to $d$, since
\[ |B(z, 2r)| = 2^{Q+2} |B(z, r)|, \quad z \in \mathbb{R}^{N+1}, \quad r > 0. \]

Therefore, the space $(\mathbb{R}^{N+1}, dz, d)$ is a space of homogenous type. Recall that if $f$ and $g$ are functions on $\mathbb{R}^{N+1}$, their convolution $f * g$ is defined by
\[ f * g(x) = \int_{\mathbb{R}^{N+1}} f(x \circ y^{-1}) g(y) dy = \int_{\mathbb{R}^{N+1}} g(y^{-1} \circ x) f(y) dy. \]

**Lemma 2.1** ([14]). The operator $L$ possesses a fundamental solution
\begin{equation}
\Gamma(z) = \begin{cases} 0, & t \leq 0, \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp(-\frac{1}{4} (C^{-1}(t)x, x)), & t > 0, \end{cases}
\end{equation}
where $z = (x, t)$ and $C(t)$ is as in (1.2). Moreover, $\Gamma \in C^\infty(\mathbb{R}^{N+1} \setminus \{0\})$.

The authors in [8] and [21] proved a representation formula:
\begin{equation}
u(z) = -(Lu * \Gamma)(z) = -\int_{\mathbb{R}^{N+1}} \Gamma(\zeta^{-1} \circ z) Lu(\zeta) d\zeta.
\end{equation}

And the following formula was given by Bramanti in [5]:
\begin{equation}
\partial^2_{x_i x_j} u(z) = -PV(Lu * \partial^2_{x_i x_j} \Gamma)(z) + c_{i j} Lu(z)
\end{equation}
for every $u \in C^\infty_0(\mathbb{R}^{N+1})$ and some constants $c_{i j}$, $i, j = 1, 2, \ldots, p_0$. The principal value in (2.8) is understood as
\[ PV(Lu * \partial^2_{x_i x_j} \Gamma)(z) = \lim_{\varepsilon \to 0} \int_{\|\zeta^{-1} \circ z\| > \varepsilon} (\partial^2_{x_i x_j} \Gamma)(\zeta^{-1} \circ z) Lu(\zeta) d\zeta. \]

Set
\[ \Gamma_i(z) = \partial_{x_i} \Gamma(z), \quad \Gamma_{ij}(z) = \partial_{x_i} \partial_{x_j} \Gamma(z), \quad i, j = 1, 2, \ldots, p_0. \]
We also observe that $\Gamma(z)$ is homogeneous of degree $-Q$ with respect to the group $(D(\lambda))_{\lambda > 0}$ and $\Gamma_i(z)$ $(i, j = 1, 2, \ldots, p_0)$ are homogeneous of degree $-(Q + 1)$. Recall that $\Gamma_{ij}(\cdot)$ has the following properties.

**Lemma 2.2** ([4]). For $i, j = 1, 2, \ldots, p_0$, one has
(a) $\Gamma_{ij}(\cdot) \in C^\infty(\mathbb{R}^{N+1} \setminus \{0\})$;
(b) $\Gamma_{ij}(\cdot)$ is homogeneous of degree $-Q - 2$;
(c) for every $R > r > 0$,
\[ \int_{r < \|z\| < R} \Gamma_{ij}(z) dz = \int_{\|z\| = 1} \Gamma_{ij}(z) d\sigma(z) = 0. \]
Lemma 2.3 ([8, 9]). Let $K_h$ be a kernel in $C^\infty(\mathbb{R}^{N+1}\{0\})$ and homogeneous of degree $h - Q - 2$, for some integer $h$ with $0 < h < Q + 2$. Denote $T_h f = f \ast K_h$ and let $P^h$ be a homogeneous left invariant differential operator of degree $h$.

Then (d) $P^h T_h f = \text{P.V.}(f \ast P^h K_h) + \alpha f$, where $\alpha$ is a constant depending on $P^h$ and $K_h$;

(e) the singular integral operator $f \mapsto \text{P.V.}(f \ast P^h K_h)$ is continuous on $L^p(\mathbb{R}^{N+1})$ for $1 < p < \infty$.

