THE LEFSCHETZ CONDITION ON PROJECTIVIZATIONS
OF COMPLEX VECTOR BUNDLES

HIROKAZU NISHINOBU AND TOSHIHIRO YAMAGUCHI

Abstract. We consider a condition under which the projectivization $P(E^k)$ of a complex $k$-bundle $E^k \to M$ over an even-dimensional manifold $M$ can have the hard Lefschetz property, affected by [10]. It depends strongly on the rank $k$ of the bundle $E^k$. Our approach is purely algebraic by using rational Sullivan minimal models [5]. We will give some examples.

1. Introduction

A Poincaré duality space $Y$ of the formal dimension $fd(Y) = \max \{i; H^i(Y; \mathbb{Q}) \neq 0\} = 2m$ is said to be cohomologically symplectic (c-symplectic) if $u^m \neq 0$ for some $u \in H^2(Y; \mathbb{Q})$ and, furthermore, is said to have the hard Lefschetz property (or simply the Lefschetz property) with respect to the c-symplectic class $u$, if the maps $\cup u^j : H^{m-j}(Y; \mathbb{Q}) \to H^{m+j}(Y; \mathbb{Q}), \quad 0 \leq j \leq m$
are monomorphisms (then called the Lefschetz maps) [17]. For example, a compact Kähler manifold has the hard Lefschetz property [17], [6, Theorem 4.35]. Recall the Thurston-Weinstein problem [17, p. 198]: “Describe symplectic compact manifolds with no Kähler structure”. Conversely, what conditions on a symplectic manifold imply the existence of a Kähler structure or, more generally, that the manifold satisfies the hard Lefschetz property?

Let $M$ be an even-dimensional manifold and $\xi : E^k \to M$ be a complex $k$-bundle over $M$. The projectivization of the bundle $\xi$

$P(\xi) : \mathbb{C}P^{k-1} \xrightarrow{j} P(E^k) \to M$
satisfies the rational cohomology algebra condition (*):

$H^*(P(E^k); \mathbb{Q}) = H^*(M; \mathbb{Q})[x]/(x^k + c_1x^{k-1} + \cdots + c_{k-j}x^j + \cdots + c_{k-1}x + c_k)$

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where \( c_i \) are the \( i \)-th Chern classes of \( \xi \) and \( x \) is a degree 2 class generating the cohomology of the complex projective space fiber (Leray-Hirsch theorem) \([3, 10], [17, \text{p. 122}].\) The manifold \( P(E^k) \) appears as the exceptional divisor in the blow-up construction for a certain embedding of \( M \) \([11], [17, \text{Chap. 4}].\) When \( M \) is a non-toral symplectic nilmanifold of dimension \( 2n \), there is a bundle \( E^n \) such that \( P(E^n) \) is not Lefschetz \([18], [10, \text{Example 4.4}].\) In general, for a \( 2k \)-dimensional manifold \( M \) and a fibration \( CP^{k-1} \to E \to M \), the total space \( E \) is Lefschetz if and only if \( M \) is Lefschetz \([10, \text{Remark 4.2}].\) We consider the following:

**Problem 1.1.** Suppose that the projectivization \( P(E^k) \) of a \( k \)-dimensional vector bundle \( E^k \to M \) is c-symplectic with respect to \( \tilde{x} \) where \( j^*(\tilde{x}) = x \); i.e., \( \tilde{x}^m \neq 0 \) when \( \dim P(E^k) = 2m \). What rational homotopical conditions on \( M \) are necessary for \( P(E^k) \) to have the Lefschetz property with respect to \( \tilde{x} \)?

**Proposition 1.2.** Let \( M \) be an even dimensional manifold.

1. For a sufficiently large \( k \), there is a \( k \)-dimensional vector bundle \( E^k \to M \) such that \( P(E^k) \) is c-symplectic with respect to \( x \).
2. If \( P(E^k) \) is c-symplectic with respect to \( x \), then there is a vector bundle \( E^m \to M \) such that \( P(E^m) \) is c-symplectic with respect to \( x \) for any \( m > k \).

**Definition 1.3.** An even-dimensional manifold (or more general Poincaré duality space) \( M \) is said to be projective (\( k \))-Lefschetz if there exists a complex \( k \)-bundle \( E^k \) such that the projectivization \( P(E^k) \) is c-symplectic with respect to \( \tilde{x} \) and has the Lefschetz property with respect to \( \tilde{x} \). Then we often say simply that \( M \) is projective Lefschetz. In particular, we say that \( M \) is projective non-Lefschetz if \( P(E^k) \) cannot have the Lefschetz property for any \( k \) and \( E^k \).

