CODIMENSION REDUCTION FOR SUBMANIFOLDS OF UNIT \((4m + 3)\)-SPHERE AND ITS APPLICATIONS

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Abstract. In this paper we establish codimension reduction theorem for submanifolds of a \((4m+3)\)-dimensional unit sphere \(S^{4m+3}\) with Sasakian 3-structure and apply it to submanifolds of a quaternionic projective space.

1. Introduction

As is well-known, for a submanifold \(M\) of a Riemannian manifold \(\widetilde{M}\), the codimension of \(M\) is said to be reduced if there exists a totally geodesic submanifold \(\mathcal{M}\) of \(\widetilde{M}\) such that \(M \subset \mathcal{M}\).

In particular, when the ambient manifold is a complex manifold, the intermediate submanifold \(\mathcal{M}\) is requested to be not only totally geodesic, but also complex submanifold.

The codimension reduction problem was investigated by Allendoerfer [1] in the case that the ambient manifold \(\widetilde{M}\) is a Euclidean space and by Erbacher [14] in the case that \(\widetilde{M}\) is a real space form. For submanifolds of a complex projective space, Cecil [2] proved a codimension reduction theorem for complex submanifolds. Okumura [14] extended Cecil’s result to real submanifolds by using the standard submersion method established by Lawson [12] (for real submanifolds of a complex hyperbolic space, see [8]).

As a quaternionic analogue for real submanifolds of a quaternionic projective space, Kwon and the second author [11] provided a codimension reduction theorem which may correspond to Okumura’s result in [14] (for real submanifolds of a quaternionic hyperbolic space, see [9]).

On the other hand, in 1982, Okumura [13] studied submanifolds \(M\) of an odd-dimensional sphere with the canonical Sasakian structure \(\{\phi, \xi\}\) to which the structure vector field \(\xi\) is always tangent and proved that, under some

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additional conditions, if $\dim(T_x M \cap \phi T_x M^\perp) < \dim M$, where $T_x M$ and $T_x M^\perp$ denote the tangent space and normal space to $M$ at $x \in M$, respectively. Using this theorem, in his paper [13], Okumura presented a codimension reduction theorem for real submanifolds of a complex projective space by means of the standard submersion method due to Lawson [12].

In this paper we first consider a $(4m + 3)$-dimensional unit sphere with the canonical Sasakian 3-structure $\{\phi, \psi, \theta\}$ (for definition, see [7, 10, 17]). Let $M$ be a real submanifold of the space to which the structure vector fields $\xi, \eta, \zeta$ are always tangent. If at each point $x \in M$ the tangent space $T_x M$ satisfies $\phi T_x M \subset T_x M$, $\psi T_x M \subset T_x M$, $\theta T_x M \subset T_x M$, $M$ is called an invariant submanifold under $\{\phi, \psi, \theta\}$. It is well known that an invariant submanifold is a manifold with Sasakian 3-structure. We consider the more general case that at each point $x \in M$ $T_x M$ and $T_x M^\perp$ satisfy the condition that $\dim(T_x M \cap \phi T_x M \cap \psi T_x M \cap \theta T_x M^\perp)$ is independent of $x$. Such submanifolds involve invariant submanifolds as a special case.

The main purpose of the paper is to study relations between $\dim(T_x M \cap \phi T_x M \cap \psi T_x M \cap \theta T_x M^\perp)$ and the codimension of $M$, and to prove that, under some additional conditions, if $\dim(T_x M \cap \phi T_x M \cap \psi T_x M \cap \theta T_x M^\perp)$ is less than the codimension, then there exists a totally geodesic invariant submanifold $M'$ such that $M \subset M'$, which will be used in codimension reducing for submanifolds of a quaternionic projective space by using the standard submersion method established by Lawson [12].

All manifolds, submanifolds and geometric objects will be assumed to be connected, differentiable and of class $C^\infty$, and all maps also be of class $C^\infty$ if not stated otherwise.

2. Submanifolds of a $(4m + 3)$-dimensional unit sphere

Let us consider a $(4m + 3)$-dimensional unit sphere $S^{4m+3}$ as a real hypersurface of the real $(4m + 1)$-dimensional quaternionic number space $Q^{m+1}$. For any point $x$ in $S^{4m+3}$, we set

$$
\xi = E_1 x, \quad \eta = E_2 x, \quad \zeta = E_3 x,
$$

where $\{E_1, E_2, E_3\}$ denotes the canonical quaternionic Kähler structure of $Q^{m+1}$. Then $\{\xi, \eta, \zeta\}$ becomes a Sasakian 3-structure, namely, $\xi, \eta$ and $\zeta$ are mutually orthogonal unit Killing vector fields which satisfy

$$
\nabla_Y \nabla_X \xi = g(X, \xi)Y - g(Y, X)\xi,
\nabla_Y \nabla_X \eta = g(X, \eta)Y - g(Y, X)\eta,
\nabla_Y \nabla_X \zeta = g(X, \zeta)Y - g(Y, X)\zeta
$$

for any vector fields $X, Y$ tangent to $S^{4m+3}$, where $g$ denotes the canonical metric on $S^{4m+3}$ induced from that of $Q^{m+1}$ and $\nabla$ the Riemannian connection.
with respect to \( g \). In this case, putting
\[
(2.2) \quad \phi X = \nabla_X \xi, \quad \psi X = \nabla_X \eta, \quad \theta X = \nabla_X \zeta,
\]
it follows that
\[
(2.3) \quad \phi \xi = 0, \quad \psi \eta = 0, \quad \theta \zeta = 0,
\]
and
\[
(2.4) \quad \phi^2 = -I + f_\xi \otimes \xi, \quad \psi^2 = -I + f_\eta \otimes \eta, \quad \theta^2 = -I + f_\zeta \otimes \zeta,
\]
where \( I \) denotes the identity transformation and
\[
(2.5) \quad f_\xi(X) = g(\xi, X), \quad f_\eta(X) = g(\eta, X), \quad f_\zeta(X) = g(\zeta, X).
\]
Moreover, from (2.1) and (2.2), we have
\[
(2.6) \quad (\nabla_Y \phi)X = g(X, \xi)Y - g(Y, X)\xi, \quad (\nabla_Y \psi)X = g(X, \eta)Y - g(Y, X)\eta, \quad (\nabla_Y \theta)X = g(X, \zeta)Y - g(Y, X)\zeta
\]
for any vector fields \( X, Y \) tangent to \( S^{4m+3} \) (cf. [7, 10, 15, 17]).

Let \( M \) be an \((n + 3)\)-dimensional submanifold isometrically immersed in \( S^{4m+3} \) and denote by \( TM \) and \( TM^\perp \) the tangent and normal bundle of \( M \), respectively. We shall delete the isometric immersion \( \iota : M \rightarrow S^{4m+3} \) and its differential \( i_* \) in our notations. Let \( \nabla \) and \( \nabla^\perp \) denote the covariant differentiation in \( M \) and the normal connection of \( M \) in \( S^{4m+3} \), respectively. To each \( N_x \in T_x M^\perp \), we extend \( N_x \) to a normal vector field \( N \) defined in a neighborhood of \( x \). Given an orthonormal basis \( \{(N_1)_x, \ldots, (N_p)_x\} \) of \( T_x M^\perp \), we denote by \( H_A \) the Weingarten map with respect to \( N_A \), which will be called the \textit{second fundamental tensor} associated to \( N_A \). If the second fundamental tensors \( H_A \) \((A = 1, \ldots, p)\) vanish identically on \( M \), \( M \) is called a \textit{totally geodesic submanifold}. The first normal space \( N^1_x \) is defined to be the orthogonal complement of \( \{N_x \in T_x M^\perp \mid H_N = 0\} \) in \( T_x M^\perp \) (cf. [4]). If \( N_1, \ldots, N_p \) are orthonormal normal vector fields in a neighborhood of \( x \in M \), they determine normal connection forms \( s_{AB} \) in a neighborhood of \( x \) by
\[
\nabla^\perp_X N_A = \sum_{B=1}^{p} s_{AB}(X)N_B
\]
for \( X \) tangent to \( M \). Then we have the following Gauss and Weingarten formulas:
\[
(2.7) \quad \tilde{\nabla}_X Y = \nabla_X Y + \sum_{A=1}^{p} g(H_A X, Y)N_A, \quad g(H_A X, Y) = g(X, H_A Y),
\]
\(\nabla_X N_A = -H_A X + \sum_{B=1}^{p} s_{AB}(X)N_B, \quad s_{AB}(X) = -s_{BA}(X).\)  

The mean curvature vector field \(\mu\) of \(M\) is defined by

\[\mu = \frac{1}{n+3} \sum_{A=1}^{p} (\text{trace} H_A)N_A.\]

The submanifold \(M\) is said to be **minimal** if \(\mu\) vanishes identically on \(M\).