3. Proof of Theorem 1.1

In this section, we first establish global second order Morrey estimates and the higher order Morrey estimates by applying Lemmas 2.2 and 2.3 and then give the relation between the global Morrey space and the global weak Morrey space. Based on these, Theorem 1.1 is proved.

Lemma 3.1. Let $a \in \mathbb{R}^1$ and $\gamma \in C(\mathbb{R}^{N+1}\{0\})$ be a homogeneous function with degree $a$ with respect to the group $(D(\lambda))_{\lambda > 0}$. Then $|\gamma(z)| \leq c\|z\|^a$.

Proof. Since $\gamma$ is a homogeneous function with degree $a$, one has $|\gamma(z)| = \|z\|^a|\gamma(D(\|z\|^{-1})z)|, z \in \mathbb{R}^{N+1}\{0\}$.

It is clear that $\|D(\|z\|^{-1})z\| = 1$, hence $|\gamma(D(\|z\|^{-1})z)| \leq c, z \in \mathbb{R}^{N+1}\{0\}$. Therefore, $|\gamma(z)| \leq c\|z\|^a$. The proof is completed. $\Box$

Let us define a singular integral operator

\begin{equation}
T_{ij} g(z) = \lim_{\varepsilon \to 0} \int_{\|\zeta - oz\| > \varepsilon} (\partial_{\xi_i, x_j} \Gamma)(\zeta^{-1} \circ z) g(\zeta) d\zeta, \ i, j = 1, \ldots, p_0
\end{equation}

for every measurable function $g$.

Lemma 3.2. Let $p \in (1, \infty), \lambda \in [0, Q + 2)$. For every $g \in L^{p, \lambda}(\mathbb{R}^{N+1})$, there exists a positive constant $c$ depending on $p, \lambda$ and the operator $L$, such that $\|T_{ij} g\|_{L^{p, \lambda}(\mathbb{R}^{N+1})} \leq c\|g\|_{L^{p, \lambda}(\mathbb{R}^{N+1})}$.

Proof. Fix $g$ and $r > 0$. For any measurable set $E \subset \mathbb{R}^{N+1}$, its characteristic function is denoted by $\chi_E$. Then we set $g_0 = g \chi_{B_r(y)}$, $g_k = g \chi_{B_{2^k r}(y) \setminus B_{2^{k-1} r}(y)}$, $k = 1, 2, \ldots$, where $B_r(y)$ is the ball with radius $r$ centered at $y$.
and \( \delta = 1/(2c_1^2c_2) \), where \( c_1 \) and \( c_2 \) are the constants in (2.3) and (2.4). It immediately follows from Lemma 2.3 that

\[
\|T_{ij}g\|_{L^p(\mathbb{R}^{N+1})} \leq c\|g\|_{L^p(\mathbb{R}^{N+1})},
\]

where \( c \) is a positive constant. Hence,

\[
(3.2) \quad \|T_{ij}g_0\|_{L^p(B_r(y))} \leq \|T_{ij}g_0\|_{L^p(\mathbb{R}^{N+1})} \leq c\|g\|_{L^p(B_r(y))} \leq c\|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},
\]

where the last inequality is obtained from the definition of \( L^{p,\lambda}(\mathbb{R}^{N+1}) \). We now consider \( g_k(z) \), where \( k \in \mathbb{N}^+ \) and \( z \in B_{\delta r}(y) \). By Lemma 3.1, there exists \( c > 0 \) such that

\[
(3.3) \quad |T_{ij}(\zeta^{-1} \circ z)| \leq c\|\zeta^{-1} \circ z\|^{-(Q+2)}.
\]

Moreover, if \( \zeta \in B_{2\delta r}(y) \setminus B_{2\delta r}(y) \), then by (2.3) and (2.4),

\[
\frac{2^{k-1}r}{c_1} \leq \|\zeta^{-1} \circ y\|
\]

