EXISTENCE AND REGULARITY FOR SEMILINEAR NEUTRAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

Jin-Mun Jeong

Abstract. In this paper, we construct some results on the existence and regularity for solutions of neutral functional differential equations with unbounded principal operators in Hilbert spaces. In order to establish the existence and regularity for solutions of the neutral system by using fractional power of operators and the local Lipschitz continuity of nonlinear term without using many of the strong restrictions considering in the previous literature.

1. Introduction

Let $H$ and $V$ be real Hilbert spaces such that $V$ is a dense subspace in $H$. In this paper, we are concerned with the global existence of solution and the approximate controllability for the following abstract neutral functional differential system in a Hilbert space $H$:

$$
\begin{cases}
\frac{d}{dt}[x(t) + (Bx)(t)] = Ax(t) + f(t, x(t)) + k(t), & t \in (0, T], \\
x(0) = x_0, & (Bx)(0) = y_0,
\end{cases}
$$

(1.1)

where $A$ is an operator associated with a sesquilinear form on $V \times V$ satisfying Gårding’s inequality, $f$ is a nonlinear mapping of $[0, T] \times V$ into $H$ satisfying the local Lipschitz continuity, $B : L^2(0, T; V) \to L^2(0, T; H)$ is a bounded linear mapping.

Recently, the existence of solutions for mild solutions for neutral differential equations with state-dependence delay has been studied in the literature in [1] and references therein. As for partial neutral integro-differential equations, we refer to [2]. However there are few papers treating the regularity for neutral systems with local Lipschitz continuity, we can just find a recent article Wang [3] in case semilinear systems.

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In this paper, we construct some results on the regularity of solutions for neutral functional differential equations with unbounded principal operators in Hilbert spaces. In order to establish the existence and regularity of solutions of the neutral system by using fractional power of operators and the local Lipschitz continuity of nonlinear term without using many of the strong restrictions considering in the previous literature.

2. preliminaries

If $H$ is identified with its dual space we may write $V \subset H \subset V^* \subset V^*$ densely and the corresponding injections are continuous. The norm on $V$, $H$ and $V^*$ will be denoted by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$, respectively. For brevity, we may regard that $|u| \leq \|u\|$, $\forall u \in V$.

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding’s inequality

$$\text{Re } a(u, u) \geq \delta \|u\|^2, \quad \delta > 0.$$

From (2.2) we may think that there exists a constant $C_0 > 0$ such that

$$\|u\| \leq C_0 \|u\|_{D(A)}^{1/2} |u|^{1/2}. \quad \text{(2.3)}$$

Thus we have the following sequence:

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \quad \text{(2.4)}$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. With the notations (2.3), (2.4), we have

$$(V, V^*)_{1/2, 2} = H, \quad (D(A), H)_{1/2, 2} = V,$$

where $(V, V^*)_{1/2, 2}$ denotes the real interpolation space between $V$ and $V^*$ (Sec. 1.3 of [4]).

It is also well known that $A$ generates an analytic semigroup $S(t)$ in both $H$ and $V^*$. By virtue of (2.2), we have that $0 \in \rho(A)$ the closed half plane $\{\lambda : \text{Re } \lambda \geq 0\}$ is contained in the resolvent set of $A$. In this case, $A^{-\alpha}$ is a bounded operator. So we can assume that there is a constant $M_0 > 0$ such that

$$\|A^{-\alpha}\|_{L(H)} \leq M_0, \quad \|A^{-\alpha}\|_{L(V^*, V)} \leq M_0. \quad \text{(2.5)}$$

For each $\alpha \geq 0$ we can define the fractional power $A^\alpha (\alpha > 0)$ of $A$ and collect some simple properties of the fractional power of $A$. 

Lemma 2.2. (a) $A^\alpha$ is a closed operator with its domain dense.
(b) If $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$.
(c) For any $T > 0$, there exists a positive constant $C_\alpha$ such that the following inequalities hold for all $t > 0$ ([5, Lemma 3.6.2]):
\[
||A^\alpha S(t)||_{L^2} \leq \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{L^2(V,H)} \leq \frac{C_\alpha}{t^{3\alpha/2}}. \tag{2.6}
\]
By a simple calculation, we obtain the following.