Differentiating (2.9) covariantly, we have

\[(n+3)\nabla_X^\perp \mu = \sum_{A=1}^{p} \{(X\text{trace} H_A)N_A + \sum_{B=1}^{p} (\text{trace} H_A)s_{AB}(X)N_B\}.\]

Hence the mean curvature vector field is parallel with respect to the normal connection \(\nabla^\perp\) if and only if

\[X\text{trace} H_A = \sum_{B=1}^{p} (\text{trace} H_B)s_{AB}(X).\]  

Let us denote by \(R\) and \(R^N\) the curvature tensors for \(\nabla\) and \(\nabla^\perp\), respectively.

Since the curvature tensor \(\bar{R}\) for \(\bar{\nabla}\) on \(S^{4m+3}\) is given by

\[\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y,\]

we have the following relations:

\[(\nabla_X H_A)Y - (\nabla_Y H_A)X = \sum_{B=1}^{p} \{s_{AB}(X)H_B Y - s_{AB}(Y)H_B X\},\]

\[R^N(X,Y)N_A = \sum_{B=1}^{p} g((H_A H_B - H_B H_A)X, Y).\]

If \(R^N\) vanishes identically on \(M\), the normal connection of \(M\) in \(S^{4m+3}\) is said to be **flat**. The normal connection of \(M\) is flat if and only if \(H_A H_B = H_B H_A\) for all \(A, B = 1, 2, \ldots, p\) (cf. [3]).

For any \(X \in TM\) and for \(N_A, A+1, \ldots, p\), the transforms \(\phi X, \psi X, \theta X\) and \(\phi N_A, \psi N_A, \theta N_A\) are, respectively, written in the following forms:

\[(i) \phi X = F_X + \sum_{A=1}^{p} u^A(X)N_A, \quad (ii) \psi X = G_X + \sum_{A=1}^{p} v^A(X)N_A,\]

\[\text{(iii) } \theta X = H_X + \sum_{A=1}^{p} w^A(X)N_A,\]

\[(i) \phi N_A = -U_A + \sum_{B=1}^{p} F_A^\phi N_B, \quad (ii) \psi N_A = -V_A + \sum_{B=1}^{p} F_A^\psi N_B,\]
(iii) $\theta N_A = -W_A + \sum_{A=1}^{p} P^\phi_{AB} N_B$, 

where $\{F, G, H\}$ and $\{P^\phi, P^\psi, P^\theta\}$ define endomorphisms of $TM$ and $TM^\perp$, respectively, and $\{U_A, V_A, W_A\}$ and $\{v^A, w^A, u^A\}$ are local tangent vector fields and local 1-forms on $M$. They satisfy

(2.15) $g(FX, Y) = -g(X, FY)$, $g(GX, Y) = -g(X, GY)$, $g(HX, Y) = -g(X, HY)$, 

(2.16) $P^\phi_{AB} = -P^\phi_{BA}$, $P^\psi_{AB} = -P^\psi_{BA}$, $P^\theta_{AB} = -P^\theta_{BA}$, 

(2.17) $u^A(X) = g(U_A, X)$, $v^A(X) = g(V_A, X)$, $w^A(X) = g(W_A, X)$

for tangent vectors $X, Y$ to $M$. If $U_A = 0, V_A = 0, W_A = 0, A = 1, 2, \ldots, p$ identically, the submanifold is called an invariant submanifold because the structure vector field $\xi$ is always tangent to $M$ and use the same notations as appeared in the case of ambient manifold. Then, from (2.3), (2.4) and (2.13), we have

(2.18) $F \xi = 0$, $G \eta = 0$, $H \zeta = 0$, 

(2.19) $F \eta = \zeta$, $F \zeta = -\eta$, $G \zeta = \xi$, $G \xi = -\zeta$, $H \xi = \eta$, $H \eta = -\xi$, 

(2.20) $u^A(\xi) = u^A(\eta) = u^A(\zeta) = 0$, $v^A(\xi) = v^A(\eta) = v^A(\zeta) = 0$, $w^A(\xi) = w^A(\eta) = w^A(\zeta) = 0$, $A = 1, 2, \ldots, p$.

Applying $\phi$ to both sides of (2.13)(i) and (2.14)(i), it follows from (2.4), (2.5), (2.13)-(2.14) and (2.16)-(2.17) that

(2.21) $F^2 X = -X + \sum_{A=1}^{p} u^A(X) U_A + g(\xi, X) \xi$, $FU_A = -\sum_{B=1}^{p} P^\phi_{AB} U_B$, 

$g(U_A, U_B) = \delta_{AB} + \sum_{C=1}^{p} P^\phi_{AC} P^\phi_{CB}$

because the structure vector field $\xi$ is tangent to $M$. Similarly, from (2.13)(iii), (2.13)(iii), (2.14)(ii) and (2.14)(iii), we get

(2.22) $G^2 X = -X + \sum_{A=1}^{p} v^A(X) V_A + g(\eta, X) \eta$, $GV_A = -\sum_{B=1}^{p} P^\psi_{AB} V_B$, 

$g(V_A, V_B) = \delta_{AB} + \sum_{C=1}^{p} P^\psi_{AC} P^\psi_{CB}$,

(2.23) $H^2 X = -X + \sum_{A=1}^{p} w^A(X) W_A + g(\zeta, X) \zeta$, $HW_A = -\sum_{B=1}^{p} P^\theta_{AB} W_B$, 

$g(W_A, W_B) = \delta_{AB} + \sum_{C=1}^{p} P^\theta_{AC} P^\theta_{CB}$.
Applying $\psi$ and $\theta$ to both sides of (2.13)(i), respectively, and using (2.3)-(2.5), (2.13)-(2.14) and (2.16)-(2.17), we have

\begin{align}
GFX &= -HX + \sum_{A=1}^{p} u^A(X)V_A + g(\xi, X)\eta, \\
v^A(FX) &= -w^A(X) + \sum_{B=1}^{p} P_{AB}^\psi u^B(X),
\end{align}

\begin{align}
HFX &= GX + \sum_{A=1}^{p} u^A(X)W_A + g(\xi, X)\zeta, \\
w^A(FX) &= v^A(X) + \sum_{B=1}^{p} P_{AB}^\theta u^B(X).
\end{align}

Similarly, it follows from (2.13)(ii) and (2.13)(iii) that

\begin{align}
HGX &= -FX + \sum_{A=1}^{p} v^A(X)W_A + g(\eta, X)\zeta, \\
w^A(GX) &= -u^A(X) + \sum_{B=1}^{p} P_{AB}^\theta v^B(X),
\end{align}

\begin{align}
FGX &= HX + \sum_{A=1}^{p} v^A(X)U_A + g(\eta, X)\xi, \\
u^A(GX) &= w^A(X) + \sum_{B=1}^{p} P^\phi_{AB} v^B(X),
\end{align}

\begin{align}
FHX &= -GX + \sum_{A=1}^{p} w^A(X)U_A + g(\zeta, X)\xi, \\
u^A(HX) &= -v^A(X) + \sum_{B=1}^{p} P^\phi_{AB} w^B(X),
\end{align}

\begin{align}
GHX &= FX + \sum_{A=1}^{p} w^A(X)V_A + g(\zeta, X)\eta, \\
v^A(HX) &= u^A(X) + \sum_{B=1}^{p} P_{AB}^\psi w^B(X).
\end{align}

Applying $\psi$ and $\theta$ to both sides of (2.14)(i), respectively, and using (2.4)-(2.5), (2.13)-(2.14) and (2.17), we have

\begin{align}
GU_A &= -W_A - \sum_{B=1}^{p} P_{AB}^\phi V_B, \\
g(U_A, V_B) &= P_{AB}^\theta + \sum_{C=1}^{p} P_{AC}^\phi P_{CB}^\psi.
\end{align}
(2.31) \[ HU_A = V_A - \sum_{B=1}^{p} P_{AB}^{\phi} W_B, \quad g(U_A, W_B) = -P_{AB}^{\phi} + \sum_{C=1}^{p} P_{AC}^{\psi} P_{CB}^{\phi}. \]

Similarly, it follows from (2.14)(ii) and (2.14)(iii) that

(2.32) \[ HV_A = -U_A - \sum_{B=1}^{p} P_{AB}^{\psi} W_B, \quad g(V_A, W_B) = P_{AB}^{\phi} + \sum_{C=1}^{p} P_{AC}^{\psi} P_{CB}^{\phi}. \]

(2.33) \[ FV_A = W_A - \sum_{B=1}^{p} P_{AB}^{\phi} U_B, \quad g(V_A, U_B) = -P_{AB}^{\psi} + \sum_{C=1}^{p} P_{AC}^{\psi} P_{CB}^{\phi}. \]