In this paper, we recall D. Sullivan’s rational model in §2 and we give some examples that indicate how the rational cohomology algebra of \( M \) determines the projective (\( n \))-Lefschetzness of \( M \) when \( M \) is the product of at most four spheres in §3.

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### 2. Sullivan model

Let \( \mathcal{M}(Y) = (\Lambda Y, d) \) be the Sullivan minimal model of a nilpotent space \( Y \). It is a freely generated \( \mathbb{Q} \)-commutative differential graded algebra (abbr. DGA) with a \( \mathbb{Q} \)-graded vector space \( V = \bigoplus_{i \geq 1} V^i \) where \( \dim V^i < \infty \), \( V \) admits a basis \( \{v_\alpha \} \) indexed by a well-ordered set \( \{\alpha\} \) such that \( \deg(v_\alpha) \leq \deg(v_\beta) \) if \( \alpha < \beta \) and \( d(v_\alpha) \in \Lambda(v_\beta)_{\beta \leq \alpha} \). The differential \( d \) is a decomposable; i.e., \( d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^i+1 \). Here \( \Lambda^+ V \) is the ideal of \( \Lambda V \) generated by elements of positive degree. Denote the degree of a homogeneous element \( f \) of a graded algebra as \( |f| \). Then \( xy = (-1)^{|x||y|}yx \) and \( d(xy) = d(x)y + (-1)^{|x|}xd(y) \). Note
that $\mathcal{M}(Y)$ determines the rational homotopy type of $Y$. In particular, it is known that

$$H^*(\Lambda Y, d) \cong H^*(Y; \mathbb{Q})$$

and $V_i \cong \text{Hom}(\pi_i(Y), \mathbb{Q})$.

See [5, §12–§15] for details. When $\pi_*(Y) \otimes \mathbb{Q} < \infty$ and $\dim H^*(Y; \mathbb{Q}) < \infty$, $Y$ is said to be \textit{elliptic}. It is known that

$$fd(Y) = fd(\Lambda Y, d) = \sum_i |y_i| - \sum_i (|x_i| - 1)$$

for $V^{\text{odd}} = \mathbb{Q}(y_i)$ and $V^{\text{even}} = \mathbb{Q}(x_i)$, when $Y$ is elliptic [5, §32].

**Proposition 2.1.** Let $M$ be an even dimensional manifold. Then there is a graded algebra $A_0 = H^*(M; \mathbb{Q})[x]/(x^k + c_1x^{k-1} + \cdots + c_{k-1}x + c_k)$ with $|x| = 2$ and $c_i \in H^{2i}(M; \mathbb{Q})$ if and only if there is a complex $k$-bundle $\xi : E^k \to M$ such that $c_i$ are the Chern classes of $\xi$ by suitable scalar multiplying and $A$ is the rational cohomology of $P(E^k)$.

**Proof.** The set of equivalence classes of complex $k$-vector bundles over $M$ is identified as the homotopy set from $M$ to the complex Grassmanian $G(k, N)$ of $k$-planes in $\mathbb{C}^N$ for a sufficiently large $N$ [2, IV]. Then the Chern classes of a $k$-bundle are given as $f^*(c_1(\gamma)), \ldots, f^*(c_k(\gamma))$ for the classifying map $f$ and the universal bundle $\gamma$ over $G(k, N)$. Conversely, for given elements $c_1, \ldots, c_k$, a rational map $M \to M(0) \to G(k, N)(0)$ induced by $\Pi_i c_i : M \to \Pi_i K(Q, 2i) \simeq BU(k)_0$ is factored through a map $f : M \to G(k, N)$ [12, Theorem 5.3] because $G(k, N) = U(N)/U(k) \times U(N - k)$ is 0-universal [1, Proposition 3.7]. Here $BU(k)$ is the classifying space of the unitary group $U(k)$ and $Y(0)$ is the rationalization of a space $Y$ [8]. Thus we obtain the appropriate $k$-bundle as the pullback of $\gamma$ by $f$.

**Corollary 2.2.** The projective Lefschetzness of an even-dimensional manifold $M$ depends only on the graded algebra $H^*(M; \mathbb{Q})$.

