CERTAIN CLASSES OF ANALYTIC FUNCTIONS AND DISTRIBUTIONS WITH GENERAL EXPONENTIAL GROWTH

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Abstract. Let $K'_M$ be the generalized tempered distributions of $e^{M(t)}$, growth, where the function $M(t)$ grows faster than any linear functions as $|t| \to \infty$, and let $K'_M$ be the Fourier transform spaces of $K'_M$. We obtain the relationship between certain classes of analytic functions in tubes, $K'_M$ and $K'_M$.

1. Introduction

In his book [9], V. S. Vladimirov has considered the relationship between the class of analytic functions in tubes $H(A; C)$ and tempered distributions with polynomial growth $S'$. Later R. D. Carmichael [3] has introduced two different types of classes of analytic functions in tubes $G_p(A; C)$ and $F_p(A; C)$ both of which are extensions of $H(A; C)$ and has obtained the relationship between $G_p(A; C)$ (and $F_p(A; C)$) and tempered distributions with exponential growth of polynomial powers $K'_p$, $p > 1$, and the Fourier transform spaces $K'_p$, $p > 1$.

In this paper, we introduce two different types of classes of analytic functions in tubes $G_M(A; C)$ and $F_M(A; C)$ which are extensions of $G_p(A; C)$ and $F_p(A; C)$, respectively, and obtain the relationship between $G_M(A; C)$ (and $F_M(A; C)$) and tempered distributions with general exponential powers growth $K'_M$ and the Fourier transform $K'_M$ of $K'_M$.

In the main sections, we show that elements of $G_M(A; C)$ and $F_M(A; C)$ can be represented as the Fourier-Laplace transform of distributions $K'_M$. Also we present representations of $G_M(A; C)$ and $F_M(A; C)$ as elements in $K'_M$ in terms of Fourier transforms in $K'_M$ of certain elements in $K'_M$ and strong boundedness for $G_M(A; C)$ and $F_M(A; C)$ as elements in $K'_M$. In particular, we show that elements of $F_M(A; C)$ obtain distributional boundary values in $K'_M$.
2. Notation and preliminaries

We denote the points of $\mathbb{R}^n$ spaces by $t = (t_1, t_2, \ldots, t_n)$ and $s = (s_1, s_2, \ldots, s_n)$. The letter $n$ always denotes the dimension. In $\mathbb{C}^n$ the points are denoted by $z = x + iy$, $x, y \in \mathbb{R}^n$. We define $\langle t, s \rangle = t_1s_1 + t_2s_2 + \cdots + t_ns_n$ and similarly define $\langle t, z \rangle$, $t \in \mathbb{R}^n$, $z \in \mathbb{C}^n$. $\alpha$ denotes $n$ tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers. $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$.

If $k = (k_1, k_2, \ldots, k_n)$ is an $n$ tuples of integers, $t^k = t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n}$, $t \in \mathbb{R}^n$, with similar definition for $z^k$, $z \in \mathbb{C}^n$. If $a \in \mathbb{R}$, then $at = (at_1, at_2, \ldots, at_n)$.

We write $D_j = -\frac{1}{\gamma(\alpha)} \left( \frac{a}{\gamma(\alpha)} \right)$, $j = 1, 2, \ldots, n$, and $D^a = D_1^aD_2^a\cdots D_n^a$ and similarly write $D^a$.

**Definition 1.** A set $C \subset \mathbb{R}^n$ is a cone with vertex at zero if $y \in C$ implies $\lambda y \in C$ for all positive real scalar $\lambda$.

**Definition 2.** Let $C$ be a cone. $C \cap \{y \in \mathbb{R}^n : |y| = 1\}$ is called the projection of $C$ and is denoted $\text{pr}(C)$.

**Definition 3.** If $C'$ and $C$ are cones such that $\text{pr}(C') \subset \text{pr}(C)$, then $C'$ is called a compact subcone of $C$.

For a cone $C$, $\mathcal{O}(c)$ will denote the convex hull or envelop of $C$ and $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ is a tube in $\mathbb{C}^n$.

**Definition 4.** If $C$ is open, $T^C$ is called a tubular cone. If $C$ is both open and connected, $T^C$ is called a tubular radial domain.

**Definition 5.** The set $C^* = \{t \in \mathbb{R}^n : \langle t, y \rangle \geq 0, \ y \in C\}$ is the dual cone of the cone $C$ and $C_* = \mathbb{R}^n \setminus C^*$.

**Definition 6.** The function

$$u_C = \sup_{y \in \text{pr}(C)} (-\langle t, y \rangle)$$

is the indicatrix of the cone $C$.

It follows that $C^* = \{t \in \mathbb{R}^n : u_C(t) \leq 0\}$. Further $u_C(t) \leq u_{\mathcal{O}(C)}(t)$ and if $t \in C^*$, then $u_C(t) = u_{\mathcal{O}(C)}(t)$ [9, p. 219]. To characterize the nonconvexity of the cone, we have the following: for a cone $C$, let

$$\rho_C = \sup_{t \in C^*} \frac{u_{\mathcal{O}(C)}}{u_C(t)}$$

A cone $C$ is convex if and only if $\rho_C = 1$ [9, Sec. 25.1, Lemma 2] and if a cone is open and consists of finite number of components, then $\rho_C < 1$ [9, Sec. 25.1, Lemma 3]. In this paper we shall be considering the case $1 \leq \rho_C < +\infty$ for all cones $C$.

Now we present four important facts what will be used frequently later.

**Lemma 1** ([9, Sec. 25.1]). Let $C$ be a cone. Then

$$-\langle t, y \rangle \leq |y|u_{\mathcal{O}(C)}^C, \ u_{\mathcal{O}(C)} \leq \rho_Cu_C(t), \ t \in C^*, y \in \mathcal{O}(C).$$
Lemma 2 ([9, Eq.(28), p. 241]). Let $C$ be an open connected cone and let $C'_*$ be a compact subcone of $C_*$. Then there exist $\xi = \xi(C'_*)$, depending on $C'_*$, such that

$$\xi|t| \leq u_C(t) \leq |t|, \quad t \in C'_*.$$

Lemma 3 ([9, Sec. 25.2, Lemma 2]). Let $C$ be an open cone and $C'$ that is an arbitrary subcone of $\partial \mathcal{O}(C)$. Then there exist a number $\delta = \delta(C')$ and open cone $(C')'$ both depending on $C'$ such that $C'^* \subset (C'^*)_*$ and

$$(y, t) \geq \delta |y| |t|, \quad y \in C' \subset \partial \mathcal{O}(C), \quad t \in (C'^*)_'.$$

Lemma 4 ([9, Lemma, p. 241]). Let $C'_*$ be cone that is compact in the cone $C_*$. For an arbitrary number $\eta \in (0, 1)$, there exists a compact subcone $C' = C'(C'_*, \eta)$ depending on $C'_*$ and on $\eta$ such that for any $t \in C'_*$, there exists a point $y^0_t \in \text{Pr}(C')$ at which

$$-(t, y^0_t) \geq (1 - \eta)u_C(t).$$

3. The distribution spaces $\mathcal{K}_M'$ and $\mathcal{K}_M''$

Let $\mu(\xi)$, $0 \leq \xi \leq \infty$, denote a continuous increasing function such that $\mu(0) = 0$, $\mu(\infty) = \infty$. For $t \geq 0$ we define

$$M(t) = \int_0^t \mu(\xi)d\xi.$$

The function $M(t)$ is an increasing, convex, and continuous function with $M(0) = 0$ and $M(\infty) = \infty$. Further we define $M(t)$ for negative $t$ by $M(-t) = M(t)$. Since the derivative $\mu'(t)$ of $M(t)$ is unbounded in $R$, the function $M(t)$ will grow faster than any linear function as $|t| \to \infty$.

The function $M(t)$ can be defined on $\mathbb{R}^n$ by $M(t_1 + t_2 + \cdots + t_n) = M(t_1) + M(t_2) + \cdots + M(t_n)$ for all $t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. (Refer to Sec. 4.1 of Chapter 1 in [4] or p. 130 in [5].)

Definition 7. Let $M(x)$ and $\Omega(y)$ be the functions corresponding to $\mu(\xi)$ and $\omega(\eta)$ as above, respectively. Then $M(x)$ and $\Omega(y)$ are called to be dual in the sense of Young if $\mu(\omega(\eta)) = \eta$ and $\omega(\mu(\xi)) = \xi$.

We have two examples of dual functions in the sense of Young as follow;

1. $M(s) = \frac{s^p}{p}$, $\Omega(t) = \frac{t^q}{q}$, $\frac{1}{p} + \frac{1}{q} = 1$, $s, t \geq 0$.
2. $M(s) = e^s - s - 1$, $\Omega(t) = (t + 1)\log(t + 1) - t$, $s, t \geq 0$.

We list some properties of function $M(x)$, $x \in \mathbb{R}^n$.

Lemma 5. For $t \geq 0$ we define $M(t) = \int_0^t \mu(\xi)d\xi$, where $\mu(\xi)$ ($0 \leq \xi \leq \infty$) is a continuous increasing function such that $\mu(0) = 0$ and $\mu(\infty) = \infty$. Then we have that

$$M(s) + M(t) \leq M(s + t) \text{ for all } st \geq 0.$$
where $s, t$ and the equality holds if and only if $t$.