Lemma 2.3. For every $k \in L^2(0,T;H)$, let $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \leq t \leq T$. Then there exists a constant $C_2$ such that
\[
||x||_{L^2(0,T;V)} \leq C_2 \sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.7}
\]

3. Neutral differential equations

In this section, we will show that the initial value problem (1.1) has a solution by solving the integral equation:
\[
x(t) = S(t)[x_0 + y_0] - (Bx)(t) + \int_0^t AS(t-s)Bx(s)ds
+ \int_0^t S(t-s)[f(s,x(s)) + k(s)]ds.
\]

Now we give the basic assumptions on the system (1.1)

**Assumption (B).** Let $B : L^2(0,T;V) \rightarrow L^2(0,T;H)$ be a bounded linear mapping such that there exists constants $\beta > 2/3$ and $L > 0$ such that
\[
||A^\beta Bx||_{L^2(0,T;H)} \leq L||x||_{L^2(0,T;V)}, \quad \forall x \in L^2(0,T;V).
\]

**Assumption (F).** $f$ is a nonlinear mapping of $[0,T] \times V$ into $H$ satisfying following:

(i) There exists a function $L_1 : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that for $||x|| \leq r$ and $||y|| \leq r$,
\[
|f(t,x) - f(t,y)| \leq L_1(r)||x - y||, \quad t \in [0,T].
\]

(ii) The inequality
\[
|f(t,x)| \leq L_1(r)(||x|| + 1)
\]
holds for every $t \in [0,T]$ and $x \in V$.

From now on, we establish the following results on the solvability of the equation (1.1).

**Theorem 3.1.** Let Assumptions (B) and (F) be satisfied. Assume that $x_0 \in H$, $k \in L^2(0,T;V^*)$ for $T > 0$. Then, there exists a solution $x$ of the equation (1.1) such that
\[
x \in W_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \subset C([0,T];H).
\]
Moreover, there is a constant $C_2$ independent of $x_0$ and the forcing term $k$ such that

$$||x||_{W_1(T)} \leq C_2(1 + |x_0| + ||k||_{L^2(0,T;V')}}. \quad (3.1)$$

One of the main useful tools is the following Sadovskii’s fixed point theorem.

**Lemma 3.2.** Suppose that $\Sigma$ is a closed convex subset of a Banach space $X$. Assume that $K_1$ and $K_2$ are mappings from $\Sigma$ into $X$ such that the following conditions are satisfied:

(i) $(K_1 + K_2)(\Sigma) \subset \Sigma$,

(ii) $K_1$ is a completely continuous mapping,

(iii) $K_2$ is a contraction mapping.

Then the operator $K_1 + K_2$ has a fixed point in $\Sigma$.

**Proof of Theorem.**

Let $r_0 = 2(C_1|x_0 + y_0| + r_0 M_0 L)$, where $C_1$ is constant satisfying

$$||x||_{W_1(T)} \leq C_1(||x_0|| + ||k||_{L^2(0,T;H)}). \quad (3.2)$$

Let $\gamma = \max\{1/2, (3\beta - 2)^{1/2}\}$, choose $0 < T_1 < T$ such that

$$T_1^\gamma \{C_2 L_1(r_0)(r_0 + 1) + C_2 ||k||_{L^2(0,T_1,V')}\} + (3\beta - 2)^{-1/2}r_0 LC_{1-\beta} \leq C_1|0 + y_0| + r_0 M_0 L,$n

where $C_2$ is constant in (2.7) and

$$\tilde{M} \equiv T_1^\gamma \{C_2 L_1(r_0) + (3\beta - 2)^{-1/2}C_{1-\beta}L\} < 1. \quad (3.4)$$