Let $\mathcal{M}(\mathbb{C}P^{k-1}) = (\mathbb{Q}[x] \otimes \Lambda(y), d)$ with $d(y) = x^k$ and $d(x) = 0$. From Corollary 2.2, the information of $P(E^k)$ that we need in this note is given as the relative Sullivan model [5, §14] :

$$(H^*(M; \mathbb{Q}), 0) \to (H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D) \to (\mathbb{Q}[x] \otimes \Lambda(y), d)$$

with $D(f) = 0$ for $f \in H^*(M; \mathbb{Q})$, $D(x) = 0$ and

$$(**) \quad D(y) = x^k + c_1x^{k-1} + \cdots + c_{k-1}x + c_k,$$

where $c_i \in H^{2i}(M; \mathbb{Q})$ are the Chern classes of $\xi$. Especially, we don’t need the assumption that $M$ is nilpotent. Remark that $H^*(P(E^k); \mathbb{Q}) \cong H^*(H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ as a $\mathbb{Q}$-graded algebra and then

$$H^1(P(E^k); \mathbb{Q}) = H^1(M; \mathbb{Q}) \oplus H^{j-2}(M; \mathbb{Q})x \oplus \cdots \oplus H^{j-2k}(M; \mathbb{Q})x^{k-1}.$$

Notice that $(**)$ is equivalent to $(*)$ of §1 and also equivalent to

$$[x^k] = -[c_1x^{k-1} + \cdots + c_{k-1}x + c_k].$$
in $H^*(P(E^k); \mathbb{Q})$, which is the only relation between the elements of $H^*(M; \mathbb{Q})$ and $x$. Then, for example, $[x^{k+1}] = -[c_1 x^k + \cdots + c_{k-1} x^{j+1} + \cdots + c_k x] = [c_1 x^{k-1} + \cdots + (c_1 c_k - c_{k-j+1}) x^j + \cdots + (c_1 c_{k-1} - c_k) x + c_1 c_k]$. In particular,

\[ (**) \quad [a] \neq 0 \text{ in } H^*(M; \mathbb{Q}) \text{ if and only if } [ax^j] \neq 0 \text{ in } H^*(P(E^k); \mathbb{Q}) \text{ for any } 0 \leq j < k. \]

**Lemma 2.3.** Let $A = (H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ with $D(y) = x^k + c_1 x^{k-1} + \cdots + c_{k-1} x + c_k$ and let $B = (H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y'), D')$ with $D'(y') = x^{m-k} D(y) = x^m + c_1 x^{m-1} + \cdots + c_{k-1} x^{m-k+1} + c_k x^{m-k}$ for $k < m$. If $[f] \neq 0$ in $H^*(A)$, then $[fx^{m-k}] \neq 0$ in $H^*(B)$.

**Proof.** Notice that an element of $H^*(A)$ is identified as one of $H^*(B)$ since $H^*(A)$ is a submodule of $H^*(B)$ over $H^*(M; \mathbb{Q})$. Suppose that $[f] = [a_1 x^{k-1} + \cdots + a_{k-1} x + a_k] \neq 0$ in $H^*(A)$ for $[a_i] \in H^*(M; \mathbb{Q})$. Then there is an index $i$ with $[a_i] \neq 0$ in $H^*(M; \mathbb{Q})$. Thus, in $H^*(B)$, $[fx^{m-k}] = [a_1 x^{m-1} + \cdots + a_{k-1} x^{m-k+1} + a_k x^{m-k}] = [a_1 x^{m-1} + \cdots + [a_{k-1}] x^{m-k+1} + [a_k] x^{m-k}] \neq 0$ from $H^{**}$. \( \square \)

**Proof of Proposition 1.2.** From Proposition 2.1, it is sufficient to construct a certain DGA $(H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$. Let $\dim M = 2n$.

1. Let $\Omega$ be the fundamental class of $M$. Then we can define $(H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ by $D(y) = \Omega x^{k-n} + x^k$ for $k \geq n$. Notice $\dim P(E^k) = \dim M + \dim \mathbb{Q}^k - 2 = 2n + 2k - 2$. Then we have $[x^{n+k-1}] = -[(\Omega x^{k-n}) x^{n-1}] = -[\Omega x^{k-n}] \neq 0$ from $H^{**}$.

2. Suppose that the DGA $(H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y), D)$ makes $P(E^k)$ $c$-symplectic; i.e., $[x^{n+k-1}] \neq 0$. Then, for $m > k$, the DGA $(H^*(M; \mathbb{Q}) \otimes \mathbb{Q}[x] \otimes \Lambda(y'), D')$ with $|g'| = 2m - 1$ and $D'(y') := x^{m-k} D(y)$ makes a $2n + (2m - 2)$-dimensional manifold $P(E^m)$ $c$-symplectic. Indeed, $[x^{n+m-1}] = [x^{n+k-1}]$, $x^{m-k}] \neq 0$ in cohomology from Lemma 2.3. \( \square \)

In (2) in Proposition 1.2, the bundle $E^m$ is geometrically realized as the Whitney sum $E^k \oplus \theta^{m-k}$ where $\theta^{m-k}$ is the trivial $m - k$-bundle over $M$, in the manner of Proposition 2.1. Thus, if $P(E^k)$ is $c$-symplectic with respect to $x$, then $P(E^k \oplus \theta^m)$ is $c$-symplectic with respect to $x$ for any $m > 0$.