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Lemma 6. Let $M(s)$ and $\Omega(t)$ be defined as in Definition 7, where $s, t \in R$. Then

\[ st \leq M(s) + \Omega(t) \] for any $s, t \geq 0$

and the equality holds if and only if $t = \mu(s)$ or $s = \omega(t)$.

Hence if we let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$, then

\[ \langle x, y \rangle \leq M(x) + \Omega(y) \] for any $x_i, y_i \geq 0$, \( i = 1, 2, \ldots, n \)

and the equality holds if and only if $y_i = \mu(x_i)$ or $x_i = \omega(y_i)$.

For more details about the function $M(x)$ and $\Omega(y)$, we can refer to [4, Chapter 1].

Using the function $M(t)$, we define the space $K_M$ as the space of all functions $\varphi(t)$ in $C^\infty$ such that

\[ v_k(\varphi) = \sup_{t \in \mathbb{R}^n, |\alpha| \leq k} e^{M(kt)}|D_\alpha^k \varphi(t)| < \infty, \quad k = 1, 2, \ldots, \]

where $D_\alpha^k = D_1^{\alpha_1}D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ and $D_\alpha^k = \frac{\partial^{|\alpha|}}{\partial t_1^{\alpha_1} \cdots \partial t_n^{\alpha_n}}$. The topology in $K_M$ is defined by the countably family of semi-norms $\{v_k\}_{k=1}^\infty$. It follows that the space $K_M$ becomes a Fréchet space [5] and the identity mapping $\mathcal{D} \hookrightarrow K_M \hookrightarrow \mathcal{E}$ are continuous when $\mathcal{E}$ denotes the space of all $C^\infty$ functions on $\mathbb{R}^n$ and $\mathcal{D}$ the space of all $C^\infty$ functions with compact support in $\mathbb{R}^n$.

Lemma 7. $K_M$ is a Montel space.

Proof. If $B$ is a bounded set of $K_M$, then $B$ is a bounded set in $C^\infty$ since the imbedding $K_M \hookrightarrow C^\infty$ is continuous. Since $C^\infty$ is a Montel space, it suffices to show that $B$ is a relatively compact set in $C^\infty$. Let $(\phi_j)$ be a sequence of elements of $B$ such that $(\phi_j)$ converges to $\phi$ in $C^\infty$. Since $B$ is a bounded set of $K_M$, for all $k \in N$ and all $\alpha \in N^n$, there exists a constant $C_{k, \alpha}$ such that

\[ \sup_{t \in \mathbb{R}^n} |e^{M(kt)}D_\alpha^k \phi_j(t)| \leq C_{k, \alpha}, \quad \phi_j \in B. \]  

The inequality (1) implies that, given $\epsilon > 0$ there is a constant $M > 0$ such that for $t$ with $|t| > M$,

\[ |e^{M(kt)}D_\alpha^k \phi_j(t)| \leq \epsilon, \quad \phi_j \in B. \]
Since \( \phi_j \to \phi \) in \( C^\infty \), (2) implies that
\[
|e^{M(kt)}D^\alpha \phi(t)| \leq \epsilon, \quad |t| > M.
\]

Hence \( \phi \in \mathcal{K}_M \). On the other hand, since \( \phi_j \to \phi \) in \( C^\infty \), \( (D^\alpha \phi_j) \) converges uniformly to \( D^\alpha \phi \) on the compact set \( \{ t \in \mathbb{R}^n : |t| \leq M \} \). This implies that given \( \epsilon > 0 \), we can find an integer \( j_0 \) such that
\[
e^{M(kt)}|D^\alpha \phi_j(t) - D^\alpha \phi(t)| \leq \epsilon
\]
for all \( t \) with \( |t| \leq M \) and all \( j \geq j_0 \). Last three inequalities imply that
\[
\sup_{t \in \mathbb{R}^n} e^{M(kt)}|D^\alpha \phi_j(t) - D^\alpha \phi(t)| \leq \epsilon
\]
for all \( j \geq j_0 \); therefore \( \phi_j \to \phi \) in \( \mathcal{K}_M \). The proof is completed. \( \square \)

We denote by \( \mathcal{K}'_M \) the space of all continuous linear functional on \( \mathcal{K}_M \).
Clearly when \( M(t) = \log(1 + |t|) \), \( \mathcal{K}'_M \) is the space of Schwartz’s tempered distributions. When \( M(t) = |t| \), \( \mathcal{K}'_M \) is the space of tempered distributions of \( \mathcal{K}'_1 \) which is introduced and characterized by J. Sevastião E. Silva [8]. When \( M(t) = |t|^p \), \( p > 1 \), \( \mathcal{K}'_M \) is the space of tempered distributions of \( \mathcal{K}'_p \), \( p > 1 \), which is introduced and characterized by Sampson and Zielezny [6].

The restriction \( \tilde{T} = T|_{\mathcal{D}} \) of a functional \( T \in \mathcal{K}'_M \) to \( \mathcal{D} \) is a distribution. Since \( \mathcal{D} \) is dense in \( \mathcal{K}_M \), \( T \) is determined by its values on \( \mathcal{D} \). We characterize the distributions in \( \mathcal{K}'_M \) by their growth at infinity.

**Lemma 8** (5, Theorem 2.3]). A distribution \( T \in \mathcal{D} \) is in \( \mathcal{K}'_M \) if and only if there exist positive integers \( k \), \( \alpha \) and a bounded continuous function \( f(t) \) on \( \mathbb{R}^n \) such that
\[
T = D^\alpha \left[ e^{M(kt)} f(t) \right].
\]

Let \( \phi(t) \in L^1(\mathbb{R}^n) \). We define the Fourier transform of \( \phi(t) \) by
\[
\hat{\phi}(x) = \mathcal{F}[\phi(t); x] = \int_{\mathbb{R}^n} \phi(t) e^{2\pi i (x,t)} dt
\]
and the inverse Fourier transform of \( \phi(t) \) by
\[
\mathcal{F}^{-1}[\phi(t); x] = \int_{\mathbb{R}^n} \phi(t) e^{-2\pi i (x,t)} dt.
\]

Now we have a Paley-Wiener type theorem for the space \( \mathcal{K}_M \) from [5, Theorem 4.1]; an entire function \( F(\zeta) \) is a Fourier transform of a function \( \varphi \) in \( \mathcal{K}_M \) if and only if, for every integer \( N \geq 0 \) and every \( \epsilon > 0 \), there exists a constant \( C \) such that
\[
|F(\xi + i\eta)| \leq C(1 + |\zeta|)^{-N} e^{\Omega(\zeta)}, \quad \zeta = \xi + i\eta \in \mathbb{C}^n.
\]

Let \( \mathcal{K}_M \) be the space of Fourier transform of functions in \( \mathcal{K}_M \). We define in \( \mathcal{K}_M \) a locally convex topology by means of the seminorms
\[
\omega_k(\varphi) = \sup_{\zeta \in \mathbb{C}^n} (1 + |\zeta|)^k e^{-\Omega(\zeta)} |\hat{\varphi}(\zeta)|, \quad k = 1, 2, \ldots, \varphi \in \mathcal{K}_M.
\]
Lemma 9 ([5, Coro. 4.2]). The Fourier transform is a topological isomorphism of $K_M$ onto $K'_M$.

Let $K'_M$ be the space of continuous linear functional on $K_M$ which equipped with the topology of uniform convergence on all bounded set in $K_M$. Each distribution $T$ in $K'_M$ has a Fourier transform $\hat{T}$ in $K'_M$ defined by Parseval’s formula

$$\langle \hat{T}, \varphi \rangle = (2\pi)^n \langle T, \varphi \rangle, \quad \varphi \in K_M.$$ 

Moreover, we have:

Lemma 10 ([5, Coro. 4.3]). The Fourier transform is a topological isomorphism of $K'_M$ onto $K'_M$.

For further detailed structure theories about $K'_M$ and $K'_M$, we can refer to [4] and [5].

4. The analytic spaces $G_M(A; C)$ and $F_M(A; C)$

To find the relations between the increase in certain classes of analytic functions and the properties of their spectral functions, Vladimirov [9, Sec. 26.4] introduced the following class of analytic functions;

Let $C$ be an open cone in $\mathbb{R}^n$ and $C'$ be an arbitrary compact subcone of $C$. $p \geq 1$ and $A \geq 0$ are real numbers. A function $f(z)$ belongs to the class $H_p(A; C)$ if $f(z)$ is analytic in the tubular cone $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ and satisfies

$$|f(z)| \leq K(C') (1 + |z|)^{N} (1 + |y|)^{-M} e^{A|y|^p}, \quad z = x + iy \in T^C,$$

where $K(C')$ is a constant depending on $C'$, and $N$ and $M$ are nonnegative real numbers which do not depend on $C'$.