(2.34) \[ FW_A = -V_A - \sum_{B=1}^{p} P_{AB}^{\phi} U_B, \quad g(W_A, U_B) = P_{AB}^{\psi} + \sum_{C=1}^{p} P_{AC}^{\psi} P_{CB}^{\phi}. \]

(2.35) \[ GW_A = U_A - \sum_{B=1}^{p} P_{AB}^{\phi} V_B, \quad g(W_A, V_B) = -P_{AB}^{\phi} + \sum_{C=1}^{p} P_{AC}^{\psi} P_{CB}^{\phi}. \]

Differentiating (2.13)(i) covariantly and making use of (2.6)-(2.8), (2.13)-(2.14) and (2.16), we obtain

(2.36) \[ (\nabla_Y F)X = g(X, \xi)Y - g(X, Y)\xi - \sum_{A=1}^{p} g(H_A X, Y)U_A \]
\[ + \sum_{A=1}^{p} u^A(X)H_A Y, \]

(2.37) \[ (\nabla_Y u^A)X = -g(H_A FX, Y) - \sum_{B=1}^{p} P_{AB}^{\phi} g(H_B X, Y) \]
\[ + \sum_{B=1}^{p} s_{AB}^{\phi}(Y)u^B(X). \]

Similarly, from (2.13)(ii) and (2.13)(iii), we also get

(2.38) \[ (\nabla_Y G)X = g(X, \eta)Y - g(X, Y)\eta - \sum_{A=1}^{p} g(H_A X, Y)V_A \]
\[ + \sum_{A=1}^{p} v^A(X)H_A Y, \]

(2.39) \[ (\nabla_Y v^A)X = -g(H_A GX, Y) - \sum_{B=1}^{p} P_{AB}^{\psi} g(H_B X, Y) \]
\[ + \sum_{B=1}^{p} s_{AB}^{\psi}(Y)v^B(X), \]
\begin{equation}
(\nabla_Y H)X = g(X, \zeta)Y - g(X, Y)\zeta - \sum_{A=1}^{p} g(H_AX, Y)W_A \\
+ \sum_{A=1}^{p} w^A(X)H_AY,
\end{equation}

\begin{equation}
(\nabla_Y w^A)X = -g(H_A H X, Y) - \sum_{B=1}^{p} P_{AB}^\phi g(H_B X, Y) \\
+ \sum_{B=1}^{p} s_{AB}(Y) w^B(X).
\end{equation}

Differentiating (2.14)(i) covariantly and taking account of (2.6)-(2.8), (2.13)-(2.14) and (2.16), we obtain

\begin{equation}
\nabla_X U_A = F H_A X - \sum_{B=1}^{p} P_{AB}^\phi H_B X + \sum_{B=1}^{p} s_{AB}(X)U_B,
\end{equation}

\begin{equation}
\nabla_X P_{AB}^\phi = g(U_A, H_B X) - u^B(H_A X) - \sum_{C=1}^{p} P_{AC}^\psi s_{CB}(X) \\
+ \sum_{C=1}^{p} P_{BC}^\phi s_{CA}(X).
\end{equation}

Similarly, from (2.13)(ii) and (2.13)(iii), we also get

\begin{equation}
\nabla_X V_A = G H_A X - \sum_{B=1}^{p} P_{AB}^\psi H_B X + \sum_{B=1}^{p} s_{AB}(X)V_B,
\end{equation}

\begin{equation}
\nabla_X P_{AB}^\psi = g(V_A, H_B X) - v^B(H_A X) - \sum_{C=1}^{p} P_{AC}^\phi s_{CB}(X) \\
+ \sum_{C=1}^{p} P_{BC}^\psi s_{CA}(X),
\end{equation}

\begin{equation}
\nabla_X W_A = H H_A X - \sum_{B=1}^{p} P_{AB}^\theta H_B X + \sum_{B=1}^{p} s_{AB}(X)W_B,
\end{equation}

\begin{equation}
\nabla_X P_{AB}^\theta = g(W_A, H_B X) - w^B(H_A X) - \sum_{C=1}^{p} P_{AC}^\psi s_{CB}(X) \\
+ \sum_{C=1}^{p} P_{BC}^\theta s_{CA}(X).\end{equation}

Moreover, it is clear from (2.2) that

\begin{equation}
\nabla_X \xi = F X, \quad \nabla_X \eta = G X, \quad \nabla_X \zeta = H X,
\end{equation}
(2.49) \[ H_A \xi = U_A, \quad H_A \eta = V_A, \quad H_A \zeta = W_A. \]

3. Laplacian for a global function defined on \( M \)

We define a function \( f \) on \( M \) by

\[
\sum_{A=1}^{p} \{ u^A(U_A) + v^A(V_A) + w^A(W_A) \}.
\]

Then, since \( \xi, \eta, \zeta \) are mutually orthogonal unit vector fields, (2.21)-(2.23) yield

\[
f = \text{tr} F^2 + \text{tr} G^2 + \text{tr} H^2 + 3(n-1), \quad (\text{tr} := \text{trace})
\]

which means that \( f \) is independent of the choice of \( N_A \)'s and thus \( f \) is a global function defined on \( M \). \( f \) vanishes identically on \( M \) if and only if \( M \) is an invariant submanifold under \( \{ \phi, \psi, \theta \} \).

From now on we compute the Laplacian \( \Delta f \). For any vector field \( X \) on \( M \) it follows from (2.15), (2.17)-(2.18), (2.36), (2.38) and (2.40) that

\[
\frac{1}{2} X f = \frac{1}{2} X (\text{tr} F^2 + \text{tr} G^2 + \text{tr} H^2)
\]

\[
= \text{tr} (\nabla_X F) F + \text{tr} (\nabla_X G) G + \text{tr} (\nabla_X H) H
\]

\[
= 2 \sum_{A=1}^{p} \{ g(FH_A X, U_A) + g(GH_A X, V_A) + g(HH_A X, W_A) \},
\]

from which together with (2.20)-(2.23), (2.42), (2.44), (2.46) and (2.49), we get

\[
\frac{1}{4}(\nabla_Y \nabla_X f - \nabla_{\nabla_Y X} f) = \frac{1}{4} \{ \nabla_Y (X f) - (\nabla_Y X) f \}
\]

\[
= \sum_{A=1}^{p} \{ g((\nabla_Y F) H_A X, U_A) + g(F(\nabla_Y H_A) X, U_A) + g(FH_A X, \nabla_Y U_A)
\]

\[
+ g((\nabla_Y G) H_A X, V_A) + g(G(\nabla_Y H_A) X, V_A) + g(GH_A X, \nabla_Y V_A)
\]

\[
+ g((\nabla_Y H) H_A X, W_A) + g(H(\nabla_Y H_A) X, W_A) + g(HH_A X, \nabla_Y W_A)
\]

\[
= \sum_{A=1}^{p} \{ g(U_A, X) g(U_A, Y) + g(V_A, X) g(V_A, Y) + g(W_A, X) g(W_A, Y)
\]

\[
- g((\nabla_Y H_A) FU_A, X) - g((\nabla_Y H_A) GV_A, X) - g((\nabla_Y H_A) HW_A, X)
\]

\[
- g(H_A F^2 H_A X, Y) - g(H_A G^2 H_A X, Y) - g(H_A H^2 H_A X, Y)
\]

\[
+ \sum_{B=1}^{p} \{ g(HA UB, X) g(HB UA, Y) + g(HA VB, X) g(HB VA, Y)
\]

\[
+ g(HA WB, X) g(HB WA, Y) - g(HB HA X, Y) g(UB, U_A)
\]

\[
- g(HB HA X, Y) g(VB, V_A) - g(HB HA X, Y) g(WB, W_A)
\]

\[
- PAB g(HB FH_A X, Y) - PAB g(HB GH_A X, Y)
\]

\[ \]
which together with (3.2) yield

\[ \begin{align*}
- P_{AB}^0 g(H_B H_A X, Y) + s_{AB}(Y) g(F H_A X, U_B) \\
+ s_{AB}(Y) g(G H_A X, V_B) + s_{AB}(Y) g(H H_A X, W_B)) \end{align*} \]