Thus, from (3.1) and (3.3), it follows

\[
|T_{ij}g_k(z)| \leq \int_{2^{k-1}r \leq \|\zeta^{-1} \circ y\| \leq 2^{k+1}r} \frac{c}{\|\zeta^{-1} \circ z\|^{Q+2}}|g(\zeta)|d\zeta
\]

\[
\leq \frac{c(4c_1c_2)^{Q+2}}{(2^{k+2})^{Q+2}} \int_{B_{2\delta r}(y)} |g(\zeta)|d\zeta
\]

and by integration over \( B_{\delta r}(y) \),

\[
(3.4) \quad \|T_{ij}g_k\|_{L^p(B_r(y))} \leq c''\|g\|_{L^p(\mathbb{R}^{N+1})} (2^{k})^{\frac{\lambda-(Q+2)}{p}},
\]

where \( c'' \) is a positive constant depending only on \( p \) and the constants \( c_1 \) and \( c_2 \). By (3.2) and (3.4),

\[
\frac{1}{(\delta r)^{Q+2}} \|T_{ij}g\|_{L^p(B_r(y))} \leq \delta^{-\frac{Q+2}{p}} \|g\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \left( c + c'' \sum_{k=1}^{\infty} (2^{-\frac{(Q+2)}{p}})^k \right),
\]

which plainly proves the conclusion, since the above series is convergent. \( \square \)

**Theorem 3.1.** For every \( p \in (1, \infty), \lambda \in [0, Q + 2) \), there exists a constant \( C > 0 \), depending on \( p, p_0 \), the matrix \( B \) and the number \( \nu \) in (1.7) such that for every \( u \in C_0^\infty(\mathbb{R}^{N+1}) \),

\[
\|\partial_{x,\xi}^2 u\|_{L^p,\lambda(\mathbb{R}^{N+1})} \leq C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}, \quad i, j = 1, 2, \ldots, p_0;
\]

\[
\|Y_0 u\|_{L^p,\lambda(\mathbb{R}^{N+1})} \leq C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})},
\]
where \( Y_0 = \langle x, B\nabla \rangle - \partial_t \).

**Proof.** It follows from (2.8) that
\[
\partial^2_{x_i x_j} u(z) = -PV(Lu \ast \partial^2_{x_i x_j} \Gamma)(z) + c_{ij} Lu(z).
\]

By Lemma 3.2, there exists a constant \( C \) such that
\[
\|\partial^2_{x_i x_j} u\|_{L^p,\lambda(R^{N+1})} \leq C \|Lu\|_{L^p,\lambda(R^{N+1})}, \quad i, j = 1, 2, \ldots, p_0.
\]

The estimate of \( \|Y_0 u\|_{L^p,\lambda(R^{N+1})} \) yields from
\[
(3.5) \quad Y_0 u = Lu - \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} u.
\]

This ends the proof. \( \square \)

**Lemma 3.3.** Let
\[
P^k = Y_{j_1} \cdots Y_{j_l}, \quad Y_{j_i} = \partial x_{j_i}, \quad i = 1, \ldots, l
\]
for \( k = 1, 2, \ldots \) and \( 0 \leq j_i \leq p_0, \quad i = 1, 2, \ldots, l \), where \( Y_{j_1} \cdots Y_{j_l} \) is homogeneous of degree \( k \). Then for any test function \( \tau \), we have
\[
(P^k \Gamma) \ast \tau = \Gamma \ast P^k \tau.
\]

**Proof.** It is easy to check that for any test function \( \tau \),
\[
(3.7) \quad (Y_i \Gamma) \ast \tau = \Gamma \ast Y_i \tau, \quad i = 1, 2, \ldots, p_0.
\]

By (3.5),
\[
Y_0 = L - \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j}.
\]