Define a mapping $J : L^2(0,T_1;V) \to L^2(0,T_1;V)$ as

$$(Jx)(t) = S(t)(x_0 + y_0) - (Bx)(t)$$

$$+ \int_0^t A S(t-s)(Bx)(s)ds + \int_0^t S(t-s)\{f(s,x(s)) + k(s)\}ds.$n

It will be shown that the operator $J$ has a fixed point in the space $L^2(0,T_1;V)$. By assumptions (B) and (F), we know that $J$ is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma = \{x \in L^2(0,T_1;V) : ||x||_{L^2(0,T_1,V)} \leq r_0, x(0) = x_0\},$$

which is a bounded closed subset of $L^2(0,T_1;V)$. By (2.5) and Assumption (B) we have

$$||Bx||_{L^2(0,T_1,V)} \leq ||A^{-\beta}||_{L(H,V')} ||A^\beta Bx||_{L^2(0,T_1,H)} \leq r_0 M_0 L. \quad (3.5)$$

By virtue of (2.7), for $0 < t < T_1$, it holds

$$||\int_0^t S(t-s)\{f(s,x(s)) + k(s)\}ds||_{L^2(0,T_1,V)} \leq C_2 \sqrt{T_1} ||f(\cdot,x) + k||_{L^2(0,T_1,H)} \quad (3.6)$$

$$\leq C_2 \sqrt{T_1} \{L_1(r_0)(r_0 + 1) + ||k||_{L^2(0,T_1,V')}\}.$$
Since (2.6) and Assumption (F) the following inequality holds:
\[
\|AS(t-s)Bx(s)\| = \|A^{1-\beta}S(t-s)A^{\beta}Bx(s)\| \leq \frac{C_{1-\beta}}{(t-s)^{\beta(1-\beta)/2}}r_0L,
\]
there holds
\[
\|\int_0^t AS(t-s)Bx(s)ds\|_{L^2(0,T;V)} \leq (3\beta - 2)^{-1/2}r_0LC_{1-\beta}T_1^{\sqrt{3\beta-2}}. \tag{3.7}
\]
Therefore, from (3.2), (3.4)-(3.7) it follows that
\[
\|Jx\|_{L^2(0,T;V)} \leq C_1|x_0 + y_0| + r_0M_0L
+ T_1^2 \left( \|C_2L_1(r_0)(r_0 + 1) + C_2\|_{L^2(0,T;V)} \right)
+ (3\beta - 2)^{-1/2}r_0LC_{1-\beta} \leq r_0,
\]
and hence \( J \) maps \( \Sigma \) into \( \Sigma \). Define mapping \( J = K_1 + K_2 \) on \( L^2(0,T;V) \) by the formula
\[
(K_1x)(t) = - (Bx)(t)
\]
\[
(K_2x)(t) = S(t)(x_0 + y_0) + \int_0^t AS(t-s)(Bx)(s)ds
\]
\[
+ \int_0^t S(t-s)[f(s,x(s)) + k(s)]ds.
\]
We can now employ Lemma 3.1 with \( \Sigma \). Assume that a sequence \( \{x_n\} \) of \( L^2(0,T;V) \) converges weakly to an element \( x_\infty \in L^2(0,T;V) \), i.e., \( w - \lim_{n \to \infty} x_n = x_\infty \). Then we will show that
\[
\lim_{n \to \infty} \|K_1x_n - K_1x_\infty\| = 0, \tag{3.8}
\]
which is equivalent to the completely continuity of \( K_1 \) since \( L^2(0,T;V) \) is reflexive. For a fixed \( t \in [0,T] \), let \( x^*_n(t) = (K_1x)(t) \) for every \( x \in L^2(0,T;V) \). Then \( x^*_n \in L^2(0,T;V^*) \) and we have \( \lim_{n \to \infty} x^*_n(x_n) = x^*_n(x_\infty) \) since \( w - \lim_{n \to \infty} x_n = x_\infty \). Hence,
\[
\lim_{n \to \infty} (K_1x_n)(t) = (K_1x_\infty)(t), \quad t \in [0,T].
\]
By (2.5) and Assumption (B) we have
\[
\|(K_1x)(t)\| \leq \|A^{-\beta}\|_{L(H,V)} \|A^{\beta}Bx\|_{L^2(0,T;H)} \leq \infty.
\]
Therefore, by Lebesgue’s dominated convergence theorem it holds
\[
\lim_{n \to \infty} \|K_1x_n\|_{L^2(0,T;V)} = \|K_1x_\infty\|_{L^2(0,T;V)}.
\]
Since \( L^2(0,T;V) \) is a Hilbert space, it holds (3.8).