Motivated by the works of Vladimirov, R. D. Carmichael introduced two different types of classes of analytic functions in tubes both of which are more general spaces than the class $H_p(A; C)$ as follow;

Let $C$ be an open cone in $\mathbb{R}^n$ and $C'$ be an arbitrary compact subcone of $C$. $p \geq 1$ and $A \geq 0$ are real numbers. Let $m > 0$. $T(C'; m)$ denotes the set $T(C'; m) = \mathbb{R}^n + i(C' \cap (C' \cap N(0,m)))$ where $N(0,m)$ is a closed ball in $\mathbb{R}^n$ of radius $m > 0$ with center at the origin. A function $f(z)$ belongs to the class $G_p(A; C)$ if, for each compact subcone $C' \subset C$, there exists a fixed $m = m(C') > 0$ depending on $C'$ such that $f(z)$ is analytic in $T(C'; m)$ and satisfies

$$|f(z)| \leq K(C'; m)(1 + |z|)^{N} e^{2\pi A|y|^p}, \quad z = x + iy \in T(C'; m),$$

where $K(C'; m)$ is a constant depending on $C'$ and on $m$ and $N$ is a nonnegative real number which does not depend on $C'$ and on $m$.

A function $f(z)$ belongs to $F_p(A; C)$ if, for each compact subcone $C' \subset C$, $f(z)$ is analytic in $T^{C'} = \mathbb{R}^n + iC'$ and satisfies

$$|f(z)| \leq K(C'; m)(1 + |z|)^{N} e^{2\pi A|y|^p}, \quad z = x + iy \in T(C'; m),$$
where $K(C'; m)$ is a constant depending on $C'$ and on $m$ and $N$ is a nonnegative real number which does not depend on $C'$ and on $m$.

Cramichael studied the relationship between $G_p(A; C)$ (and $F_p(A; C)$), the distributions $K_p$, $p \geq 1$, and the Fourier transform $K_p'$, $p \geq 1$, of $K_p$, $p \geq 1$, in [3]. Since $K_p' \subset K_p'$, $p \geq 1$, we need more general classes of analytic functions than $G_p(A; C)$ or $F_p(A; C)$ to find the relationship between the classes of analytic functions, $K_p'$, and $K_p'$ as follow:

For $t \geq 0$, let $M(t) = \int_0^t \mu(\xi)d\xi$, where $\mu(\xi) (0 \leq \xi \leq \infty)$ is a continuous increasing function such that $\mu(0) = 0$ and $\mu(\infty) = \infty$. Let $C'$ be an open cone in $\mathbb{R}^n$ and $C'$ be an arbitrary compact subcone of $C$. $A \geq 0$ are real numbers. Let $m > 0$. $T(C'; m)$ denote the set $T(C'; m) = \mathbb{R}^n + i(C' \cap N(0, m))$ where $N(0, m)$ is a closed ball in $\mathbb{R}^n$ of radius $m > 0$ with center at the origin.

A function $f(z)$ belongs to the class $G_{M}(A; C)$ if, for each compact subcone $C' \subset C$, there exists a fixed $m = m(C') > 0$ depending on $C'$ such that $f(z)$ is analytic in $T(C'; m)$ and satisfies

$$|f(z)| \leq K(C'; m)(1 + |z|)^N e^{2\pi M(A \nu)}, \quad z = x + iy \in T(C'; m),$$

where $K(C'; m)$ is a constant depending on $C'$ and on $m$ and $N$ is a nonnegative real number which does not depend on $C'$ and on $m$.

A function $f(z)$ belongs to $F_{M}(A; C)$ if, for each compact subcone $C' \subset C$, $f(z)$ is analytic in $T(C') = \mathbb{R}^n + iC'$ and satisfies

$$|f(z)| \leq K(C'; m)(1 + |z|)^N e^{2\pi M(A \nu)}, \quad z = x + iy \in T(C'; m),$$

where $K(C'; m)$ is a constant depending on $C'$ and on $m$ and $N$ is a nonnegative real number which does not depend on $C'$ and on $m$.

The $2\pi$ in the exponential term in the definition of $G_M(A; C)$ and $F_M(A; C)$ simply reflects the way we have defined the Fourier transform in this paper. Obviously we have the following inclusion relation:

$$F_{M}(A; C) \subset G_{M}(A; C), \quad G_p(A; C) \subset G_{M}(A; C), \quad F_p(A; C) \subset F_{M}(A; C).$$

We need three lemmas which will be useful to obtain main results in the next two sections.

**Lemma 11.** For $t \geq 0$, let $\Omega(t) = \int_0^t \omega(\xi)d\xi$, where $\omega(\xi) (0 \leq \xi \leq \infty)$ is a continuous increasing function such that $\omega(0) = 0$ and $\omega(\infty) = \infty$. Let $C$ be an open connected cone and let $C'$ be an arbitrary compact subcone of $C_* = \mathbb{R}^n \setminus C_*$. Let $\gamma$ be an $n$-tuple of nonnegative integers. Let $n \geq 1$ be an integer and let $R > 0$. Then we have

$$|1 + |t|^{n+1+|\gamma|} \leq M_1 \exp[2\pi R \Omega(u_C(t))],$$

where $M_1 = M_1(C')$ depends on $C' \subset C_*$. Hence for $A > 0$

$$|1 + |t|^{n+1+|\gamma|} \leq M_2 \exp \left[2\pi R \Omega \left(\frac{u_C(t)}{A}\right)\right],$$
where \( M_2 = M_2(C_*, A) \) depends on \( C_* \subset C_* \) and on \( A \).

**Proof.** From Lemma 2, given \( C_* \subset C_* \) there exists \( \xi = \xi(C_*) \), depending on \( C_* \), such that

\[
(5) \quad \xi |t| \leq u_C(t) \leq |t|, \quad t \in C_*.
\]

Hence, for any \( R > 0 \),

\[
0 < \exp[2\pi R\Omega(t)] \leq \exp[2\pi R\Omega(u_C(t))], \quad t \in C_*.
\]

Since the function \( \Omega(t) \) in the hypothesis grows faster than any linear function as \( |t| \to \infty \) for \( t \in \mathbb{R}^n \),

\[
(1 + |t|)^{-n-1-\gamma} \exp[2\pi R\Omega(t)] \to \infty \quad \text{as} \quad |t| \to \infty
\]

for \( t \in \mathbb{R}^n \), hence

\[
(6) \quad (1 + |t|)^{-n-1-\gamma} \exp[2\pi R\Omega(u_C(t))] \to \infty \quad \text{as} \quad |t| \to \infty
\]

for \( t \in C_* \subset C_* \). Let \( N(0, m) \) be a closed ball of the origin in \( \mathbb{R}^n \) of radius \( m > 0 \). We can find \( O_m > 1 \), depending on \( m \), such that

\[
(7) \quad Q_m (1 + |t|)^{-n-1-\gamma} \exp[2\pi R\Omega(t)] \geq 1, \quad t \in N(0, m).
\]

By Lemma 2 and (7),

\[
(8) \quad Q_m (1 + |t|)^{-n-1-\gamma} \exp[2\pi R\Omega(u_C(t))] \geq 1, \quad t \in N(0, m) \cap C_*.
\]

Thus we have (3) from (6) and (8). Now if \( A > 0 \), we have from (5) that given \( C_* \subset C_* \), there exists \( \xi = \xi(C_*) \), depending on \( C_* \), such that

\[
(9) \quad \frac{\xi |t|}{A} \leq \frac{u_C(t)}{A} \leq \frac{|t|}{A}, \quad t \in C_*.
\]

If we replace (5) by (9), we have from the same process as above that

\[
(10) \quad \left(1 + \frac{|t|}{A} \right)^{n+1+\gamma} \leq M \exp \left[2\pi R\Omega \left(\frac{u_C(t)}{A}\right)\right].
\]

But since \((1 + |t|) \leq C \left(1 + |t|/A\right)\) when \( C = C(A) \), depending on \( A \), equals \( A \) if \( A \geq 1 \) and equals 1 if \( 0 < A < 1 \), we have (4) from (10).

\[ \square \]

**Lemma 12.** For \( t \geq 0 \), let \( \Omega(t) = \int_0^t \omega(\xi)d\xi \), where \( \omega(\xi) \) \((0 \leq \xi \leq \infty)\) is a continuous increasing function such that \( \omega(0) = 0 \) and \( \omega(\infty) = \infty \). Let \( C \) be an open connected cone and let \( C' \) be an arbitrary open compact subcone of \( O(C) \). Let \( C_* \) be an arbitrary compact subcone of \( C_* = \mathbb{R}^n \setminus C_* \). Let \( A > 0 \). Let \( g(t) \) be a continuous function of \( t \in \mathbb{R}^n \) which satisfies

\[
|g(t)| \leq K(C_*, \eta) \exp \left[-2\pi(1-2\eta)\Omega \left(\frac{u_C(t)}{A}\right)\right], \quad t \in C_* \subset C_*.
\]

for any \( \eta \in (0, 1) \) with \( 1 - 3\eta > 0 \), where \( K(C_*, \eta) \) is a constant, depending on \( C_* \) and on \( \eta \). Let \( z_0 \in T^{C'} = \mathbb{R}^n + ic' \) be an arbitrary but fixed point and let
$z \in N'(z_0, r) \subset T^C$, where $N'(z_0, r)$ is an open neighborhood of $z_0$ with radius $r > 0$ whose closure is in $T^C$. Then for any $n$-tuple $\gamma$ of nonnegative integer,

$$h^{\gamma, g}_{C^*}(z) = \int_{C^*_z} \Gamma g(t) e^{2\pi i \langle z, t \rangle} dt$$

converges absolutely and uniformly for $z \in N'(z_0, r)$.