Hence we have

On the other hand, substituting \( FU_A, GV_A \) and \( HW_A \) for \( X \) into (2.11), respectively, we have

\[ \begin{align*}
(\nabla Y H_A) FU_A &= (\nabla FU_A) H_A Y + \sum_{B=1}^{p} \{ s_{AB}(Y) H_B FU_A - s_{AB}(FU_A) H_B Y \}, \\
(\nabla Y H_A) GV_A &= (\nabla GV_A) H_A Y + \sum_{B=1}^{p} \{ s_{AB}(Y) H_B GV_A - s_{AB}(GV_A) H_B Y \}, \\
(\nabla Y H_A) HW_A &= (\nabla HW_A) H_A Y + \sum_{B=1}^{p} \{ s_{AB}(Y) H_B HW_A - s_{AB}(HW_A) H_B Y \},
\end{align*} \]

which together with (3.2) yield

\[ \begin{align*}
\frac{1}{4} (\nabla Y \nabla_X f - \nabla_Y \nabla_X f) \\
= \sum_{A=1}^{p} \left[ g(U_A, X) g(U_A, Y) + g(V_A, X) g(V_A, Y) + g(W_A, X) g(W_A, Y) \\
- g((\nabla FU_A) H_A Y, X) - g((\nabla GV_A) H_A Y, X) - g((\nabla HW_A) H_A Y, X) \\
- g(H_A F^2 H_A X, Y) - g(H_A G^2 H_A X, Y) - g(H_A H^2 H_A X, Y) \\
+ \sum_{B=1}^{p} \{ s_{AB}(FU_A) g(H_B Y, X) + s_{AB}(GV_A) g(H_B Y, X) \\
+ s_{AB}(HW_A) g(H_B Y, X) + g(H_A U_B, X) g(H_B U_A, Y) \\
+ g(H_B V_B, X) g(H_B V_A, Y) + g(H_A W_B, X) g(H_B W_A, Y) \\
- g(H_B H_A X, Y) g(U_B, U_A) - g(H_B H_A X, Y) g(V_B, V_A) \\
- g(H_B H_A X, Y) g(W_B, W_A) - P_{AB}^0 g(H_B F H_A X, Y) \\
- P_{AB}^0 g(H_B G H_A X, Y) - P_{AB}^0 g(H_B H H_A X, Y)) \right].
\end{align*} \]

Hence we have

(3.3)

\[ \begin{align*}
\frac{1}{4} \Delta f &= \sum_{A=1}^{p} \left[ g(U_A, U_A) + g(V_A, V_A) + g(W_A, W_A) - \text{tr} F^2 H_A^2 - \text{tr} G^2 H_A^2 \\
- \text{tr} H^2 H_A^2 - \nabla FU_A (\text{tr} H_A) - \nabla GV_A (\text{tr} H_A) - \nabla HW_A (\text{tr} H_A) \\
+ \sum_{B=1}^{p} \{ s_{AB}(FU_A) \text{tr} H_B + s_{AB}(GV_A) \text{tr} H_B + s_{AB}(HW_A) \text{tr} H_B \\
+ g(H_A U_B, H_B U_A) + g(H_A V_B, H_B V_A) + g(H_A W_B, H_B W_A) \\
- (\text{tr} H_B H_A) g(U_B, U_A) - (\text{tr} H_B H_A) g(V_B, V_A) \right].
\end{align*} \]
\[ - (\text{tr } H_B H_A) g(W_B, W_A) - P^\phi_{AB} (\text{tr } F H_A H_B) \]
\[ - P^\psi_{AB} (\text{tr } G H_A H_B) - P^\theta_{AB} (\text{tr } H H_A H_B) \].

On the other hand, (2.21)-(2.23) and (2.49) imply
\[ \text{tr } F^2 H^2_A = - \text{tr } H^2_A + g(U_A, U_A) + \sum_{B=1}^{p} g(H_A U_B, H_A U_B), \]
\[ \text{tr } G^2 H^2_A = - \text{tr } H^2_A + g(V_A, V_A) + \sum_{B=1}^{p} g(H_A V_B, H_A V_B), \]
\[ \text{tr } H^2 H^2_A = - \text{tr } H^2_A + g(W_A, W_A) + \sum_{B=1}^{p} g(H_A W_B, H_A W_B), \]

from which combined with (3.3) it follows that
\[ \frac{1}{4} \Delta f = \sum_{A=1}^{p} \left[ 3 \text{tr } H^2_A - (F U_A) \text{tr } H_A - (G V_A) \text{tr } H_A - (H W_A) \text{tr } H_A \right. \]
\[ + \left. \sum_{B=1}^{p} \{ s_{AB} (F U_A) \text{tr } H_B + s_{AB} (G V_A) \text{tr } H_B + s_{AB} (H W_A) \text{tr } H_B \right. \]
\[ + g(H_A U_B, H_B U_A - H_A U_B) + g(H_A V_B, H_B V_A - H_A V_B) \]
\[ + g(H_A W_B, H_B W_A - H_A W_B) - (\text{tr } H_B H_A) g(U_B, U_A) \]
\[ - (\text{tr } H_B H_A) g(V_B, V_A) - (\text{tr } H_B H_A) g(W_B, W_A) \]
\[ - P^\phi_{AB} (\text{tr } F H_A H_B) - P^\psi_{AB} (\text{tr } G H_A H_B) - P^\theta_{AB} (\text{tr } H H_A H_B) \].

Now we prepare some lemmas for later use.

**Lemma 3.1.** Let \( M \) be a submanifold of a unit \((4m + 3)\)-sphere \( S^{4m+3} \) to which the Sasakian 3-structure vector fields \( \xi, \eta, \zeta \) are always tangent. If the normal connection of \( M \) in \( S^{4m+3} \) is flat, then
\[ \sum_{A=1}^{p} u^A(U_A), \sum_{A=1}^{p} v^A(V_A), \sum_{A=1}^{p} w^A(W_A) \]
are constant and consequently the function \( f \) is also constant.

**Proof.** For any vector field \( X \) tangent to \( M \), it follows from (2.7), (2.15), (2.21) and (2.42) that
\[ \frac{1}{2} X \left( \sum_{A=1}^{p} u^A(U_A) \right) = \sum_{A=1}^{p} g(\nabla_X U_A, U_A) \]
\[ = \sum_{A=1}^{p} [g(F H_A X, U_A) - \sum_{B=1}^{p} P^\phi_{AB} g(X, H_B U_A)] \]
\[ \sum_{A,B=1}^{p} P_{AB}^p g(X, H_AU_B - H_BU_A). \]

On the other hand, if the normal connection is flat, then by means of (2.49) we obtain
\[ H_AU_B - H_BU_A = (H_AH_B - H_BH_A)\xi = 0, \]
(3.5)
\[ H_AV_B - H_VA = (H_AH_B - H_BH_A)\eta = 0, \]
\[ H_AW_B - H_BW_A = (H_AH_B - H_BH_A)\zeta = 0, \]
which together with the above equation yield \( X(\sum_{A=1}^{p} u^A(U_A)) = 0 \), namely \( \sum_{A=1}^{p} u^A(U_A) \) is constant. Similarly we can prove that \( \sum_{A=1}^{p} v^A(V_A) \) and \( \sum_{A=1}^{p} w^A(W_A) \) are also constant. \( \square \)

**Lemma 3.2.** Let \( M \) be as in Lemma 3.1. If the normal connection of \( M \) in \( S^{4m+3} \) is flat and the mean curvature vector field \( \mu \) is parallel with respect to the normal connection, then
\[ 3 \sum_{A=1}^{p} \text{tr} H_A^2 = \sum_{A,B=1}^{p} \{(\text{tr} \ H_AH_B)g(U_A,U_B) + (\text{tr} \ H_AH_B)g(V_A,V_B) + (\text{tr} \ H_AH_B)g(W_A,W_B)\}. \]

**Proof.** Owing to Lemma 3.1, it follows from (2.10), (3.4) and (3.5) that
\[ 3 \sum_{A=1}^{p} \text{tr} H_A^2 = \sum_{A,B=1}^{p} \{(\text{tr} \ H_BH_A)g(U_B,U_A) + (\text{tr} \ H_BH_A)g(V_B,V_A) + (\text{tr} \ H_BH_A)g(W_B,W_A) + P_{AB}^p(\text{tr} \ FH_AH_B) + P_{AB}^p(\text{tr} \ GH_AH_B) + P_{AB}^p(\text{tr} \ HH_AH_B)\}, \]
from which combined with (2.16) and \( H_AH_B = H_BH_A \), we get (3.6). \( \square \)