3. Examples

In this section, let $M$ be a 2-connected even-dimensional manifold and $\dim P(E^k) = 2m$.

**Theorem 3.1.** The $2n$-dimensional sphere $S^{2n}$ is projective $(k)$-Lefschetz for any $k \geq n$.

**Proof.** Let $H^*(S^{2n}; \mathbb{Q}) = \mathbb{Q}[v]/(v^2)$ with $|v| = 2n$. Consider $P(E^k)$ such that $\dim P(E^k) = 2m$ and $D(y) = vx^{k-n} + x^k$ for $k \geq n$. Then $m = n + k - 1$ from $2m = \dim \mathbb{C} P^{k-1} + \dim S^{2n} = 2n + 2k - 2$. Since $[x^m] = -[vx^{k-n}]$. 

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\[ x^{m-k} = -[ex^{k-1}] \neq 0 \text{ from (***), } P(E^k) \text{ is c-symplectic with respect to } x. \]

Furthermore, \[ \cup x^{k-n-1-2i}(v^x) = ex^{k-n-1-i} \neq 0 \text{ in } \]

\[ \cup x^{k-n-1-2i} : H^{m-(k-n-1-2i)}(P(E^k); \mathbb{Q}) \to H^{m+(k-n-1-2i)}(P(E^k); \mathbb{Q}) \]

for \( i \geq 0 \) Thus \( S^{2n} \) is projective \((k)\)-Lefschetz. \( \square \)

**Proposition 3.2.** When \( M \) has the rational homotopy type of the product of odd spheres such that \( H^*(M; \mathbb{Q}) \cong \Lambda(v_1, v_2, \ldots, v_n) \) with all \( |v_i| \) odd and \( 1 < |v_1| \leq |v_2| \leq \cdots \leq |v_n| \) \((n \text{ even})\), then there exists a bundle \( E^k \) such that \( P(E^k) \) is c-symplectic if and only if \( |v_1| + |v_n| \leq 2k, |v_2| + |v_{n-1}| \leq 2k, \ldots, |v_{n/2}| + |v_{n/2+1}| \leq 2k \).

**Proof.** (sketch) The minimal DGA \((\mathbb{Q}[x] \otimes \Lambda(v_1, v_2, \ldots, v_n, y), D)\) with \( |y| = 2k - 1 \) is c-symplectic if \( D(v_1) = \cdots = D(v_{n/2}) = 0 \) and

\[ D(y) = v_1v_nx^{a_1} + v_2v_{n-1}x^{a_2} + \cdots + v_{n/2}x^{a_{n/2}} + x^k \]

for \( a_i = (2k - |v_i| - |v_{n-i+1}|)/2 \geq 0 \). Then we have the “if” part from Proposition 2.1 and [14, Theorem 1.2]. The “only if” part is obvious from [14, Theorem 1.2]. \( \square \)

**Theorem 3.3.** Let \( M = S^a \times S^b \) with \( a \leq b \).

(i) When \( a = b \), it is projective \((k)\)-Lefschetz for \( k \geq b \).

(ii) When \( a \) and \( b \) are even, it is projective \((\frac{k}{2})\)-Lefschetz.

(iii) When \( a \) and \( b \) are odd with \( a < b \), it is projective non-Lefschetz.

**Proof.** Note that \( H^*(M; \mathbb{Q}) = \mathbb{Q}[v_1, v_2]/(v_1^2, v_2^2) = \mathbb{Q}(1, v_2, v_1v_2) \) as a \( \mathbb{Q} \)-graded vector space with \( |1| = 0, |v_1| = a, |v_2| = b \) and \( |v_1v_2| = a + b \). Consider \( P(E^k) \) such that \( \dim P(E^k) = 2m \) and

\[ D(y) = v_1v_2x^{k-\frac{a+b}{2}} + x^k \]

for \( k \geq (a + b)/2 \). Then \( m = \frac{a+b}{2} + k - 1 \) from \( 2m = a + b + 2k - 2 \) and

\[ \cup x^{m-a}(v_1) = v_1x^{m-a} = v_1x^{\frac{a+b}{2}+k-1-a} \text{ in } \]

\[ \cup x^{m-a} : H^a(P(E^k); \mathbb{Q}) \to H^{2m-a}(P(E^k); \mathbb{Q}) \]

for \( 0 \leq a \leq m \). In cohomology, this element has the form \( v_1x^{\geq k} = 0 \) if and only if \( a < b \). Thus, when \( a < b \), \( \cup x^{m-a}(v_1) = 0 \); i.e., \( \cup x^{m-a} \) is not the Lefschetz map. On the other hand, when \( a = b \), we have from (***)