**Proof.** From Lemma 1 and assumption about the estimation of $g(t)$, for $z = x + iy \in N'(z_0, r)$, there exists a real number $T$ with $|y| = |\text{Im}(z)| \leq T$ such that for $A > 0$ and any $\eta > 0$ with $1 - 2\eta > 0$

$$|h^{\gamma, g}_{C^*}(t)| \leq K(C^*, \eta) \int_{C^*_z} |t|^e^{-2\pi |y, t|}$$

where

By Lemma 2 and Lemma 11 with $R = \eta$, for $t \in C^*_z \subset C_*$, there exists a $\xi = \xi(C^*)$ such that

$$f(x) = 2\pi T \rho C x - 2\pi (1 - 3\eta) \Omega \left( \frac{\xi x}{A} \right), \quad x > 0.$$
Hence, for \( t \in C'_* \subset C_* \), if we take \( \eta \in (0,1) \) with \( 1 - 3\eta > 0 \),

\[
\exp \left[ 2\pi T \rho_C |t| - 2\pi (1 - 3\eta) \Omega \left( \frac{\xi t}{A} \right) \right] \\
\leq \exp \left[ 2\pi T \rho_C \frac{A}{\xi} \cdot \omega^{-1} \left( \frac{T \rho_C}{1 - 3\eta} \right) - 2\pi (1 - 3\eta) \Omega \left( \omega^{-1} \left( \frac{T \rho_C}{1 - 3\eta} \right) \right) \right] .
\]

Thus we have from (11), (12), and (13) that

\[
|h_{C'_*}^{\gamma, g}(z)| \leq M(C'_*, A) K(C'_*, \eta) \exp \left[ 2\pi T \rho_C \frac{A}{\xi} \cdot \omega^{-1} \left( \frac{T \rho_C}{1 - 3\eta} \right) - 2\pi (1 - 3\eta) \Omega \left( \omega^{-1} \left( \frac{T \rho_C}{1 - 3\eta} \right) \right) \right] \\
\quad \exp \left[ \frac{1}{(1 + |t|)^{n+1}} dt \right] \\
\leq K'(C'_*, A, \eta) \exp \left[ 2\pi T \rho_C \frac{A}{\xi} \cdot \omega^{-1} \left( \frac{T \rho_C}{1 - 3\eta} \right) \\
- 2\pi (1 - 3\eta) \Omega \left( \omega^{-1} \left( \frac{T \rho_C}{1 - 3\eta} \right) \right) \right] .
\]

for all \( z \in N'(z_0, r) \), where \( K'(C'_*, A, \eta) \) is a constant depending on fixed \( C'_* \), on fixed \( A > 0 \), and on fixed \( \eta \in (0,1) \) with \( 1 - 3\eta > 0 \). Since the last term of (14) is independent of \( z \in N'(z_0, r) \), the function \( h_{C'_*}^{\gamma, g}(z) \) converges absolutely and uniformly for \( z \in N'(z_0, r) \).

\[ \square \]

Remark. The estimation of inequalities in (14) will be continued under some additional conditions in Theorem 2 of the next section.

**Lemma 13.** Let \( C \) be an open connected cone and let \( C' \) be an arbitrary open compact subcone of \( \mathcal{O}(C) \). Let \( (C^*)' \) be an open cone as in Lemma 3 and let \( C'_* = \mathbb{R}^n \setminus (C^*)' \subset C_* \). Let \( z_0 \in T(C'; m) = \mathbb{R}^n + i(C' \cap N(0, m)) \) be arbitrary but fixed and let \( z \in N'(z_0, r) \subset T(C'; m) \), where \( N'(z_0, r) \) is an open neighborhood of \( z_0 \) with radius \( r > 0 \) whose closure is in \( T(C'; m) \). Let \( g(t) \) satisfies

\[
|g(t)| \leq Ke^{k|t|}, \quad t \in (C^*)'
\]

for some constants \( K \) and \( k \geq 0 \). Then for any \( n \)-tuple \( \gamma \) of nonnegative integers,

\[
h_{(C, \gamma)}^{\gamma, g}(z) = \int_{(C^*)'} g(t) e^{2\pi i (\gamma, t)} dt
\]

converges absolutely and uniformly for \( z \in N'(z_0, r) \).

**Proof.** By Lemma 3, there exist a number \( \delta = \delta(C') \) and an open cone \( (C^*)' \) both depending on \( C' \) such that \( C^* \subset (C^*)' \) and

\[
\langle y, t \rangle \geq \delta |y| |t|, \quad y \in C' \subset \mathcal{O}(C), \quad t \in (C^*)'.
\]

We choose the real number \( m = m(C') > 0 \) depending on \( C' \) such that

\[
m = \frac{k}{(2\pi \delta)} + 1,
\]
where \( k \geq 0 \) is as in (15). Then if \( y \in C' \) with \( |y| > m, k - 2\pi\delta|y| < -2\pi\delta < 0. \)
For the chosen \( m > 0 \) in (17), let \( z_0 \) be an arbitrary but fixed point in \( T(C'; m) \).
Choose \( N'(z_0, r) \) whose closure is in \( T(C'; m) \). Then we have from (16) and (17) that for \( z \in N'(z_0, r), \)
(18) \[
\left| h_{C}^{\gamma, g}(z) \right| \leq K \int_{(C')'} \left| f(t) e^{2\pi i (z,t)} dt \right| \leq K \int_{(C')'} \left| f(t) e^{2\pi i (y,t)} dt \right| \leq K Z_n \int_{0}^{\infty} s^{\gamma+n-1} \exp[-2\pi\delta s] ds = K Z_n (|\gamma| + n - 1)! (2\pi\delta)^{-|\gamma|-n}.
\]
Here we have used [7, Theorem 32, p. 39] in the second to last step in (18) and integration by parts \(|\gamma| + n - 1 \) times in the last step in (18), where \( K \)
is the constant as in (15) and \( Z_n \) is the area of the unit sphere in \( \mathbb{R}^n \). Since the last term of (18) is independent of \( z \in N'(z_0, r) \), the function \( h_{C}^{\gamma, g}(z) \) converges absolutely and uniformly for \( z \in N'(z_0, r) \).

5. The relationship \( G_M(A; C), K'_M, \) and \( K'_M \)

In this section, we show that elements of \( G_M(A; C) \) can be represented as the Fourier-Laplace transform of distributions \( K'_M \). Also we present representations of \( G_M(A; C) \) as elements in \( K'_M \) in terms of Fourier transforms in \( K'_M \) of certain elements in \( K'_M \) and strong boundedness for \( G_M(A; C) \) as elements in \( K'_M \).

**Theorem 1.** Let \( M(x) \) and \( \Omega(y) \) be the functions as in Definition 7. For the open connected cone \( C \), let \( f(z) \in G_M(A; C) \). For any compact subcone \( C' \subset C \), let \( m = m(C') \) be a fixed real number which depends on \( C' \) as in the definition of \( G_M(A; C) \). Then there exist a unique element \( V = D_{t}^{\alpha} (g(t)) \in K'_M \) where \( \alpha \) is an \( n \)-tuple of nonnegative integers and \( g(t) \) is a continuous function of \( t \in \mathbb{R}^n \) such that the following are hold.

(I) For \( A \geq 0 \)
(19) \[
f(z) = z^{\alpha} F[e^{-2\pi i (y,t)} g(t); x], \quad z = x + iy \in T(C'; m),
\]
where the Fourier transform is taken in the \( L^2 \) sense.

(II) For \( A \geq 0 \), \( g(t) \) satisfies
(20) \[
|g(t)| \leq K(C', m) \exp[2\pi (M(Ay) + |y||t|)], \quad t \in \mathbb{R}^n,
\]
where \( C' \subset C \) is arbitrary and \( K(C', m) \) depends on \( C' \) and on \( m \). Inequality (20) is independent of \( y \in (C' \setminus (C' \cap N(0, m))) \) and \( \text{supp}(g) = \text{supp}(V) \subseteq \{ t : u_C(t) \leq A \}. \)
(III) For $A > 0$ and any compact subcone of $C' \subset C_* = \mathbb{R}^n \setminus C^*$, $g(t)$ satisfies
\begin{equation}
|g(t)| \leq M(C', \eta) \exp \left[-2\pi(1 - 2\eta)\Omega \left(\frac{u_C(t)}{A}\right)\right], \quad t \in C',
\end{equation}
where any $\eta \in (0, 1)$ is such that $1 - 2\eta > 0$ and $M(C', \eta)$ is a constant depending on $C'$ and on $\eta$.

(IV) Let $A \geq 0$. If $g(t)$ satisfies that $|g(t)| \leq Ke^{k|t|}$ for any $t \in (C')'$ and for some constants $K$ and $k > 0$, then
\begin{equation}
f(z) = \langle V, e^{2\pi i(z,t)} \rangle, \quad z = x + iy \in T(C'; m).
\end{equation}

(V) For $A \geq 0$,
\begin{equation}
f(z) = \mathcal{F}[e^{-2\pi i(y,t)}V], \quad z = x + iy \in T(C'; m),
\end{equation}
where the equality in (23) holds in $K_M'$.

(VI) \begin{equation}
\{f(z) : y = \text{Im}(z) \in (C' \setminus (C' \cap N(0, m))), |y| \leq Q_m\}
\end{equation}
is strongly bounded in $K_M'$, where $Q_m > m > 0$.