### 4. Submanifolds with \( \dim(TM \cap \phi TM^\perp \cap \psi TM^\perp \cap \theta TM^\perp) < p \)

Suppose that at a point \( x \in M \)
\[ \dim(T_xM \cap \phi T_xM^\perp \cap \psi T_xM^\perp \cap \theta T_xM^\perp) = q. \]
Then we can choose in \( TM^\perp \) \( 3q \) orthonormal normal vector fields \( N_\alpha(\alpha = 1, \ldots, 3q) \) in such a way that
\[ \phi_x(N_\alpha)_x, \psi_x(N_\alpha)_x, \theta_x(N_\alpha)_x \in T_xM \oplus \text{Span}\{N_\alpha\}_{\alpha=1,\ldots,3q}, \]
and further (4.1)
\[ \phi_x(N_1)_x = \psi_x(N_{q+1})_x = \theta_x(N_{2q+1})_x, \ldots, \phi_x(N_q)_x = \psi_x(N_{2q})_x = \theta_x(N_{3q})_x. \]
In fact, if \{(X_1)_x, \ldots, (X_q)_x\} is an orthonormal basis of \(T_x M \cap \phi T_x M^\perp \cap \psi T_x M^\perp \cap \theta T_x M^\perp\), then there exist 3\(q\) normal vector fields \(N_\alpha\) such that
\[
\begin{align*}
(X_1)_x &= -\psi_x(N_1)_x = -\psi_x(N_{q+1})_x = -\theta_x(N_{2q+1})_x, \\
(X_q)_x &= -\psi_x(N_q)_x = -\psi_x(N_{2q})_x = -\theta_x(N_{3q})_x
\end{align*}
\]
and consequently all of \((X_i)_x\) are mutually orthogonal to \(\xi, \eta\) and \(\zeta\) because of (2.3). With such a choice of \(N_\alpha(\alpha = 1, \ldots, 3q)\), it follows from (2.14) that
\[
\begin{align*}
(X_1)_x &= (U_1)_x = (V_{q+1})_x = (W_{2q+1})_x, \\
&\vdots \quad \vdots \quad \vdots \quad \vdots \\
(U_1)_x &= \cdots = (U_{3q})_x = 0, \\
(V_1)_x &= \cdots = (V_{q})_x = (V_{2q+1})_x = \cdots = (V_{3q})_x = 0, \\
(W_1)_x &= \cdots = (W_{2q})_x = 0,
\end{align*}
\]
(4.3b) \(P^{\phi}_{(q+1)(2q+1)} = -P^{\phi}_{(3q+1)(q+1)} = 1, \ldots, P^{\phi}_{(3q)(3q)} = -P^{\phi}_{(3q)(2q)} = 1,\)
\(P^{\psi}_{(1)(2q+1)} = -P^{\psi}_{(2q+1)(1)} = 1, \ldots, P^{\psi}_{(q)(3q)} = -P^{\psi}_{(3q)(q)} = -1,\)
\(P^{\theta}_{(1)(q+1)} = -P^{\theta}_{(q+1)(1)} = 1, \ldots, P^{\theta}_{(q)(2q)} = -P^{\theta}_{(2q)(q)} = 1,\)
\(P^{\phi}_{\alpha\nu} = 0, P^{\psi}_{\alpha\nu} = 0, P^{\theta}_{\alpha\nu} = 0, (\alpha = 1, \ldots, 3q, \nu = 3q + 1, \ldots, p),\)
(4.3c) \(\phi_x(N_\nu)_x = -(U_\nu)_x + \sum_{\delta=3q+1}^{p} P^{\phi}_{\nu\delta}(x)(N_\delta)_x,\)
\(\psi_x(N_\nu)_x = -(V_\nu)_x + \sum_{\delta=3q+1}^{p} P^{\psi}_{\nu\delta}(x)(N_\delta)_x,\)
\(\theta_x(N_\nu)_x = -(W_\nu)_x + \sum_{\delta=3q+1}^{p} P^{\theta}_{\nu\delta}(x)(N_\delta)_x,\)
where we have used (2.4) and (4.2). Furthermore, it is clear from (2.4), (4.1) and (4.2) that
\[
\begin{align*}
g_x((X_i)_x, (U_\nu)_x) = 0, \quad g_x((X_i)_x, (V_\nu)_x) = 0, \quad g_x((X_i)_x, (W_\nu)_x) = 0, \quad i = 1, \ldots, q, \quad \nu = 3q + 1, \ldots, p.
\end{align*}
\]
**Lemma 4.1.** If the normal connection is flat, \(q\) is constant over \(M\).

**Proof.** We put
\[
\begin{align*}
f_1 &= \sum_{A=1}^{p} u^A(U_A), \quad f_2 = \sum_{A=1}^{p} v^A(V_A), \quad f_3 = \sum_{A=1}^{p} w^A(W_A).
\end{align*}
\]
Assume that at \(y \in M\)
\[
\dim(T_y M \cap \phi T_y M^\perp \cap \psi T_y M^\perp \cap \theta T_y M^\perp) = q'.
\]
and let say \( q < q' \). At \( x \) and \( y \) the function \( f_1 \) can be rewritten as the following:

\[
\begin{align*}
(4.5) \quad f_1(x) &= \sum_{\alpha=1}^{3q} u^\alpha(U_\alpha)(x) + \sum_{\nu=3q+1}^{3q'} u^\nu(U_\nu)(x) + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(x), \\
 f_1(y) &= \sum_{\alpha=1}^{3q} u^\alpha(U_\alpha)(y) + \sum_{\nu=3q+1}^{3q'+1} u^\nu(U_\nu)(y).
\end{align*}
\]

By means of Lemma 3.1, the function \( f_1 \) is constant and consequently (4.3) and (4.5) imply

\[
3q + \sum_{\nu=3q+1}^{3q'} u^\nu(U_\nu)(x) + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(x) = 3q' + \sum_{\nu=3q'+1}^{p} u^\nu(U_\nu)(y),
\]

or equivalently,

\[
(4.6) \quad 3(q - q') + \sum_{\nu=3q+1}^{3q'} u^\nu(U_\nu)(x) + \sum_{\nu=3q'+1}^{p} \{u^\nu(U_\nu)(x) - u^\nu(U_\nu)(y)\} = 0.
\]

On the other hand, it follows from (2.21) that \( u^\nu(U_\nu) = 1 - \sum_{A=1}^{p} (P_{\nu A}^\phi)^2 \) and thus

\[
\sum_{\nu=3q+1}^{3q'} u^A(U_A)(x) = 3(q' - q) - \sum_{\nu=3q'+1}^{p} \sum_{A=1}^{p} (P_{\nu A}^\phi)^2(x),
\]

from which, inserting back into (4.6), we have

\[
(4.7) \quad - \sum_{\nu=3q+1}^{3q'} \sum_{A=1}^{p} (P_{\nu A}^\phi)^2(x) + \sum_{\nu=3q'+1}^{p} \{u^\nu(U_\nu)(x) - u^\nu(U_\nu)(y)\} = 0.
\]

Since \( u^\nu(U_\nu) \) and \( P_{\nu A}^\phi \) are differentiable functions, we obtain

\[
\lim_{x \to y} \{u^\nu(U_\nu)(x) - u^\nu(U_\nu)(y)\} = 0.
\]

Hence it is clear from (4.7) that

\[
\sum_{A=1}^{p} (P_{\nu A}^\phi)^2(y) = 0, \text{ i.e., } P_{\nu A}^\phi(y) = 0, \quad \nu = q + 1, \ldots, q',
\]

which is a contradiction because of (4.3b). By using the functions \( f_2 \) or \( f_3 \) we can derive the same conclusion. \( \square \)

In the following we assume that \( 3q < p \) and that the mean curvature vector field \( \mu \) is parallel with respect to the normal connection. Then (2.21)-(2.23),
(3.6), (4.3) and (4.4) yield
\[
\sum_{\nu=3q+1}^{p} (\text{tr} H^{2}_{\nu})\left[\{1 - g(U_{\nu}, U_{\nu})\} + \{1 - g(V_{\nu}, V_{\nu})\} + \{1 - g(W_{\nu}, W_{\nu})\}\right]
\]
\[
= \sum_{\nu=3q+1}^{p} (\text{tr} H^{2}_{\nu}) \sum_{A=1}^{p} \left\{(P^{\phi}_{\nu A})^{2} + (P^{\psi}_{\nu A})^{2} + (P^{\theta}_{\nu A})^{2}\right\} = 0,
\]
which implies \(\text{tr} H^{2}_{\nu} = 0\) for \(\nu = 3q + 1, \ldots, p\). Thus \(H_{\nu} = 0, \nu = 3q + 1, \ldots, p\) and \(U_{\nu} = V_{\nu} = W_{\nu} = 0, \nu = 3q + 1, \ldots, p\) by means of (2.49). Particularly, when \(q = 0\), we have the following.

**Theorem 4.2.** Let \(M\) be an \((n+3)\)-dimensional complete submanifold isometrically immersed in a unit \((4m+3)\)-sphere \(S^{4m+3}\) to which the structure vector fields \(\xi, \eta, \zeta\) are always tangent. Suppose that the normal connection of \(M\) in \(S^{4m+3}\) is flat and that the mean curvature vector field is parallel with respect to the normal connection. If \(\dim(T_x M \cap \phi T_x M^\perp \cap \psi T_x M^\perp \cap \theta T_x M^\perp) = 0\) at some point \(x \in M\), then \(M\) is a totally geodesic, invariant submanifold and consequently a great sphere.

**Corollary 4.3.** Let \(M\) be an \((n+3)\)-dimensional complete, minimal submanifold isometrically immersed in a unit \((4m+3)\)-sphere \(S^{4m+3}\) to which the structure vector fields \(\xi, \eta, \zeta\) are always tangent. Suppose that the normal connection of \(M\) in \(S^{4m+3}\) is flat and that the mean curvature vector field is parallel with respect to the normal connection. If \(\dim(T_x M \cap \phi T_x M^\perp \cap \psi T_x M^\perp \cap \theta T_x M^\perp) = 0\) at some point \(x \in M\), then \(M\) is a totally geodesic, invariant submanifold and consequently a great sphere.