\[ \cup x^{m-2k}(x^i) = x^{m-i} \]
\[ \cup x^{m-a-2k}(v_1x^i) = v_1x^{m-a-i} \]
\[ \cup x^{m-b-2k}(v_2x^i) = v_2x^{m-b-i} \]
\[ \cup x^{m-a-b-2k}(v_1v_2x^i) = v_1v_2x^{m-a-b-i}, \]

whose linear combination can not be zero in cohomology. Thus \( M \) is projective \((k)\)-Lefschetz for \( k \geq b \) when \( a = b \).
Let $a \leq b$ be even. Consider $P(E^k)$ such that $\dim P(E^k) = 2m$ and
\[ D(y) = v_1 x^{\frac{a+b}{2}} + v_2 + x^{\frac{a}{2}}, \quad (k = \frac{b}{2}) \]
Then $m = \frac{b}{2} + b - 1$ and we have from (***)
\[ \cup x^{m-2i}(x^i) = x^{m-i}, \]
\[ \cup x^{m-a-2i}(v_1 x^i) = v_1 x^{m-a-i} = \begin{cases} v_1 v_2 x^{m-a-\frac{b}{2}-i} & (i < -\frac{b}{2}) \\ v_1 x^{m-a-i} & (\frac{b}{2} \leq i < -\frac{a+2b}{2}) \end{cases} \]
\[ \cup x^{m-b-2i}(v_2 x^i) = v_2 x^{m-b-i}, \]
\[ \cup x^{m-a-b-2i}(v_1 v_2 x^i) = v_1 v_2 x^{m-a-b-i}, \]
whose linear combination cannot be zero in cohomology; i.e., $\cup x^j$ are the Lefschetz maps. Thus $M$ is projective $(\frac{b}{2})$-Lefschetz.

**Remark 3.4.** Even if $M$ is projective $(k)$-Lefschetz, it is not projective $(m)$-Lefschetz for $m > k$, in general. For example, when $M = S^4 \times S^6$, $M$ is projective $(3)$-Lefschetz from Theorem 3.3 but not projective $(4)$-Lefschetz. Indeed, in the proof of Theorem 3.3, $\cup x^2 : H^{m-2}(P(E^4)) \to H^{m+2}(P(E^4))$ is not a monomorphism since $\cup x^2([v_1 x + v_2 x^3]) = [v_1 x^3 + v_2 x^2 + x^3] = 0$, when $Dy = v_1 x^3 + v_2 x^2 + x^4 (m = 8)$.

**Theorem 3.5.** Let $M = S^a \times S^b \times S^c$ with $a \leq b \leq c$. We have the following:

(i) When $a$, $b$ and $c$ are even, $M$ is projective $(\frac{a}{2})$-Lefschetz.

(ii) When $a$ and $c$ are odd, $b$ is even, $M$ is projective non-Lefschetz.

(iii) When $a$ is even, $b$ and $c$ are odd, $M$ is projective Lefschetz if and only if $b = c$. Then $M$ is projective $(b)$-Lefschetz.

(iv) When $a$ and $b$ are odd, $c$ is even, $M$ is projective Lefschetz if and only if $a = b$. Then $M$ is projective $(\max\{a, b\})$-Lefschetz.

**Proof.** Then $\dim M = a + b + c$ and $H^*(M; \mathbb{Q}) = \Lambda(v_1, v_2, v_3)/(v_1^2, v_2^2, v_3^2)$ with $|v_1| = a$, $|v_2| = b$, $|v_3| = c$.

(i) When $k = \frac{a}{2}$, $\dim P(E^k) = a + b + 2c - 2$ and $m = \frac{a+b+2c-2}{2}$. Then $|y| = c - 1$ and $d(y) = x^\frac{c}{2}$. Let $D(y) = v_1 x^{\frac{a+b}{2}} + v_2 x^{\frac{a+c}{2}} + v_3 + x^\frac{c}{2}$. Then $P(E^k)$ is $c$-symplectic by $x$ since $[x^m] = -[v_1 v_2 v_3 x^\frac{a+b+c}{2}] \neq 0$. Moreover, we have from (***)
\[ \cup x^{m-2i}(x^i) = x^{m-i}, \]
\[ \cup x^{m-a-2i}(v_1 x^i) = \begin{cases} 2v_1 v_2 v_3 x^{\frac{a+c-2}{2}-i} & (0 \leq i < -\frac{a+b}{2}) \\ v_1 v_2 v_3 x^{\frac{a+c-2}{2}-i} - v_1 v_3 x^{\frac{a+b+c-2}{2}-i} & (\frac{a+b}{2} \leq i < -\frac{a+c}{2}) \end{cases} \]
\[ \cup x^{m-b-2i}(v_2 x^i) = \begin{cases} v_1 v_2 v_3 x^{\frac{a+b-c-2}{2}-i} - v_2 v_3 x^{\frac{a+b+c-2}{2}-i} & (0 \leq i < -\frac{b+c}{2}) \\ -v_1 v_2 v_3 x^{\frac{a+b-c-2}{2}-i} - v_2 v_3 x^{\frac{a+b+c-2}{2}-i} & (\frac{b+c}{2} \leq i < \frac{a+b+c}{2}) \end{cases} \]
\[ \cup x^{m-a-b-2i}(v_1 v_2 x^i) = \begin{cases} v_1 v_2 v_3 x^{\frac{a-b+c-2}{2}-i} & (0 \leq i < -\frac{b+c}{2}) \\ -v_1 v_2 v_3 x^{\frac{a-b+c-2}{2}-i} & (\frac{b+c}{2} \leq i < \frac{a+b+c}{2}) \end{cases} \]
whose linear combination can not be zero in cohomology. Thus $M$ is projective $(\frac{1}{2})$-Lefschetz.