**Proof.** Let $C$ be an open connected cone and let $C'$ be an arbitrary open compact subcone of $C$. For any compact subcone $C' \subset C$, let $m = m(C')$ be a fixed real number which depends on $C'$ as in the definition of $G_M(A; C)$ corresponding to $f(z)$. Since $f(z) \in G_M(A; C)$, we can choose an $n$-tuple $\alpha$ of nonnegative integers which is independent of $C'$ and of $m$ such that for $z = x + iy \in T(C'; m)$ and $\epsilon > 0$,
\begin{equation}
|z^{-\alpha}f(z)| \leq K'(C'; m)(1 + |z|)^{-n-\epsilon}e^{2\pi M(Ay)},
\end{equation}
where $K'(C'; m)$ is a constant and $n$ is a dimension. Put
\begin{equation}
g(t) = \int_{\mathbb{R}^n} z^{-\alpha}f(z)e^{-2\pi i(z,t)}dx, \quad z = x + iy \in T(C'; m),
\end{equation}
which is a continuous function of $t \in \mathbb{R}^n$. By [2, Theorem 1, p. 846] and (25), $g(t)$ is independent of $y = \text{Im}(z) \in (C' \setminus (C' \cap N(0, m)))$.

**Proof of (I).** We have from (25) that $z^{-\alpha}f(z) \in L_1 \cap L_2$ as a function of $x = \text{Re}(z)$ for an arbitrary $y \in (C' \setminus (C' \cap N(0, m)))$. Thus from (26),
\begin{equation}
e^{-2\pi i(y,t)}g(t) = \mathcal{F}^{-1}[z^{-\alpha}f(z); t], \quad z = x + iy \in T(C'; m),
\end{equation}
where the Fourier transform is taken in the $L_2$ sense. By the Plancherel theorem, $e^{-2\pi i(y,t)}g(t) \in L_2$ and
\begin{equation}
z^{-\alpha}f(z) = \mathcal{F}[e^{-2\pi i(y,t)}g(t); x], \quad z = x + iy \in T(C'; m),
\end{equation}
where the Fourier transform is taken in the $L_1$ or $L_2$ sense. This complete the proof of (I).

**Proof of (II).** From (25) and (26),
\begin{equation}
|g(t)| \leq K'(C'; m)e^{2\pi M(Ay)}e^{2\pi i(y,t)} \int_{\mathbb{R}^n} (1 + |x|)^{-n-\epsilon}dx
\end{equation}
\[ \leq K''(C'; m) \exp \left[ 2\pi (M(Ay_t) + \langle y, t \rangle) \right], \]

where \( K''(C'; m) \) is a constant. Since \( g(t) \) is independent of \( y = \Im(z) \in (C' \cap N(0, m)) \), (29) holds independently of \( y = \Im(z) \in (C' \cap N(0, m)) \). From exactly the same process in [1, pp. 846–847], we have that \( \text{supp}(g) = \text{supp}(V) \subseteq \{ t : u_C(t) \leq A \} \). This complete the proof of (II).

Consider

\[(30) \quad V = D^\delta_t (g(t)).\]

Since \( g(t) \) is a continuous function and satisfies (II), \( g(t) \in K'_M \) by Lemma 8, hence \( V = D^\delta_t (g(t)) \in K'_M \). In fact \( V = D^\delta_t (g(t)) \in K'_3 < K'_p \), \( p > 1 \).

**Proof of (III).** Let \( C'_* \) be an arbitrary but fixed compact subcone of \( C_* \). By Lemma 4, for any \( \eta \in (0, 1) \), there exists a compact subcone \( C' = C'(C'_*, \eta) \) of \( C \setminus O(C) \), depending on \( C'_* \) and on \( \eta \), such that we can find a point \( y^0_t \in Pr(C') \) where

\[ -\langle t, y^0_t \rangle \geq (1 - \eta)u_C(t) \]

for any \( t \in C'_* \). Put

\[(31) \quad y_t = \frac{1}{A} y^0_t \left| M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right) \right|.\]

Since \( C' \) is a cone and \( y^0_t \in Pr(C') \), \( y_t \in C' \subseteq C \) for any \( t \in C'_* \). Choose a real number \( R > 0 \) such that

\[(32) \quad R > A(\Omega^{-1}(M(Am))) \]

where \( m = m(C') \) is as in the definition of \( G_M(A; C) \) corresponding to \( f(z) \) and \( \xi = \xi(C'_*) \) is as in Lemma 2. Then for \( t \in C'_* \) with \( |t| > R > 0 \), we have from Lemma 2, (31), and (32) that

\[(33) \quad |y_t| = \frac{1}{A} \left| M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right) \right| \geq \frac{1}{A} \left| M^{-1} \left( \Omega \left( \frac{\xi t}{A} \right) \right) \right| \geq m.\]

Hence if \( t \in C'_* \) with \( |t| > R > 0 \), \( y_t \in (C' \setminus (C' \cap N(0, m))) \). We have from (II) that for \( t \in C'_* \) with \( |t| > R \)

\[(34) \quad |g(t)| \leq K(C', m) \exp[2\pi (M(Ay_t) + \langle y_t, t \rangle)].\]

By Lemma 4 and (31), we have for all \( t \in C'_* \) that

\[(35) \quad \langle y_t, t \rangle = \frac{1}{A} \left| M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right) \left( y^0_t, t \right) \right| \leq -(1 - \eta) \frac{1}{A} u_C(t) \left| M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right) \right|.\]
Since \(|y_0^t| = 1\), we have from (31) that
\[
M(Ay_t) = \Omega \left( \frac{u_C(t)}{A} \right).
\]

Applying (35) and (36) to (29), we have for all \(t \in C_\ast\) with \(|t| > R\) that
\[
|g(t)| \leq K(C', m) \exp \left[ 2\pi \Omega \left( \frac{u_C(t)}{A} \right) - 2\pi (1 - \eta) \frac{1}{A} u_C(t) M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right) \right].
\]

Using the Young’s inequality in Lemma 6, \(u_C(t) M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right)\)
\[
= A \frac{u_C(t)}{A} M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right)
\leq A \left( M \left( M^{-1} \left( \Omega \left( \frac{u_C(t)}{A} \right) \right) \right) + \Omega \left( \frac{u_C(t)}{A} \right) \right)
= 2A \Omega \left( \frac{u_C(t)}{A} \right).
\]

Applying (38) to (37), we have for all \(t \in C_\ast\) with \(|t| > R\) that
\[
|g(t)| \leq K(C', m) \exp \left[ -2\pi (1 - 2\eta) \Omega \left( \frac{u_C(t)}{A} \right) \right].
\]

We find the estimation like (39) for \(t \in C_\ast\) with \(|t| \leq R\) for a fixed \(R > 0\) of (32). Put
\[
y_0^t = Qy_0^t
\]
for a \(y_0^t \in Pr(C')\) corresponding to \(t \in C_\ast \subset C_\ast\) and a fixed \(Q > m > 0\).

Then since \(y_0^t \in (C'(C' \cap N(0, m)))\) and the estimation (29) of (II) holds for \(t \in C_\ast \subset C_\ast\) independently of \(y \in (C'(C' \cap N(0, m)))\), we have from (29), Lemma 4, and the fact that \(|y_0^t| = 1\) that for \(t \in C_\ast\),
\[
|g(t)| \leq K(C', m) \exp[2\pi M(AQ)] \cdot \exp[-2\pi Q(1 - \eta)u_C(t)].
\]

Since \(\eta \in (0, 1)\) and \(u_C(t) > 0\) for \(t \in C_\ast \subset C_\ast\), we have that
\[
\exp[-2\pi Q(1 - \eta)u_C(t)] \leq 1, \ t \in C_\ast \subset C_\ast.
\]

From (41) and Lemma 2, if we take \(\eta \in (0, 1)\) with \(1 - 2\eta > 0\), we have for \(t \in C_\ast\) with \(|t| \leq R\) that
\[
g(t) \leq K(C', m) \exp[2\pi M(AQ)] \cdot \exp[-2\pi Q(1 - \eta)u_C(t)]
\leq K(C', m) \exp[2\pi M(AQ)]
= K(C', m) \exp[2\pi M(AQ)]
\cdot \exp \left[ 2\pi (1 - 2\eta) \Omega \left( \frac{u_C(t)}{A} \right) \right] \cdot \exp \left[ -2\pi (1 - 2\eta) \Omega \left( \frac{u_C(t)}{A} \right) \right]
\leq K(C', m) \exp[2\pi M(AQ)]
compact subcone of \( C \) be taken in the proof of (III).