Consequently, we have
\[
(Y_0 \Gamma) \ast \tau = Y_0 (\Gamma \ast \tau) = L(\Gamma \ast \tau) - \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} (\Gamma \ast \tau)
\]
\[
= (\Gamma \ast L \tau) - (\Gamma \ast \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} \tau)
\]
\[
= \Gamma \ast (L - \sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} \tau) = \Gamma \ast Y_0 \tau.
\]

It follows from (3.7) and (3.8) that
\[
(P^k \Gamma) \ast \tau = \Gamma \ast P^k \tau.
\]

This finishes the proof. \( \square \)
Theorem 3.2. For $p \in (1, \infty)$, $\lambda \in [0, Q + 2)$, there exists a constant $C > 0$, depending on $p$, $p_0$, the matrix $B$ and the number $\nu$ in (1.7) such that for every $u \in C_0^\infty(\mathbb{R}^{N+1})$ and nonnegative integer $k$,
\begin{equation}
\|D^{k+2}u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C\|Lu\|_{S^{k,\lambda}(\mathbb{R}^{N+1})}.
\end{equation}

Proof. Due to (1.8), (3.6) and Theorem 3.1, we have
\begin{equation}
\|P^2u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C\|Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{equation}

In order to obtain the estimate of $\|D^3u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}$, it is enough to consider $\|P^3u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}$ from (1.9) and (3.6). Note that the $P^3$ can be split as $P^2Y_i$ and $Y_iY_0$, $i = 1, \ldots, p_0$. In the first case, by Lemma 3.3,
\begin{equation*}
P^2Y_iu = P^2Y_i(\Gamma \ast Lu) = P^2(\Gamma \ast Y_iLu).
\end{equation*}

Thus, on the basis of Lemma 3.2, there exists a constant $C$ such that
\begin{equation}
\|P^2Y_iu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq \|P^2(\Gamma \ast Y_iLu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C\|Y_iLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{equation}

In the second case, note that $Y_0\Gamma(z) = -\sum_{i,j=1}^{p_0} a_{ij} \partial^2_{x_i x_j} \Gamma(z)$, $z \in \mathbb{R}^{N+1} \setminus \{0\}$. By Lemmas 3.2 and 3.3, there exists a constant $C$ such that
\begin{equation}
\|Y_iY_0u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} = \|Y_iY_0(\Gamma \ast Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}
\leq \|Y_iY_0(\Gamma \ast Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}
\leq C\|Y_iLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}
\leq C\|DLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{equation}

By (3.11) and (3.12), we have
\begin{equation}
\|P^3u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C\|DLu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{equation}

For $k \geq 2$, we will consider the estimates of $D^{k+2}u$. In fact, by (1.9) and (3.6), we only to show the estimates of $P^{k+2}u$. Let us split the $P^{k+2}u$ as $P^2P^k$ (with $P^2 = Y_0$) and $P^3P^{k-1}$ (with $P^3 = Y_iY_0$ for some $i = 1, 2, \ldots, p_0$). In the first case, using Lemmas 3.2 and 3.3, leads to that there exists a constant $C$ such that
\begin{equation}
\|P^2P^k u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} = \|P^2(P^k \Gamma \ast Lu)\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}
\leq C\|P^k Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}
\leq C\|D^k Lu\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\end{equation}
In the second case, by Lemmas 3.2, 3.3 and Theorem 3.1, there exists a constant
$C$ such that
\[
\|Y_0 P^{k-1} u\|_{L^p,\lambda(\mathbb{R}^{N+1})} = \|Y_0 (P^k \Gamma \ast Lu)\|_{L^p,\lambda(\mathbb{R}^{N+1})} \\
\leq \|(Y_0 \Gamma \ast P^k Lu)\|_{L^p,\lambda(\mathbb{R}^{N+1})} \\
\leq C \|P^k Lu\|_{L^p,\lambda(\mathbb{R}^{N+1})} \\
\leq C \|D^k Lu\|_{L^p,\lambda(\mathbb{R}^{N+1})}.
\]
(3.15)
Hence, by (3.14) and (3.15),
\[
\|P^{k+2} u\|_{L^p,\lambda(\mathbb{R}^{N+1})} \leq C \|D^k Lu\|_{L^p,\lambda(\mathbb{R}^{N+1})}.
\]
(3.16)
Therefore, (3.9) is followed from (3.10), (3.13) and (3.16). The proof is completed.