(ii) For $|y| = 2k - 1$ and $m = \frac{a+b+c+2k-2}{2}$, there are two types of $c$-symplectic models as follows:

\[
\begin{align*}
\text{(1)} & \quad D(y) = v_1 v_3 x^{k - \frac{a+b}{2}} + v_2 x^{k - \frac{b}{2}} + x^k. \\
\text{(2)} & \quad D(y) = v_1 v_2 v_3 x^k + x^k.
\end{align*}
\]

Then $\cup x^{m-a}(v_1) = -v_1 v_2 x^{-\frac{a+b+2k-2}{2}} = -v_1 v_2 x^{k} = 0$ from $a < c$.

\[\text{(2)} \quad D(y) = v_1 v_2 v_3 x^{k} + x^k.\]

Then $\cup x^{m-a}(v_1) = v_1 x^{k - \frac{a+b+c+2k-2}{2}} = v_1 x^{k} = 0$. Thus the Lefschetz maps do not exist in both cases (1) and (2).

(iii) Let $b < c$. For $|y| = 2k - 1$ and $m = \frac{a+b+c+2k-2}{2}$, there are two types of $c$-symplectic models as follows:

\[
\begin{align*}
\text{(1)} & \quad D(y) = v_1 x^{k - \frac{a+b}{2}} + v_2 v_3 x^{k - \frac{b}{2}} + x^k. \\
\text{(2)} & \quad D(y) = v_1 v_2 x^{k} + x^k.
\end{align*}
\]

Then $\cup x^{m-b}(v_2) = -v_1 v_2 x^{-\frac{a+b+2k-2}{2}} = -v_1 v_2 x^{k} = 0$ from $b < c$.

\[\text{(2)} \quad D(y) = v_1 v_2 v_3 x^{k} + x^k.\]

Then $\cup x^{m-a}(v_1) = v_1 x^{k - \frac{a+b+c+2k-2}{2}} = v_1 x^{k} = 0$. Thus the Lefschetz maps do not exist in both cases (1) and (2).

Let $b = c$. Then $M = S^a \times S^b \times S^b$, dim $M = a + 2b$ and $H^*(M; \mathbb{Q}) = \mathbb{Q}[v_1]/(v_1^2) \otimes H^*(S^b, v_3)$ with $|v_1| = a$, $|v_2| = |v_3| = b$. When $k = b$, dim $P(E^k) = a + 4b - 2$ and $m = \frac{a+4b-2}{2}$. Then $|y| = 2b - 1$ and $d(y) = x^b$. Let $D(y) = v_1 x^{k - \frac{b}{2}} + v_2 v_3 + x^b$. Then $P(E^k)$ is $c$-symplectic with respect to $x$. Moreover, we have from (***)

\[
\begin{align*}
\cup x^{m-2i}(x^i) & = x^{m-i}, \\
\cup x^{m-a-2i}(v_1 x^i) & = \left\{ \begin{array}{ll}
v_1 v_2 v_3 x^{-\frac{a+b+2k-2}{2}} & (0 \leq i < \frac{a+2b}{2}) \\
v_1 x^{-\frac{a+b+c-2}{2}} & (\frac{a+2b}{2} \leq i < \frac{a+4b}{4}).
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\cup x^{m-b-2i}(v_2 x^i) & = \left\{ \begin{array}{ll}
v_1 v_2 x^{b-\frac{a+b+2k-2}{2}} & (0 \leq i < \frac{a}{2}) \\
v_2 x^{b-\frac{a+b+c-2}{2}} & \left( \frac{a}{2} \leq i < \frac{a+2b}{4} \right).
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\cup x^{m-b-2i}(v_3 x^i) & = \left\{ \begin{array}{ll}
v_1 v_3 x^{b-\frac{a+b+2k-2}{2}} & (0 \leq i < \frac{a}{2}) \\
v_3 x^{b-\frac{a+b+c-2}{2}} & \left( \frac{a}{2} \leq i < \frac{a+2b}{4} \right).
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\cup x^{m-(a+b)-2i}(v_1 v_2 x^i) & = v_1 v_2 x^{-\frac{a+b+2k-2}{2}} i,
\end{align*}
\]