Here we have used the same techniques as in (18) in the last two steps in (45), \( y_1 \leq \frac{m}{\pi} \) for all \( y \in C \). Then we have from (29) that

\[
M(45)
\]

\( C \) depends on \( \eta \) where the constant \( \kappa(C', m) \) depends on \( m \).

\[
\text{Put } \eta = \frac{T_{\rho C}}{\xi} \omega^{-1} \left( \frac{T_{\rho C}}{1 - 3\eta} \right)
\]

\[
= K(C', m)C_{A,Q}(\eta) \exp \left[ -2\pi(1 - 2\eta)\Omega \left( \frac{u_{C}(t)}{A} \right) \right],
\]

where the constant \( K(C', m) \) depends on \( C' \) and on \( m \), and the constant \( C_{A,Q}(\eta) \) depends on \( \eta \) for two fixed constants \( A \) and \( Q \). Since \( C' = C'(C', \eta) \) depends on \( C' \) and on \( \eta \) and \( m = m(C') \) depends on \( C' \), \( K(C', m)C_{A,Q}(\eta) \) depends on \( C' \) and on \( \eta \) for two fixed constants \( A \) and \( Q \). Thus we can find a constant \( M(C', \eta) \), depending on \( C' \) and on \( \eta \), such that if \( t \) is an element of \( C' \), then (21) holds for any \( \eta \in (0, 1) \) with \( 1 - 2\eta > 0 \). This completes the proof of (III).

**Proof of (IV).** Firstly, in order to show that the Fourier transform in (19) can be taken in the \( L_1 \) sense, we will show that \( (e^{-2\pi(y,t)}g(t)) \in L^p \), \( 1 \leq p < \infty \), for \( A > 0 \) and \( y \in (C' \cap N(0, m)) \). Let \( A > 0 \) and let \( p \) be arbitrary with \( 1 \leq p < \infty \). If we let \( y \) be arbitrary but fixed in \( (C' \cap N(0, m)) \), then \( y \in C' \) with \( |y| > m > 0 \). Also if we choose a positive real number \( \zeta \) such that \( 0 < m/|y| < \zeta < 1 \), then \( \zeta y \in C' \) and \( |\zeta y| > m \), hence \( \zeta y \in (C' \cap N(0, m)) \).

Then we have from (29) that

\[
|e^{-2\pi(y,t)}g(t)| \leq K(C', m) e^{-2\pi p(y,t)} \exp[2\pi p(M(A\zeta y) + (\zeta, y, t))] \leq K(C', m) \exp[2\pi p(M(A\zeta y)) \cdot \exp[-2\pi p(1 - \zeta)(y, t)]
\]

for all \( t \in \mathbb{R}^n \). We have from the fact that \( (1 - \zeta) > 0 \), Lemma 3, (40), and [7, p. 39, Theorem 3.2] that

\[
\int_{C'}|e^{-2\pi(y,t)}g(t)|^p dt
\]

\[
\leq K^p(C', m) \exp[2\pi p M(A\zeta y)] \int_{C'} \exp[-2\pi p \delta(1 - \zeta)|y||t||] dt
\]

\[
\leq K^p(C', m) Z_n \exp[2\pi p M(A\zeta y)] \int_0^{\infty} s^{n-1} \exp[-2\pi p \delta(1 - \zeta)|y||s|] ds
\]

\[
= K^p(C', m) Z_n \exp[2\pi p M(A\zeta y)] (n - 1)!(2\pi p \delta(1 - \zeta)|y|)^{-n}.
\]

Here we have used the same techniques as in (18) in the last two steps in (45), where \( Z_n \) is the area of the unit sphere in \( \mathbb{R}^n \).

Put \( C' = \mathbb{R}^n \setminus (C') \). Since \( C' \subset (C') \) and \( (C') \) is an open cone, \( C' \) is a compact subcone of \( C \), and (III) holds for \( C' \). Then we have from (14) that

\[
\int_{C'} |e^{-2\pi(y,t)}g(t)|^p dt
\]

\[
\leq K'(C', A, \eta) \exp \left[ 2\pi p T_{\rho C} \cdot \frac{A}{\xi} \cdot \omega^{-1} \left( \frac{T_{\rho C}}{1 - 3\eta} \right) \right] - 2\pi p (1 - \eta) \Omega \left( \omega^{-1} \left( \frac{T_{\rho C}}{1 - 3\eta} \right) \right).
\]

\[
\cdot \exp \left[ 2\pi(1 - 2\eta)\Omega \left( \frac{R}{A} \right) \right] \cdot \exp \left[ -2\pi(1 - 2\eta)\Omega \left( \frac{u_{C}(t)}{A} \right) \right],
\]
where \( K'(C'_r, A, \eta) \) is a constant depending on \( C'_r \), on a fixed \( \eta \in (0, 1) \), and on a fixed \( A > 0 \). Here \( \xi = \xi(C'_r) \) is the number in Lemma 2.

The open cone \((C^*)'\) in (45) is fixed depending on the compact subcone \( C' \subset \mathcal{O}(C) \). Then the compact subcone \( C'_* \subset C'_* \) defined in (46) was by \( C'_* = \mathbb{R}^n \setminus (C^*)_r \). Since \((C^*)' \cup C'_* = \mathbb{R}^n\) and \((C^*)' \cap C'_* = \emptyset\), we have from (45) and (46) that \((e^{-2\pi y, t}g(t)) \in L^p, 1 \leq p < \infty\). for \( A > 0 \) and \( y \in (C^*' \setminus (C' \cap N(0, m)))\).

Now if \( A = 0 \), then \( g(t) \) satisfies (29) and \( \text{supp } (g) \subseteq C^* \). The open cone \((C^*)'\) for which Lemma 3 holds contains \( C^* \), hence we have from (45) that \((e^{-2\pi y, t}g(t)) \in L^p, 1 \leq p < \infty\) for \( y \in (C^*' \cap N(0, m)))\).

Thus for either of the cases \( A > 0 \) or \( A = 0 \), the Fourier transform in (15) can be taken in the \( L_1 \) sense.

Secondly, we show that \( \langle V, e^{2\pi i (z, t)} \rangle \) is well defined on \( T(C'; m) \). We consider

\[
(47) \int_{\mathbb{R}^n} g(t)e^{2\pi i (z, t)} dt, \quad z \in T(C'; m).
\]

Since \((C^*)' \cap C'_* = \emptyset \) and \((C^*)' \cup C'_* = \mathbb{R}^n\), (47) can be rewrite as

\[
(48) \int_{\mathbb{R}^n} g(t)e^{2\pi i (z, t)} dt = \int_{C'_*} g(t)e^{2\pi i (z, t)} dt + \int_{(C^*)' \cap C'_*} g(t)e^{2\pi i (z, t)} dt
\]

\[
= h_{C'_*}^0 + h_{(C^*)'}^0.
\]

Here \( h_{C'_*}^0 \) and \( h_{(C^*)'}^0 \) are the functions corresponding to \( (\gamma, g) = (0, g) \) in Lemma 12 and Lemma 13, respectively. Since \( T(C'; m) \subset T' \), \( h_{C'_*}^0 \) converges absolutely and uniformly on \( T' \) by Lemma 12 and \( h_{(C^*)'}^0 \) converges absolutely and uniformly on \( T(C'; m) \) by Lemma 13, \( \langle V, e^{2\pi i (z, t)} \rangle \) is well-defined on \( T(C'; m) \).

Hence since the Fourier transform in (19) can be taken in the \( L_1 \) sense and \( \langle V, e^{2\pi i (z, t)} \rangle \) is well-defined on \( T(C'; m) \), if we use differentiation in the distributional sense, then we have that

\[
(49) \langle V, e^{2\pi i (z, t)} \rangle = (-1)^{|\alpha|} \langle g(t), D'_\alpha (e^{2\pi i (z, t)}) \rangle
\]

\[
= z^\alpha \int_{\mathbb{R}^n} g(t)e^{2\pi i (z, t)} dt = z^\alpha \mathcal{F}[e^{-2\pi y, t}g(t); x]
\]

for \( z \in T(C'; m) \) and the Fourier transform is taken in either the \( L^1 \) or \( L^2 \) sense. From (19) and (49) we have (22). This completes the proof of (IV).

**Proof of (V).** The proof of (V) follows from only replacing \( K_r, K_r', K'_r \), and \( K'_r \) in proving (7.3) of [3, pp. 1056–1057] by \( K_M, K_M, K'_M, \) and \( K'_M \), respectively.

**Proof of (VI).** Firstly, we show that \( \{e^{-2\pi y, t}V_m : y \in (C' \setminus N(0, m))\}, |y| \leq Q_m, Q_m > M > 0\), is strongly bounded set in \( K'_M \). Let \( \Phi \) be an arbitrary bounded set in \( K'_M \) and let \( \phi \in \Phi \). Since \( (e^{-2\pi y, t}V_m) \in K'_M \) for any \( y \in \mathbb{R}^n \), we have from Lemma 8 and general Leibnitz rule that for some \( n \)-tuple
\( \alpha \) of nonnegative integers, some integer \( k \geq 0 \), and some continuous function \( f \) on \( \mathbb{R}^n \) bounded by \( M > 0 \),

\[
\begin{align*}
(50) \quad \langle e^{-2\pi(y,t)}V_t, \phi(t) \rangle \\
&= \langle D_t^\alpha (\exp[M(kt)f(t)]) , e^{-2\pi(y,t)}\phi(t) \rangle \\
&= (-1)^{\|\alpha\|} \int_{\mathbb{R}^n} e^{M(kt)} f(t) \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left( \frac{1}{i} \right)^{\|\beta\|} y^\beta e^{-2\pi(y,t)} D_t^\gamma (\phi(t)) dt \\
&= (-1)^{\|\alpha\|} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \left( \frac{1}{i} \right)^{\|\beta\|} y^\beta I_y(\gamma),
\end{align*}
\]

where

\[
(51) \quad I_y(\gamma) = \int_{\mathbb{R}^n} e^{M(kt)} f(t) e^{-2\pi(y,t)} D_t^\gamma (\phi(t)) dt.
\]