On the other side, in order to consider the case where \(0 < 3q < p\), we will prepare the following two Lemmas.

**Lemma 4.4.** For \(\alpha = 1, \ldots, 3q\) and \(\nu = 3q + 1, \ldots, p\), \(s_{\nu\alpha} = 0\).

**Proof.** Since \(U_{\nu} = V_{\nu} = W_{\nu} = 0\) and \(H_{\nu} = 0\), (2.42), (2.44) and (2.46) give
\[
\sum_{B=1}^{p} s_{\nu B} U_{B} = \sum_{B=1}^{p} P^{\phi}_{\nu B} H_{B} X, \quad \sum_{B=1}^{p} s_{\nu B} V_{B} = \sum_{B=1}^{p} P^{\psi}_{\nu B} H_{B} X, \quad \sum_{B=1}^{p} s_{\nu B} W_{B} = \sum_{B=1}^{p} P^{\theta}_{\nu B} H_{B} X,
\]
from which together with \(P^{\phi}_{\alpha \nu} = P^{\psi}_{\alpha \nu} = P^{\theta}_{\alpha \nu} = 0\), it follows that
\[
\sum_{\alpha=1}^{q} s_{\nu\alpha} U_{\alpha} = 0, \quad \sum_{\alpha=q+1}^{2q} s_{\nu\alpha} V_{\alpha} = 0, \quad \sum_{\alpha=2q+1}^{3q} s_{\nu\alpha} W_{\alpha} = 0.
\]
Hence it is clear from (4.3) that \(s_{\nu\alpha} = 0\) for \(\alpha = 1, \ldots, 3q; \nu = 3q + 1, \ldots, p\). □
Lemma 4.5. The first normal space of $M$ in $S^{4m+3}$ is invariant under parallel translation with respect to the normal connection.

Proof. Since $X_i \neq 0 (i = 1, \ldots, q), U_\nu = V_\nu = W_\nu = 0$ and $H_\nu = 0 (\nu = 3q + 1, \ldots, p)$, we can see that (2.49) and (4.3) imply that the first normal space is spanned by $N_\alpha (\alpha = 1, \ldots, 3q)$. For any vector field $X$ tangent to $M$, by means of Lemma 4.4 we have

$$\nabla_X N_\alpha = \sum_{A=1}^{p} s_{\alpha A}(X) N_A = \sum_{\beta=1}^{3q} s_{\alpha \beta}(X) N_\beta,$$

which show that the first normal space is invariant under parallel translation with respect to the normal connection. \qed

Combining Lemma 4.4 with the results due to Allendoerfer [1] and Erbacher [4] yields that there exists a totally geodesic submanifold $M'$ of dimension $(n + 3 + 3q)$ such that $M \subset M'$. By means of (4.2) and (4.3) with $U_\nu = V_\nu = W_\nu = 0 (\nu = 3q + 1, \ldots, p)$, we can easily see that $M'$ is an invariant submanifold of $S^{4m+3}$ and consequently a $(4m' + 3)$-dimensional sphere for an integer $m'$.

Summing up, we may conclude:

Theorem 4.6. Let $M$ be an $(n + 3)$-dimensional submanifold isometrically immersed in a unit $(4m + 3)$-sphere $S^{4m+3}$ to which the structure vector fields $\xi, \eta, \zeta$ are always tangent. Suppose that the normal connection of $M$ in $S^{4m+3}$ is flat and that the mean curvature vector field is parallel with respect to the normal connection. If $\dim(T_x M \cap \phi T_x M^+ \cap \psi T_x M^+ \cap \theta T_x M^+) = q(3q < p)$ at some point $x \in M$, then either $M$ is a totally geodesic, invariant submanifold of $S^{4m+3}$, or there exists a totally geodesic, invariant submanifold $S^{n+3+3q}$ of $S^{4m+3}$ such that $M \subset S^{n+3+3q}$.

5. Submanifolds with $L$-flat normal connection

In this section we try to apply the results which are obtained in the previous sections to submanifolds of a quaternionic projective space.

Let $QP^m$ be a real $4m$-dimensional quaternionic projective space with quaternionic Kählerian structure $\{J, K, L\}$ and let $\tilde{g}$ be the Fubini-Study metric which satisfies the Hermitian conditions

$$\tilde{g}(J\tilde{X}, J\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \quad \tilde{g}(K\tilde{X}, K\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \quad \tilde{g}(L\tilde{X}, L\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}).$$

Then we have

$$J^2 = -I, \quad K^2 = -I, \quad L^2 = -I,$$

$$J = KL = -LK, \quad K = LJ = -JL, \quad L = JK = -KJ$$

(5.2)
and
\begin{align}
\tilde{\nabla}_X J &= r(\tilde{X}) K - q(\tilde{X}) L,
\tilde{\nabla}_X K &= -r(\tilde{X}) J + p(\tilde{X}) L,
\tilde{\nabla}_X L &= q(\tilde{X}) J - p(\tilde{X}) K,
\end{align}
(5.3)
for any vector field \( \tilde{X} \) in \( QP^m \), where \( \tilde{\nabla} \) denotes the Riemannian connection with respect to \( \tilde{g} \), and \( p, q \) and \( r \) are certain local 1-forms (cf. \[5\]). It is well known (cf. \[6, 15\]) that the quaternionic Kählerian structure \( \{J, K, L\} \) is induced from the Sasakian 3-structure \( \{\phi, \psi, \theta\} \) of a unit \((4m+3)\)-sphere \( S^{4m+3} \) by the Hopf fibration \( \tilde{\pi} : S^{4m+3} \to QP^m \). Relations between these structures are given by
\begin{align}
\phi &= J^*, \quad \psi = K^*, \quad \theta = L^*
\end{align}
(5.4)
g(\(X, Y\)) = \tilde{g}^*(\(X, Y\)) + f_\xi(\(X\))f_\xi(\(Y\)) + f_\eta(\(X\))f_\eta(\(Y\)) + f_\zeta(\(X\))f_\zeta(\(Y\)),
where \( \ast \) denotes the horizontal lift of indicated quantities. We notice that the structure vector fields \( \xi, \eta \) and \( \zeta \) are the unit vertical vector fields for the fibration.

Let \( M \) be an \( n \)-dimensional real submanifold of \( QP^m \) and construct a \( S^3 \)-bundle \( \tilde{\pi}^{-1}(M) \) over \( M \) in such a way that the following diagram is commutative :
\begin{align*}
\tilde{\pi}^{-1}(M) &\xrightarrow{\tilde{i}} S^{4m+3} \\
\pi &\downarrow \quad \downarrow \tilde{\pi} \\
M &\xrightarrow{i} QP^m
\end{align*}
where \( \tilde{i} : \tilde{\pi}^{-1}(M) \to S^{4m+3} \) and \( i : M \to QP^m \) are isometric immersions. Then \( \tilde{\pi}^{-1}(M) \) is an \((n+3)\)-dimensional submanifold of \( S^{4m+3} \) to which the structure vector fields \( \xi, \eta \) and \( \zeta \) are tangent. Given an orthonormal basis \( N_1, \ldots, N_p \) in \( TM \), horizontal lifts \( \tilde{u}^A(X)N_A \) are mutually orthonormal normal vector fields to \( \tilde{\pi}^{-1}(M) \) with respect to the Riemannian metric \( g \) of \( \tilde{\pi}^{-1}(M) \) which is induced from that of \( S^{4m+3} \). The transforms for \( X \in TM \) and for \( N_A \) by \( \{J, K, L\} \) are, respectively, written by
\begin{align}
JX &= \hat{F}X + \sum_{A=1}^{p} \hat{u}^A(X)N_A, \quad
KX &= \hat{G}X + \sum_{A=1}^{p} \hat{v}^A(X)N_A,
LX &= \hat{H}X + \sum_{A=1}^{p} \hat{w}^A(X)N_A,
\end{align}
(5.5)
\begin{align}
JN_A &= \hat{U}_A + \sum_{B=1}^{p} P^J_{AB}N_B, \quad
KN_A &= \hat{V}_A + \sum_{B=1}^{p} P^K_{AB}N_B,
LN_A &= \hat{W}_A + \sum_{B=1}^{p} P^L_{AB}N_B,
\end{align}
(5.6)
where \( \{ \hat{F}, \hat{G}, \hat{H} \} \) and \( \{ P^J, P^K, P^L \} \) define endomorphisms of \( TM \) and of \( TM^\perp \), respectively, and \( \{ \hat{U}_A, \hat{V}_A, \hat{W}_A \} \) and \( \{ \hat{u}^A, \hat{v}^A, \hat{w}^A \} \) are local tangent vector fields and local 1-forms on \( M \). Denoting by \( \hat{g} \) the Riemannian metric induced on \( M \) from that of \( Q^m \), we have