**Theorem 3.3.** For every $p \in (1, \infty)$, there exists a constant $C > 0$, such that
for every $f \in L^{p,\lambda}(\mathbb{R}^{N+1})$,
\[
\|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\]
Proof. For every $f \in L^{p,\lambda}(\mathbb{R}^{N+1})$, \[
\rho^{-\lambda_{A_{r,\rho}}} |A_{r,\rho} (f)| \leq \rho^{-\lambda} \int_{A_{r,\rho}} |f(z)|^p dz \\
\leq \rho^{-\lambda} \int_{\mathbb{R}^{N+1}} |f(z)|^p dz \\
\leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}^p,
\]
where $C$ is a constant. Then,
\[
\|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C \|f\|_{L^{p,\lambda}(\mathbb{R}^{N+1})}.
\]
It ends the proof.

**Proof of Theorem 1.1.** For every nonnegative integer $k$, by Theorem 3.2, we have
\[
\|D^{k+2} u\|_{L^p,\lambda(\mathbb{R}^{N+1})} \leq C \|Lu\|_{S^{k,\lambda}(\mathbb{R}^{N+1})}.
\]
Hence, from Theorem 3.3, there exists a constant $C$ such that
\[
\|D^{k+2} u\|_{L^{p,\lambda}(\mathbb{R}^{N+1})} \leq C \|Lu\|_{S^{k,\lambda}(\mathbb{R}^{N+1})}.
\]
The proof is completed.
4. Hölder continuity

In this section, by demonstrating Hölder estimates of two integral operators, we prove Theorem 1.2.

Lemma 4.1 ([17]). Let \( b \in \mathbb{R}^1 \) and \( K \in C^1(\mathbb{R}^{N+1}\setminus\{0\}) \) be a homogeneous function with degree \( b \) with respect to the group \( (D(\lambda))_{\lambda \geq 0} \) and there exist two constants \( c > 0 \) and \( M > 1 \) such that if \( \|z\| \geq M\|z^{-1} \circ \zeta\|. \) Then
\[
|K(\zeta) - K(z)| \leq c\|z^{-1} \circ \zeta\| \cdot \|z\|^{b-1}.
\]

Lemma 4.2. Let \( p \in (1, \infty) \) and \( \lambda \in [0, Q+2) \). Fixed \( w \in \mathbb{R}^{N+1}, \alpha \in [0, Q+2), \beta \in (0, Q+2) \) and \( \sigma > 0 \), for every \( g \in L^{p, \lambda}(\mathbb{R}^{N+1}) \), we set
\[
T'_\alpha g(z) = \int_{\|\zeta^{-1} \circ z\| \geq \sigma \|w^{-1} \circ z\|} \frac{g(\zeta)}{|\zeta^{-1} \circ z|^{Q + 2 - \alpha}} d\zeta
\]
and
\[
T''_\beta g(z) = \int_{\|\zeta^{-1} \circ z\| \leq \sigma \|w^{-1} \circ z\|} \frac{g(\zeta)}{|\zeta^{-1} \circ z|^{Q + 2 - \beta}} d\zeta.
\]
Then, if \( \lambda + \rho \alpha < Q + 2 \), then there exists \( c = c(p, \lambda, \alpha, \sigma) > 0 \) such that
\[
|T'_\alpha g(z)| \leq c\|g\|_{L^{p, \lambda}(\mathbb{R}^{N+1})} \|w^{-1} \circ z\|^{\frac{\rho \alpha + \lambda + (2 - Q)}{\rho}}.
\]
Moreover, if \( \lambda + \rho \beta > Q + 2 \), then there exists \( c = c(p, \lambda, \beta, \sigma) > 0 \) such that
\[
|T''_\beta g(z)| \leq c\|g\|_{L^{p, \lambda}(\mathbb{R}^{N+1})} \|w^{-1} \circ z\|^{\frac{\rho \beta + \lambda + (2 - Q)}{\rho}}.
\]