Let \( Q_m > 0 \) be an arbitrary but fixed real number. For \( y \in \mathbb{R}^n \) with \( |y| \leq Q_m \), if we choose \( r \geq \max\{|\alpha|, 2k + 2\pi Q_m\} \), we have from Lemma 5 and the fact that \( \phi \in \mathcal{K}_M \) that

\[
(52) \quad |I_y(\gamma)| \leq M \int_{\mathbb{R}^n} e^{M(kt)} e^{2\pi|y||t|} |D_t^\gamma (\phi(t))| dt \\
\leq M \int_{\mathbb{R}^n} e^{M(kt)} e^{2\pi Q_m|t|} |D_t^\gamma (\phi(t))| dt \\
\leq M \int_{\mathbb{R}^n} e^{M(2k+2\pi Q_m|t|)} e^{M(2\pi Q_m t)} |D_t^\gamma (\phi(t))| dt \\
\leq M \int_{\mathbb{R}^n} e^{M(2k+2\pi Q_m t)} |D_t^\gamma (\phi(t))| e^{-M(kt)} dt \\
\leq M \|\phi\|_{\mathcal{K}_M} \int_{\mathbb{R}^n} e^{-M(kt)} dt,
\]

where \( M \) is such that \( \sup_{t \in \mathbb{R}} |f(t)| \leq M \). Since \( \Phi \) is a bounded set in \( \mathcal{K}_M \), there exist a constant \( W_\gamma \), depending only \( \gamma \), such that \( \|\phi\|_{\mathcal{K}_M} \leq W_\gamma \) for all \( \phi \in \Phi \). Hence for each \( \gamma \) with \( \beta + \gamma = \alpha \),

\[
(53) \quad |I_y(\gamma)| \leq MW_\gamma \int_{\mathbb{R}^n} e^{-M(kt)} dt = W_\gamma'
\]

for all \( \phi \in \Phi \). Thus we have from (50) and (53) that

\[
(54) \quad |\langle e^{-2\pi(y,t)}V_t, \phi(t) \rangle| \leq W_\gamma' \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} Q_m^{\|\beta\|}, \quad \phi \in \Phi.
\]

Here the bound in (54) is independent of \( \phi \in \Phi \). Hence \( \{e^{-2\pi(y,t)}V_t : y \in (C'\setminus(C' \cap N(0,m))) \}, |y| \leq Q_m \}, Q_m > M > 0 \), is a bounded set in complex plane. Since \( \Phi \) be an arbitrary bounded set in \( \mathcal{K}_M \), \( \{e^{-2\pi(y,t)}V_t : y \in (C'\setminus(C' \cap \]
\[ N(0, m)), |y| \leq Q_m, \quad Q_m > M > 0, \] is strongly bounded set in \( K_M'. \) Thus we have from Lemma 10 and (23) that
\[
\{ f(z) : y = \text{Im}(z) \in (C' \cap N(0, m)), |y| \leq Q_m \} = \{ e^{-2\pi y^t V} : y = \text{Im}(z) \in (C' \cap N(0, m)), |y| \leq Q_m \}
\] is strongly bounded set in \( K_M'. \) This completes the proof of (VI).

We consider the converse of Theorem 1. We note that the inequality (20) in Theorem 1 can be rewrite as

\[ |g(t)| \leq K g e^{k |t|}, \quad t \in \mathbb{R}^n, \] for some two positive constants \( K_g \) and \( k_g \) both of which are depend on \( g. \) We will use the inequality (55) instead of the inequality (20) in the next theorem.

**Theorem 2.** Let \( C' \) be an open connected cone in \( \mathbb{R}^n \) and let \( C' \) be an arbitrary compact subcone of \( C_* = \mathbb{R}^n \setminus C_* \). Let \( A > 0 \) be such that \( A/\xi \leq 1 \), where \( \xi = \xi(C' \setminus C_* \cap N(0, m)) \) is a constant, depending on \( C' \) as in Lemma 2. Let \( V \) be a finite sum

\[ V = \sum_{\alpha} D_{\alpha}^v (g_{\alpha}(t)), \] where each \( g_{\alpha} \) are continuous function of \( t \in \mathbb{R}^n \). Assume that for each \( n \)-tuple of nonnegative integers \( \alpha \), \( g_{\alpha} \) satisfies

\[ |g_{\alpha}(t)| \leq K_{\alpha} e^{k_{\alpha} |t|}, \quad t \in \mathbb{R}^n, \] where some two positive constants \( K_{\alpha} \) and \( k_{\alpha} \) both of which are depend on \( g_{\alpha} \). Also assume that each \( g_{\alpha} \) satisfies

\[ |g_{\alpha}(t)| \leq M(C', \eta) \exp \left[ -2\pi(1 - 2\eta) \frac{u_{C'}(t)}{A} \right], \quad t \in C' \subset C_*, \] for any \( \eta \in (0, 1) \) with \( 1 - 3\eta > 0 \), where \( M(C', \eta) \) is a constant depending on \( C' \) and on \( \eta. \) Then \( V \in K_1' \subset K_M'. \) Furthermore the function

\[ f(z) = \langle V, e^{2\pi i z^t} \rangle \] and any derivative of \( f(z) \) belong to \( G_M \left( \frac{\alpha}{1 - 3\eta}; O(C) \right) \).

**Proof.** Since \( g_{\alpha}(t) \) is continuous and \( g_{\alpha}(t) \in K'_1 \subset K'_M, V \in K'_1 \subset K'_M \). Using the differentiation in the distribution sense, we write \( f(z) \) as

\[ f(z) = \langle V, e^{2\pi i z^t} \rangle = \sum_{\alpha} z^\alpha \int_{\mathbb{R}^n} g_{\alpha}(t) e^{2\pi i z^t} dt. \] To show the existence and analyticity of \( f(z) \) for a certain \( z \), consider

\[ h_{\alpha}(z) = \int_{\mathbb{R}^n} g_{\alpha}(t) e^{2\pi i z^t} dt. \]
Since $(C')' \cap C' = \emptyset$ and $(C')' \cup C' = \mathbb{R}^n$, (60) can be rewritten by

$$h_\alpha(z) = \int_{\mathbb{R}^n} g_\alpha(t) e^{2\pi i (z \cdot t)} dt$$

$$= \int_{C'} g_\alpha(t) e^{2\pi i (z \cdot t)} dt + \int_{(C')'} g_\alpha(t) e^{2\pi i (z \cdot t)} dt$$

$$= h^0,\alpha_{C'}_z + h^0,\alpha_{(C')'}_z,$$

where $h^0,\alpha_{C'}_z$ and $h^0,\alpha_{(C')'}_z$ are the functions corresponding to $(\gamma, g) = (0, g_\alpha)$ in Lemma 12 and Lemma 13, respectively.

Also we have from (59), the generalized Leibnitz rule, and the fact that $T(C'; m) \subset T'$ that

$$D_2^2(f(z)) = \sum_{\alpha} \sum_{\beta + \mu = \gamma} \frac{\gamma!}{\beta!\mu!} D_2^2(z^\alpha) \left[ D_2^2(h^0,\alpha_{C'}_z) + D_2^2(h^0,\alpha_{(C')'}_z) \right]$$

$$= \sum_{\alpha} \sum_{\beta + \mu = \gamma} \frac{\gamma!}{\beta!\mu!} D_2^2(z^\alpha) (-1)^{|\mu|} \left[ h^{\gamma,\alpha}_{C'}_z + h^{\gamma,\alpha}_{(C')'}_z \right], \quad z \in T(C'; m),$$

where $\gamma, \beta$ and $\mu$ are $n$-tuples of nonnegative integers. Here $h^{\gamma,\alpha}_{C'}_z$ and $h^{\gamma,\alpha}_{(C')'}_z$ are the functions corresponding to $(\gamma, g) = (0, g_\alpha)$ in Lemma 12 and Lemma 13, respectively.

Let $C'$ be an arbitrary compact subcone of $O(C)$. Choose $m_\alpha = m_\alpha(C')$, depending on $\alpha$ and on $C'$, such that

$$m_\alpha = (k_\alpha/(2\pi \delta)) + 1,$$

where $k_\alpha$ is as in (57) and $\delta$ is as in Lemma 3. For $m_\alpha > 0$ in (63), let $z_0$ be an arbitrary but fixed point in $T(C'; m_\alpha) = \mathbb{R}^n + i(C' \setminus N(0, m_\alpha))$. If we choose an open neighborhood $N'(z_0, r)$ of $z_0$ with radius $r > 0$ whose closure is contained in $T(C'; m_\alpha) \subset T'$, $h^{\gamma,\alpha}_{C'}_z$ and $h^{\gamma,\alpha}_{(C')'}_z$ converge absolutely and uniformly for $z \in N'(z_0, r)$ from Lemma 12 and Lemma 13, respectively. Since $z$ is an arbitrary point in $T(C'; m_\alpha)$, we have from (62) that $f(z)$ and its derivative is analytic in $T(C'; m_\alpha)$.