Next, we prove that \( K_2 \) is a contraction mapping on \( \Sigma \). Indeed, for every \( x_1 \) and \( x_2 \) in \( \Sigma \), by similar to (3.7) and (3.8), we have
\[
\|K_2x_1 - K_2x_2\|_{L^2(0,T;V)} \leq T_1^2 \left( C_2L_1(r_0) + (3\beta - 2)^{-1/2}C_{1-\beta}L \right) \|x_1 - x_2\|_{L^2(0,T;V)}.
\]
So by virtue of the condition (3.4) the contraction mapping principle gives that the solution of (1.1) exists uniquely in \( [0,T] \). So by virtue of the condition
Taking into account (3.4), there exists a constant $K$ of (3.4), and it follows that

$$K^2$$

\[ (3.4), \quad (3.5)-(3.8) \]

Thus, Lemma 3.1 gives that the equation of (1.1) has a solution in $W_1(T)$. Let $W$ be a solution in $F$ the operator $L$ is continuous. Here, we used the relation $k \in Corollary 3.4$. Let us assume that the embedding $[6]$. So we can prove the following result from Theorem 3.1.

$$Bu$$

which obtain the inequality (3.1). Since the conditions (3.3) and (3.4) are independent of initial value, we know

$$||x||_{L^2(0,T;V)} \leq (1 - \hat{M})^{-1}[C_1|x_0 + y_0| + r_0M_0L$$

which is useful for physical applications of the given equation. The proof is immediately obtained from Theorem 3.1.

\[ + T_1^2\{C_2L_1(r_0) + C_2||L_2(0,T;V^*)||\}] \leq C_3(1 + ||x_0|| + ||k||_{L^2(0,T;V^*)}) \]

Here, we used the relation $W_1(T) \leftrightarrow C([0,T;H])$, which is an easy consequence of the definition of real interpolation spaces by the trace method. So, by repeating the above process, the solution can be extended to the interval $[0,T]$.

From the following result, we obtain that the solution mapping is continuous, which is useful for physical applications of the given equation. The proof is immediately obtained from Theorem 3.1.

**Theorem 3.3.** Let Assumptions (B) and (F) be satisfied and $(x_0,y_0,k) \in H \times H \times L^2(0,T;V^*)$. Then the solution $x$ of the equation (1.1) belongs to $x \in W_1(T) \equiv L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$ and the mapping

$$H \times H \times L^2(0,T;V^*) \ni (x_0,y_0,k) \mapsto x \in W_1(T)$$

is continuous.

For $k \in L^2(0,T;V^*)$ let $x_k$ be the solution of equation (1.1) with $k$ instead of $Bu$. Here, we remark that if $V$ is compactly embedded in $H$ by assumption, the embedding $W_1(T) \subset L^2(0,T;H)$ is compact in view of Theorem 2 of Aubin [6]. So we can prove the following result from Theorem 3.1.

**Corollary 3.4.** Let us assume that the embedding $V \subset H$ is compact. For $k \in L^2(0,T;V^*)$ let $x_k$ be the solution of equation (1.1). Then the mapping $k \mapsto x_k$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$. Moreover, if we define the operator $\mathcal{F}$ by $\mathcal{F}(k) = f(\cdot,x_k)$, then $\mathcal{F}$ is also a compact mapping from $L^2(0,T;V^*)$ to $L^2(0,T;H)$. 

References


Department of Applied Mathematics, Pukyong National University, Busan 608-737, Korea
E-mail address: jmjeong@pknu.ac.kr