Proof. Observing that
\[
|T'_\alpha g(z)| \leq \sum_{k=1}^{\infty} \int_{2^k \|\zeta^{-1} \circ z\| \leq \|\zeta^{-1} \circ z\|} \left\|\frac{g(\zeta)}{|\zeta^{-1} \circ z|^{Q + 2 - \alpha}} \right\| d\zeta
\]
\[
\leq \sum_{k=1}^{\infty} \left(2/\|\zeta^{-1} \circ z\|\right)^{Q + 2 - \alpha} \int_{B_{2^k \|\zeta^{-1} \circ z\|}} |g(\zeta)| d\zeta
\]
\[
\leq c\|g\|_{L^{p, \lambda}(\mathbb{R}^{N+1})} \|w^{-1} \circ z\|^{\frac{\rho \alpha + \lambda + (2 - Q)}{\rho}} \sum_{k=1}^{\infty} \left(2/\|\zeta^{-1} \circ z\|\right)^{Q + 2 - \alpha},
\]
we know that (4.1) is true, since the above series is convergent.

Similarly, by integrating on the set
\[
\{\zeta \in \mathbb{R}^{N+1} : 2^{-k} \|\zeta^{-1} \circ z\| \leq \|\zeta^{-1} \circ z\| < 2^{1-k} \|\zeta^{-1} \circ z\|\},
\]
it yields
\[
|T''_\beta g(z)| \leq \sum_{k=1}^{\infty} \int_{2^{-k} \|\zeta^{-1} \circ z\| \leq \|\zeta^{-1} \circ z\|} \left\|\frac{g(\zeta)}{|\zeta^{-1} \circ z|^{Q + 2 - \beta}} \right\| d\zeta
\]
\[
\leq \sum_{k=1}^{\infty} \left(2/\|\zeta^{-1} \circ z\|\right)^{Q + 2 - \beta} \int_{B_{2^{1-k} \|\zeta^{-1} \circ z\|}} |g(\zeta)| d\zeta.
\]
Hence, by (4.3), (4.4) and (4.5), it is easy to obtain
\[
\beta \text{ choosing } \beta = 2 \text{ and } \sigma = M/c_1 \text{ in Lemma 4.2, there exists a positive constant } c \text{ such that }
\]
\[
|u(z) - u(w)| \leq \int_{\mathbb{R}^{N+1}} |\Gamma(\zeta^{-1} \circ z) - \Gamma(\zeta^{-1} \circ w)||Lu(\zeta)|d\zeta
\]
\[
\leq \int_{|\zeta^{-1}oz| \geq M/|z^{-1}ow|} \frac{c \cdot |z^{-1} \circ w|}{|\zeta^{-1} \circ z|^{Q+1}} |Lu(\zeta)|d\zeta
\]
\[
+ \int_{|\zeta^{-1}oz| < M/|z^{-1}ow|} \frac{c}{|\zeta^{-1} \circ z|^{Q}} |Lu(\zeta)|d\zeta
\]
\[
\equiv I_1 + I_2 + I_3.
\]

By applying Lemma 4.2 and choosing $\alpha = 2$ and $\sigma = M/c_1$, there exists a positive constant $c$ such that
\[
|I_1| \leq c\|Lu\|_{L^{p,\lambda}((\mathbb{R}^{N+1})^N)} |z^{-1} \circ w|^{2\beta+\lambda-(Q+2)};
\]

choosing $\beta = 2$ and $\sigma = Mc_1$ in Lemma 4.2, there exists a positive constant $c$ such that
\[
|I_2| \leq c\|Lu\|_{L^{p,\lambda}((\mathbb{R}^{N+1})^N)} |z^{-1} \circ w|^{2\beta+\lambda-(Q+2)};
\]

choosing $\beta = 2$ and $\sigma = c_2(1 + M)$ in Lemma 4.2, there exists a positive constant $c$ such that
\[
|I_3| \leq c\|Lu\|_{L^{p,\lambda}((\mathbb{R}^{N+1})^N)} |z^{-1} \circ w|^{2\beta+\lambda-(Q+2)}.
\]