\[
\hat{g}(FX, Y) = -\hat{g}(X, FY), \quad \hat{g}(GX, Y) = -\hat{g}(X,GY),
\]

\[
\hat{g}(HX, Y) = -\hat{g}(X, HY),
\]

\[
P^J_{AB} = -P^J_{BA}, \quad P^K_{AB} = -P^K_{BA}, \quad P^L_{AB} = -P^L_{BA},
\]

\[
\hat{u}^A(X) = \hat{g}(U_A, X), \quad \hat{v}^A(X) = \hat{g}(V_A, X), \quad \hat{w}^A(X) = \hat{g}(W_A, X)
\]

for vector fields \( X, Y \) tangent to \( M \). Applying \( J, K \) and \( L \) to (5.5) and making use of (5.2), we can easily obtain the following relations (5.10) and (5.11):

\[
\hat{F}^2X = -X + \sum_{A=1}^p \hat{u}^A(X)\hat{U}_A, \quad \hat{G}^2X = -X + \sum_{A=1}^p \hat{v}^A(X)\hat{V}_A,
\]

\[
\hat{H}^2X = -X + \sum_{A=1}^p \hat{w}^A(X)\hat{W}_A,
\]

\[
\hat{G}\hat{H}X = \hat{F}X + \sum_{A=1}^p \hat{u}^A(X)\hat{V}_A, \quad \hat{H}\hat{G}X = -\hat{F}X + \sum_{A=1}^p \hat{v}^A(X)\hat{W}_A,
\]

\[
\hat{H}\hat{F}X = \hat{G}X + \sum_{A=1}^p \hat{u}^A(X)\hat{W}_A, \quad \hat{F}\hat{H}X = -\hat{G}X + \sum_{A=1}^p \hat{w}^A(X)\hat{U}_A,
\]

\[
\hat{F}\hat{G}X = \hat{H}X + \sum_{A=1}^p \hat{v}^A(X)\hat{U}_A, \quad \hat{G}\hat{F}X = -\hat{H}X + \sum_{A=1}^p \hat{w}^A(X)\hat{V}_A.
\]

Next, applying \( J, K \) and \( L \) to (5.6) and taking account of (5.2), we have the following relations (5.12)-(5.15):

\[
\hat{F}\hat{U}_A = -\sum_{B=1}^p P^J_{AB} \hat{U}_B, \quad \hat{G}\hat{V}_A = -\sum_{B=1}^p P^K_{AB} \hat{V}_B, \quad \hat{H}\hat{W}_A = -\sum_{B=1}^p P^L_{AB} \hat{W}_B,
\]

\[
\hat{G}\hat{U}_A = -\hat{W}_A - \sum_{B=1}^p P^J_{AB} \hat{V}_B, \quad \hat{H}\hat{U}_A = \hat{V}_A - \sum_{B=1}^p P^K_{AB} \hat{W}_B,
\]

\[
\hat{H}\hat{V}_A = -\hat{U}_A - \sum_{B=1}^p P^K_{AB} \hat{W}_B, \quad \hat{F}\hat{V}_A = \hat{W}_A - \sum_{B=1}^p P^L_{AB} \hat{U}_B,
\]

\[
\hat{F}\hat{W}_A = -\hat{V}_A - \sum_{B=1}^p P^L_{AB} \hat{U}_B, \quad \hat{G}\hat{W}_A = \hat{U}_A - \sum_{B=1}^p P^J_{AB} \hat{V}_B,
\]

\[
\hat{g}(\hat{U}_A, \hat{U}_B) = \delta_{AB} + \sum_{C=1}^p P^J_{AC} P^J_{CB}, \quad \hat{g}(\hat{V}_A, \hat{V}_B) = \delta_{AB} + \sum_{C=1}^p P^K_{AC} P^K_{CB},
\]
\[ \hat{g}(\hat{W}_A, \hat{W}_B) = \delta_{AB} + \sum_{C=1}^{p} P_{AC}^L P_{CB}^L, \]

(5.15) \[ \hat{g}(\hat{U}_A, \hat{V}_B) = P_{AB}^K + \sum_{C=1}^{p} P_{AC}^J P_{CB}^K, \]

\[ \hat{g}(\hat{W}_A, \hat{U}_B) = P_{AB}^K + \sum_{C=1}^{p} P_{AC}^L P_{CB}^J. \]

Let \( \hat{\nabla} \) and \( \hat{\nabla}^\perp \) denote the Riemannian connection induced in \( M \) and the normal connection of \( M \) in \( QP^m \), respectively. Denoting by \( \hat{H}_A \) and \( \hat{s}_{AB} \) the Weingarten maps with respect to \( N_A \) and the connection forms of \( \hat{\nabla}^\perp \), respectively, we have Gauss and Weingarten formulas for \( \hat{\nabla}, \hat{\nabla} \) and \( \hat{\nabla}^\perp \) which are similar to (2.7). Differentiating (5.5) covariantly and using (5.3), we can easily obtain

(5.16)

\[ \langle \hat{\nabla}_Y \hat{F} \rangle X = r(Y) \hat{G} X - q(Y) \hat{H} X - \sum_{A=1}^{p} \hat{g}(\hat{H}_A X, Y) \hat{U}_A + \sum_{A=1}^{p} \hat{v}^A(X) \hat{H}_A Y, \]

\[ \langle \hat{\nabla}_Y \hat{G} \rangle X = -r(Y) \hat{F} X + p(Y) \hat{H} X - \sum_{A=1}^{p} \hat{g}(\hat{H}_A X, Y) \hat{V}_A + \sum_{A=1}^{p} \hat{v}^A(X) \hat{H}_A Y, \]

\[ \langle \hat{\nabla}_Y \hat{H} \rangle X = q(Y) \hat{F} X - p(Y) \hat{G} X - \sum_{A=1}^{p} \hat{g}(\hat{H}_A X, Y) \hat{W}_A + \sum_{A=1}^{p} \hat{v}^A(X) \hat{H}_A Y. \]

Differentiating (5.6) covariantly and using (5.3), we have the following relations

(5.17)

\[ \hat{\nabla}_X \hat{U}_A = r(X) \hat{V}_A - q(X) \hat{W}_A + \hat{F} \hat{H}_A X - \sum_{B=1}^{p} P_{AB}^J \hat{H}_B X + \sum_{B=1}^{p} \hat{s}_{AB}(X) \hat{U}_B, \]

\[ \hat{\nabla}_X \hat{V}_A = -r(X) \hat{U}_A + p(X) \hat{W}_A + \hat{G} \hat{H}_A X - \sum_{B=1}^{p} P_{AB}^K \hat{H}_B X + \sum_{B=1}^{p} \hat{s}_{AB}(X) \hat{V}_B, \]

\[ \hat{\nabla}_X \hat{W}_A = q(X) \hat{U}_A - p(X) \hat{V}_A + \hat{H} \hat{H}_A X - \sum_{B=1}^{p} P_{AB}^L \hat{H}_B X + \sum_{B=1}^{p} \hat{s}_{AB}(X) \hat{W}_B, \]

(5.18)

\[ \hat{\nabla}_X P_{AB}^J := \nabla_X P_{AB}^J + \sum_{C=1}^{p} P_{CB}^J s_{CA}(X) + \sum_{C=1}^{p} P_{AC}^J s_{CB}(X) \]

\[ = r(X) P_{AB}^K - q(X) P_{AB}^L + \hat{g}(\hat{U}_A, \hat{H}_B X) - \hat{u}^B(\hat{H}_A X), \]

\[ \hat{\nabla}_X P_{AB}^K := \nabla_X P_{AB}^K + \sum_{C=1}^{p} P_{CB}^K s_{CA}(X) + \sum_{C=1}^{p} P_{AC}^K s_{CB}(X). \]
\[ = -r(X)P^I_{AB} + p(X)P^L_{AB} + g(\hat{V}_A, \hat{H}_B X) - \hat{u}^B(\hat{H}_A X), \]

\[ \hat{\nabla}_X P^L_{AB} := \nabla_X P^L_{AB} + \sum_{C=1}^p P^L_{CB} \delta_C A(X) + \sum_{C=1}^p P^L_{AC} \delta_C B(X) \]

\[ = q(X)P^I_{AB} - p(X)P^K_{AB} + g(\hat{W}_A, \hat{H}_B X) - \hat{u}^B(\hat{H}_A X). \]