\[
\begin{align*}
\cup x^{m-(a+b)-2i}(v_1 v_3 x^i) & = v_1 v_3 x^{-\frac{a+b+2k-2}{2}} i,
\end{align*}
\]

\[
\begin{align*}
\cup x^{m-(a+b)-2i}(v_2 v_3 x^i) & = v_2 v_3 x^{-\frac{a+b+2k-2}{2}} i,
\end{align*}
\]
Theorem 3.6. Let \( M = S^a \times S^b \times S^c \times S^d \) with \( a \leq b \leq c \leq d \). When \( a, b, c \) and \( d \) are odd, \( M \) is projective Lefschetz if and only if \( a = b \) and \( c = d \). Then \( M \) is projective (c)-Lefschetz.

Proof. Let \( a < b \). For \( |y| = 2k - 1 \) and \( m = \frac{a+b+c+d+2k-2}{2} \), there are four types of \( c \)-symplectic models as follows:

\[
\begin{align*}
(1) & \quad D(y) = v_1v_2x^k - \frac{a+b}{2} + v_3v_4x^k - \frac{c+d}{2} + x^k. \\
\text{Then } \cup x^{m-a}(v_1) & = -v_1v_3v_4x^{\frac{a+b+2k-2}{2}} = -v_1v_3v_4x^{\geq k} = 0.
\end{align*}
\]

(2) \( D(y) = v_1v_3x^k - \frac{a+c}{2} + v_2v_4x^k - \frac{b+d}{2} + x^k. \)

Then \( \cup x^{m-a}(v_1) = -v_1v_2v_4x^{\frac{a+c+2k-2}{2}} = -v_1v_2v_4x^{\geq k} = 0. \)

(3) \( D(y) = v_1v_4x^k - \frac{a+d}{2} + v_2v_3x^k - \frac{b+c}{2} + x^k. \)

Then \( \cup x^{m-a}(v_1) = -v_1v_2v_3x^{\frac{a+d+2k-2}{2}} = -v_1v_2v_3x^{\geq k} = 0. \)

(4) \( D(y) = v_1v_2v_3v_4x^{k - \frac{a+b+c+d+2k-2}{2}}. \)

Then \( \cup x^{m-a}(v_1) = v_1x^{\geq k} = 0. \) Thus, when \( a < b \), \( M \) is projective non-Lefschetz.

Let \( c < d \). For \( |y| = 2k - 1 \) and \( m = \frac{a+b+c+d+2k-2}{2} \), there are four types of \( c \)-symplectic models as follows:

\[
\begin{align*}
(1) & \quad D(y) = v_1v_2x^k - \frac{a+b}{2} + v_3v_4x^k - \frac{c+d}{2} + x^k. \\
\text{Then } \cup x^{m-c}(v_1) & = -v_1v_2v_3x^{\frac{a+b+c+2k-2}{2}} = -v_1v_2v_3x^{\geq k} = 0.
\end{align*}
\]

(2) \( D(y) = v_1v_3x^k - \frac{a+c}{2} + v_2v_4x^k - \frac{b+d}{2} + x^k. \)

Then \( \cup x^{m-b}(v_2) = v_1v_2v_3x^{\frac{b+d+2k-2}{2}} = v_1v_2v_3x^{\geq k} = 0. \)

(3) \( D(y) = v_1v_4x^k - \frac{a+d}{2} + v_2v_3x^k - \frac{b+c}{2} + x^k. \)

Then \( \cup x^{m-a}(v_1) = -v_1v_2v_3x^{\frac{a+d+2k-2}{2}} = -v_1v_2v_3x^{\geq k} = 0. \)

(4) \( D(y) = v_1v_2v_3v_4x^{k - \frac{a+b+c+d+2k-2}{2}}. \)

Then \( \cup x^{m-a}(v_1) = v_1x^{\geq k} = 0. \) Thus, when \( c < d \), \( M \) is projective non-Lefschetz.