Hence, by (4.3), (4.4) and (4.5), it is easy to obtain
\[
\frac{|u(z) - u(w)|}{|z^{-1} \circ w|^\theta} \leq C\|Lu\|_{L^{p,\lambda}((\mathbb{R}^{N+1})^N)},
\]

where $C$ is a positive constant, $z, w \in \mathbb{R}^{N+1}$, $z \neq w$.

By (2.7), we write
\[
\partial_{x_j} u(z) = -\int_{\mathbb{R}^{N+1}} \Gamma_j(\zeta^{-1} \circ z) Lu(\zeta) d\zeta
\]
for every $z \in \mathbb{R}^{N+1}$ and $j = 1, 2, \ldots, p_0$. Analogously, by Lemmas 3.1 and 4.1, we get that there exist $M, c > 0$ such that
\[
|\partial_{x_j} u(z) - \partial_{x_j} u(w)| \leq \int_{\mathbb{R}^{N+1}} |\Gamma_j(\zeta^{-1} \circ z) - \Gamma_j(\zeta^{-1} \circ w)||Lu(\zeta)|d\zeta
\]
\[
\leq \int_{|\zeta^{-1}oz| \geq M/|z^{-1}ow|} \frac{c \cdot |z^{-1} \circ w|}{|\zeta^{-1} \circ z|^{Q+2}} |Lu(\zeta)|d\zeta
\]
\[
+ \int_{\|\zeta^{-1} \circ z - w\| < M \|z^{-1} \circ w\|} \left\| \zeta^{-1} \circ z \right\|^{\ell} |L u(\zeta)| d\zeta \\
+ \int_{\|\zeta^{-1} \circ z - w\| < M \|z^{-1} \circ w\|} \left\| \zeta^{-1} \circ w \right\|^{\ell} |L u(\zeta)| d\zeta \\
= I_1' + I_2' + I_3'.
\]

By applying Lemma 4.2 and choosing \(\alpha = 1\) and \(\sigma = M/c_1\), there exists a positive constant \(c\) such that

\[
|I_1'| \leq c \|L u\|_{L^p,\lambda}(\mathbb{R}^{N+1}) \|z^{-1} \circ w\|^{p \lambda - (Q + 2)}.
\]

choosing \(\beta = 1\) and \(\sigma = M c_1\) in Lemma 4.2, there exists a positive constant \(c\) such that

\[
|I_2'| \leq c \|L u\|_{L^p,\lambda}(\mathbb{R}^{N+1}) \|z^{-1} \circ w\|^{p \lambda - (Q + 2)}.
\]

choosing \(\beta = 1\) and \(\sigma = c_2 (1 + M)\) in Lemma 4.2, there exists a positive constant \(c\) such that

\[
|I_3'| \leq c \|L u\|_{L^p,\lambda}(\mathbb{R}^{N+1}) \|z^{-1} \circ w\|^{p \lambda - (Q + 2)}.
\]

Hence, by (4.6), (4.7) and (4.8), we derive

\[
\left| \frac{\partial x_j u(z) - \partial x_j u(w)}{\|z^{-1} \circ w\|^\delta} \right| \leq C \|L u\|_{L^p,\lambda}(\mathbb{R}^{N+1}), \quad j = 1, 2, \ldots, p_0,
\]

where \(C\) is a positive constant, \(z, w \in \mathbb{R}^{N+1}, \ z \neq w\). This ends the proof. \(\square\)

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