On the other hand, \( QF^m \) is of constant \( Q \)-sectional curvature 4 and so the curvature tensor \( \bar{R} \) of \( QF^m \) has the following form (cf. [5]):

\[ \bar{R}(\bar{X}, \bar{Y}) \bar{Z} = g(\bar{Y}, \bar{Z}) \bar{X} - g(\bar{X}, \bar{Z}) \bar{Y} \]

\[ + g(J \bar{Y}, \bar{Z}) J \bar{X} - g(J \bar{X}, \bar{Z}) J \bar{Y} - 2g(J \bar{X}, \bar{Y}) J \bar{Z} \]

\[ + g(K \bar{Y}, \bar{Z}) K \bar{X} - g(K \bar{X}, \bar{Z}) K \bar{Y} - 2g(K \bar{X}, \bar{Y}) K \bar{Z} \]

\[ + g(L \bar{Y}, \bar{Z}) L \bar{X} - g(L \bar{X}, \bar{Z}) L \bar{Y} - 2g(L \bar{X}, \bar{Y}) L \bar{Z}. \]

Thus, using (5.5) and (5.6), we have the following Codazzi and Ricci equations (5.19) and (5.20), respectively:

\[ (\hat{\nabla}_X \hat{H}_A) Y - (\hat{\nabla}_Y \hat{H}_A) X \]

\[ = \sum_{B=1}^p \{ \delta_{AB}(X) \hat{H}_B Y - \delta_{AB}(Y) \hat{H}_B X \} \]

\[ - g(\hat{U}_A, Y) \hat{F} X + g(\hat{U}_A, X) \hat{F} Y - 2g(\hat{F} X, Y) \hat{U}_A \]

\[ - g(\hat{V}_A, Y) \hat{G} X + g(\hat{V}_A, X) \hat{G} Y - 2g(\hat{G} X, Y) \hat{V}_A \]

\[ - g(\hat{W}_A, Y) \hat{H} X + g(\hat{W}_A, X) \hat{H} Y - 2g(\hat{H} X, Y) \hat{W}_A, \]

\[ \bar{R}^L(X, Y) N_A \]

\[ = \sum_{B=1}^p \{ g((\hat{H}_A \hat{H}_B - \hat{H}_B \hat{H}_A) X, Y) \}

\[ + g(\hat{U}_A, Y) g(\hat{U}_B, X) - g(\hat{U}_A, X) g(\hat{U}_B, Y) - 2g(\hat{F} X, Y) P^I_{AB} \]

\[ + g(\hat{V}_A, Y) g(\hat{V}_B, X) - g(\hat{V}_A, X) g(\hat{V}_B, Y) - 2g(\hat{G} X, Y) P^K_{AB} \]

\[ + g(\hat{W}_A, Y) g(\hat{W}_B, X) - g(\hat{W}_A, X) g(\hat{W}_B, Y) - 2g(\hat{H} X, Y) P^L_{AB} \} N_B, \]

where \( \bar{R}^L \) denotes the curvature tensor of the normal connection \( \hat{\nabla}^\perp \). Here we notice that if \( M \) is an invariant submanifold of \( QF^m \), then \( M \) is totally geodesic (cf. [6]) and \( \hat{U}_A = \hat{V}_A = \hat{W}_A = 0 \) \((A = 1, \ldots, p)\). If \( R^L \) satisfies

\[ \hat{R}^L(X, Y) N_A \]

\[ = \sum_{B=1}^p \{ -2g(\hat{F} X, Y) P^I_{AB} - 2g(\hat{G} X, Y) P^K_{AB} - 2g(\hat{H} X, Y) P^L_{AB} \} N_B \]
and
\begin{align}
\hat{\nabla}_X^\perp P_{AB}^I &= r(X)P_{AB}^K - q(X)P_{AB}^L, \\
\hat{\nabla}_X^\perp P_{AB}^K &= -r(X)P_{AB}^I + p(X)P_{AB}^K, \\
\hat{\nabla}_X^\perp P_{AB}^L &= q(X)P_{AB}^I - p(X)P_{AB}^K,
\end{align}
then the normal connection of \( M \) is said to be lift-flat or briefly \( L \)-flat. It is well known ([17, Theorem 3.5, p. 431]) that the normal connection of \( M \) is \( L \)-flat if and only if the normal connection of \( \tilde{\pi}^{-1}(M) \) is flat. In [17], when (5.22) is satisfied, the structure induced in the normal bundle of \( M \) in \( QP^m \) is said to be parallel.

Let \( H_A, \mu \) and \( \bar{\mu} \) be the Weingarten map with respect to \( N_A \), the mean curvature vector field of \( \tilde{\pi}^{-1}(M) \) and of \( M \), respectively. Then the following relations are known (cf. [16]):
\begin{align}
H_A(X) &= (H_A^\perp + \hat{\nabla}_X^\perp)\psi + \hat{\nabla}_X^\perp\eta + \hat{\nabla}_X^\perp\zeta, \\
\text{tr } H_A &= (\text{tr } H_A^\perp, (A = 1, \ldots, p)) \\
\nabla_X^\perp\mu &= \frac{n}{n+3}(\nabla_X^\perp\bar{\mu})^*, \\
P_{AB}^I \ast &= s_{AB}(\xi), \quad P_{AB}^K \ast = s_{AB}(\eta), \quad P_{AB}^L \ast = s_{AB}(\zeta).
\end{align}
It is clear from (5.23) that \( M \) is minimal if and only if \( \tilde{\pi}^{-1}(M) \) is minimal (cf. [16]). Finally we verify

**Theorem 5.1.** Let \( M \) be an \( n \)-dimensional real minimal submanifold of \( QP^m \). If the normal connection of \( M \) in \( QP^m \) is \( L \)-flat and at some point of \( x \in M \),
\[ \dim(T_xM \cap JT_xM^\perp \cap KT_xM^\perp \cap LT_xM^\perp) = q(3q < p := 4m - n), \]
then either \( M \) is a totally geodesic, invariant submanifold of \( QP^m \) or there exist a real \((n+3q)-\text{dimensional totally geodesic, invariant submanifold } QP^{(n+3q)/4} \) of \( QP^m \) such that \( M \subset QP^{(n+3q)/4} \).

**Proof.** Since \( \dim(T_xM \cap JT_xM^\perp \cap KT_xM^\perp \cap LT_xM^\perp) = q \) and the Riemannian metric \( \hat{g} \) satisfies the Hermitian conditions, there exist mutually orthonormal normal vectors \( n_1, \ldots, n_{3q} \) such that

\[ \begin{align*}
J_xn_1 &= K_xn_{q+1} = L_xn_{2q+1}, \ldots, J_xn_q &= K_xn_{2q} = L_xn_{3q},
\end{align*} \]
constitute an orthonormal basis for \( T_xM \cap JT_xM^\perp \cap KT_xM^\perp \cap LT_xM^\perp \). We extend \( n_1, \ldots, n_{3q} \) to local fields \( N_1, \ldots, N_{3q} \) in \( TM^\perp \) and choose \( N_{3q+1}, \ldots, N_p \) in \( TM^\perp \) so that \( N_1, \ldots, N_{3q}, N_{3q+1}, \ldots, N_p \) are mutually orthonormal. Then \( N_1^\ast, \ldots, N_q^\ast, N_{3q+1}^\ast, \ldots, N_p^\ast \) are orthonormal vector fields in \( T\tilde{\pi}^{-1}(M)^\perp \). Let \( y \in \tilde{\pi}^{-1}(x) \), then
\[ \dim(T_y\tilde{\pi}^{-1}(M) \cap \phi_yT_y\tilde{\pi}^{-1}(M)^\perp \cap \psi_yT_y\tilde{\pi}^{-1}(M)^\perp \cap \theta_yT_y\tilde{\pi}^{-1}(M)^\perp) = q \]
because of (5.4). Furthermore, \( \tilde{\pi}^{-1}(M) \) is minimal in \( S^{4m+3} \) because of (5.24) and the normal connection of \( \tilde{\pi}^{-1}(M) \) is flat. Thus, by means of Theorem 4.6, \( \tilde{\pi}^{-1}(M) \) is a totally geodesic invariant submanifold \( S^{n+3} \) of \( S^{4m+3} \).
or there exists a totally geodesic invariant submanifold $S^{n+3+3q}$ such that $\tilde{\pi}^{-1}(M) \subseteq S^{n+3+3q}$. $S^{n+3+3q}$ is a $S^3$-bundle over a quaternionic projective space $QP^{(n+3q)/4}$ of a real $(n + 3q)$-dimension and $\{\xi, \eta, \zeta\}$ are the unit vertical vector fields of the $S^3$-bundle. Thus the immersion $: QP^{(n+3q)/4} \rightarrow QP^m$ is compatible with the Hopf fibration $\tilde{\pi} : S^{4m+3} \rightarrow QP^m$. Since $S^{n+3+3q}$ is a totally geodesic submanifold in $S^{4m+3}$, (5.23) implies that $QP^{(n+3q)/4}$ is a totally geodesic, invariant submanifold of $QP^m$. This completes the proof.

□

References


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