Let \( a = b \) and \( c = d \). Then \( M = S^a \times S^a \times S^c \times S^c \), \( \text{dim } M = 2a + 2c \) and \( H^*(M; Q) = \Lambda(v_1, v_2, v_3, v_4) \) with \( |v_1| = |v_2| = a, |v_3| = |v_4| = c. \) When \( k = c \), \( \text{dim } P(E^k) = 2a + 4c - 2 \) and \( m = a + 2c - 1 \). Then \( |y| = 2k - 1 \) and \( d(y) = x^c. \)
Let \( D(y) = v_1v_2x^{c-a} + v_3v_4 + x^c \). Then \( P(E^k) \) is \( c \)-symplectic with respect to \( x \). Moreover, we have from (***)
\[
\begin{align*}
\cup x^{m-2i}(x^i) &= x^{m-i}, \\
\cup x^{m-a-2i}(v_1x^i) &= -v_1v_3v_4x^{c-1-i}, \\
\cup x^{m-a-2i}(v_2x^i) &= -v_2v_3v_4x^{c-1-i}, \\
\cup x^{m-c-2i}(v_3x^i) &= \begin{cases} 
-v_1v_2v_3x^{c-1-i} & (0 \leq i < a) \\
 v_3x^{a+c-1-i} & (a \leq i < \frac{a+c}{2}), 
\end{cases} \\
\cup x^{m-c-2i}(v_4x^i) &= \begin{cases} 
-v_1v_2v_4x^{c-1-i} & (0 \leq i < a) \\
 v_4x^{a+c-1-i} & (a \leq i < \frac{a+c}{2}), 
\end{cases}
\end{align*}
\]
whose linear combination can not be zero in cohomology. Thus \( M \) is projective (\( c \))-Lefschetz. \( \square \)

A nilpotent space is said to be formal if there is a quasi-isomorphism from its Sullivan minimal model to its rational cohomology algebra thought of as a DGA with zero differential [15]([5]). For example, compact Kähler manifolds are formal [4]. Finally we give a non-formal example.

**Theorem 3.7.** Let \( M \) be a simply connected 16-dimensional manifold such that \( \mathcal{M}(M) = (\Lambda(v_1, v_2, v_3, v_4), d) \) with \( |v_1| = |v_2| = 3, |v_3| = |v_4| = 5 \), \( d(v_1) = d(v_2) = 0 \), \( d(v_3) = v_1v_2 \) and \( d(v_4) = 0 \). Then \( M \) is projective non-Lefschetz.

**Proof.** There are only two cases for which \( P(E^k) \) is \( c \)-symplectic.

First, let \( D(y) = v_1v_4x^i + v_2v_3x^i + x^{i+4} \) with \( |y| = 7 + 2i \). Then
\[ \dim P(E^k) = 22 + 2i \]
and \( m = 11 + i \).

Then \( P(E^k) \) is \( c \)-symplectic from \( [x^{11+i}] = -[v_1v_2v_3v_4x^{i+3}] \neq 0 \). But \( P(E^k) \) does not have the Lefschetz property since \( [v_1x^{8+i}] = [v_1(-v_1v_4x^i - v_2v_3x^i)x^i] = -[v_1v_2v_4x^{i+4}] = [v_1v_2v_3(v_1v_4x^i + v_2v_3x^i)] = 0. \)
Secondly, let $D(y) = v_1 v_2 v_3 v_4 x^i + x^{i+8}$ with $|y| = 15 + 2i$. Then
$$\dim P(E_k) = 30 + 2i$$ and $m = 15 + i$.

Then $P(E_k)$ is $c$-symplectic from $|x^{15+i}| = |v_1 v_2 v_3 v_4 x^{i+7}| \neq 0$. But $P(E_k)$ does not have the Lefschetz property since $|v_1 x^{15+i}| = |v_1 (v_2 v_3 v_4 x') x| = 0$.

Note that the manifold $M$ of Theorem 3.7 is the product of $S^5$ with the pullback of the sphere bundle of the tangent bundle of $S^6$ by the canonical degree 1 map $S^3 \times S^3 \to S^6$. It is not formal since $H^*(M; \mathbb{Q})$ contains an indecomposable element $[v_1 v_3]$ (or $[v_2 v_3]$), which corresponds to a non-trivial Massey product $\langle v_1, v_2, v_1 \rangle$ (or $\langle v_2, v_1, v_2 \rangle$) [4]. Recall that $Y = (S^3 \times S^5)\sharp (S^3 \times S^8)$ is formal and has the same rational cohomology as $M$. From Corollary 2.2, we see that $Y$ is projective non-Lefschetz.

Remark 3.8. We know that $S^3 \times S^3 \times S^5 \times S^5$ is projective (5)-Lefschetz from Theorem 3.6. It has the same rational homotopy groups as the manifold $M$ of Theorem 3.7. Thus projective Lefschetzness is not determined by the rational homotopy groups.

**References**


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