We put

$$m = \max_{\alpha} \{ m_\alpha \},$$

where $m_\alpha$ is as in (63) for each $\alpha$. Since $T(C'; m) \subset T(C'; m_\alpha)$, $h_\alpha(z)$ in (61) is analytic in $T(C'; m)$ for each $\alpha$, hence $\sum_{\alpha} z^\alpha h_\alpha(z)$ is also analytic in $T(C'; m)$. Thus $f(z) = \langle V, e^{2\pi i (z \cdot t)} \rangle$ and any derivative of $f(z)$ are analytic in $T(C'; m), C' \subset O(C)$, for the fixed $m$ in (64).

Now we will obtain a growth of $f(z)$ and any derivative of $f(z)$ like the inequality in the definition of $G_{\mu}(A, C)$ for any compact subcone $C' \subset O(C)$ and the corresponding $m > 0$ taken in (64).
In order to estimate integral representations of \( f(z) \) and any derivative of \( f(z) \) of the form (59) on \( C_* \), we will continue the estimation of inequalities in (14) under the additional condition of \( A \) and \( \xi \) in this theorem.

Let \( A > 0 \) be such that \( A/\xi \leq 1 \), where \( \xi = \xi(C'_*) \) is a constant, depending on \( C'_* \), as in Lemma 2 and let \( z \in N'(z_0, r) \subset T^{C'_*} \), where \( N'(z_0, r) \) is an open neighborhood of \( z_0 \) with radius \( r > 0 \) whose closure is in \( T^{C'_*} \).

If we replace \( T \) by \( |y| \) in (11), (12), and (13), we have from (14) that

\[
|h_{C'_*}^{\gamma, \alpha}(z)| = \left| \int_{C'_*} t' y g(t) e^{2\pi i (z, t)} \, dt \right|
\]

\[
\leq K'(C'_*, A, \eta) \exp \left[ 2\pi |y| \rho C \frac{A}{\xi} \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) \right. \\
- \left. 2\pi (1 - 3\eta) \Omega \left( \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) \right) \right]
\]

for \( n \) tuples \( \gamma \) and \( \alpha \) of nonnegative integers and all \( z \in N'(z_0, r) \), where \( K'(C'_*, A, \eta) \) is a constant depending on fixed \( C'_* \), on fixed \( A > 0 \), and on fixed \( \eta \in (0, 1) \) with \( 1 - 3\eta > 0 \). By Lemma 6,

\[
\frac{|y| \rho C}{1 - 3\eta} \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) = M \left( \frac{|y| \rho C}{1 - 3\eta} \right) + \Omega \left( \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) \right).
\]

Since \( A/\xi \leq 1 \) and \( 0 < 1 - 3\eta < 1 \), we have from (66) that

\[
|y| \rho C \frac{A}{\xi} \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) - (1 - 3\eta) \Omega \left( \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) \right)
\]

\[
\leq |y| \rho C \cdot \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) - (1 - 3\eta) \Omega \left( \omega^{-1} \left( \frac{|y| \rho C}{1 - 3\eta} \right) \right)
\]

\[
= (1 - 3\eta) M \left( \frac{|y| \rho C}{1 - 3\eta} \right) \leq M \left( \frac{|y| \rho C}{1 - 3\eta} \right).
\]

Applying (67) to (65), since \( z \) is an arbitrary point in \( T^{C'_*} \), we have that for \( n \) tuples \( \gamma \) and \( \alpha \) of nonnegative integers

\[
|h_{C'_*}^{\gamma, \alpha}(z)| \leq K'(C'_*, A, \eta) \exp \left[ M \left( \frac{|y| \rho C}{1 - 3\eta} \right) \right], \quad z \in T^{C'_*},
\]

where \( K'(C'_*, A, \eta) \) is a constant depending on fixed \( C'_* \), on fixed \( A > 0 \) with \( A/\xi \leq 1 \), and on fixed \( \eta \in (0, 1) \) with \( 1 - 3\eta > 0 \).

We now consider the integral representations of \( f(z) \) and any derivative of \( f(z) \) in (59) on \( (C'_*)' \). Let \( m_\alpha \) and \( m \) be as in (63) and (64), respectively. Since \( m \geq m_\alpha \) for each \( m_\alpha \), \( k_\alpha - 2\pi \delta |y| < -2\pi \delta < 0 \) when \( |y| > m \) and \( y \in C' \). Then we have from Lemma 13 that for \( n \) tuples \( \gamma \) and \( \alpha \) of nonnegative integers,

\[
|h_{(C'_*)'}^{\gamma, \alpha}(z)| = \left| \int_{(C'_*)'} \gamma|y g(t) e^{2\pi i (z,t)} \, dt \right|
\]
which does not depend on $m$ and $\alpha$.

Theorem 3. Let $m(C')$ be a fixed real number which depends on $C'$ as in the

\[ C = C(C', A, \eta) = \text{constant depending on } m \text{ chosen in (64), on } C', \text{ on fixed } A > 0, \text{ with } A/\xi \leq 1, \text{ and on fixed } \eta \in (0, 1) \text{ with } 1 - 3\eta > 0. \]

We can extend the results that are described from last paragraph of [3, p. 1060] to Corollary 7.1 of [3, p. 1061] to the results in the context of spaces $\mathcal{K}_M$ or spaces $G_M(A; C)$ by the exactly same line there as follow:

(i) Under the hypothesis of Theorem 2, (V) and (VI) are hold for $z \in T(C'; m)$, $C' \subset \mathcal{O}(C)$. Let $A > 0$ be such that $A/\xi \leq 1$, where $\xi = \xi(C'_*)$ is a constant, depending on $C'_*$, as in Lemma 2. If $f(z) \in G_M(A; C)$, then $f(z)$ and any derivative of $f(z)$ can be extended to an element of $G_M\left(\frac{A/\xi}{1-3\eta}; \mathcal{O}(C)\right)$ for a constant $\eta \in (0, 1)$ with $1 - 3\eta > 0$.

6. The relationship between $F_M(A; C)$, $\mathcal{K}_M'$, and $\mathcal{K}_M$ and
distributional boundary values of the spaces $F_M(A; C)$

In this section, we only state without proof the relationship between $F_M(A; C)$, $\mathcal{K}_M'$, and $\mathcal{K}_M'$ since the ideas, methods, and any others needed to obtain the relationship between $F_M(A; C)$, $\mathcal{K}_M'$, and $\mathcal{K}_M'$ are the same as that of obtaining the relationship between $G_M(A; C)$, $\mathcal{K}_M'$ and $\mathcal{K}_M'$ in the previous section.

Exceptionally, we show that the elements of the spaces $F_M(A; C)$ can obtain distributional boundary values in $K_M'$.

Theorem 3. Let $M(x)$ and $\Omega(y)$ be the functions as in Definition 7. For
the open connected cone $C$, let $f(z) \in F_M(A; C)$. For any compact subcone $C' \subset C$, let $m = m(C')$ be a fixed real number which depends on $C'$ as in the
For any \( x \) the Fourier transform is taken in the \( L^2 \) sense.

(II) For \( A \geq 0 \), \( g(t) \) satisfies

\[
|g(t)| \leq K(C', m) \exp[2\pi(M(Ay) + |y||t|)], \quad t \in \mathbb{R}^n,
\]

where \( C' \subset C \) is arbitrary and \( K(C', m) \) depends on \( C' \) and on \( m \). Inequality (71) is independent of \( y \in (C' \cap (C' \cap N(0, m})) \) and \( \text{supp}(g) \) = supp \((V) \subseteq \{ t : u_C(t) \leq A \} \).

(III) For \( A > 0 \) and any compact subcone of \( C_0 \subset C \), \( g(t) \) satisfies

\[
|g(t)| \leq M(C', \eta) \exp \left[ -2\pi(1 - 2\eta)\Omega\left(\sum_{k \in \Lambda} c_k(t)\right) \right], \quad t \in C',
\]

for any \( \eta \in (0, 1) \) with \( 1 - 2\eta > 0 \), where \( M(C', \eta) \) is a constant depending on \( C' \) and on \( \eta \).

(IV) For \( A \geq 0 \), if \( g(t) \) satisfies that \( |g(t)| \leq K e^{k|t|} \), \( t \in (C^\ast)' \), for some constant \( K \) and \( k > 0 \), then

\[
f(z) = \langle V, e^{2\pi i(x,t)} \rangle, \quad z = x + iy \in C'.
\]

(V) For \( A \geq 0 \),

\[
f(z) = \mathcal{F}[e^{-2\pi i(y,t)}V], \quad z = x + iy \in C'.
\]

where the equality in (72) holds in \( K_M' \).

(VI)

\[
\{ f(z) : y = \text{Im}(z) \in (C' \cap (C' \cap N(0, m))), |y| \leq Q_m \}
\]

is strongly bounded in \( K_M' \), where \( Q_m > m > 0 \).

(VII) \( f(z) \rightarrow \mathcal{F}[V] \in \mathbf{K}_M \) in the strong and weak topology of \( \mathbf{K}_M' \) as \( y = \text{Im}(z) \rightarrow 0 \), \( y \in C' \subset C \), where this boundary value is obtained independently of how \( y \rightarrow 0 \) in \( C' \subset C \).

Proof. It suffices to prove only (VII). Since \( V \in \mathbf{K}_M' \), if we replace \( \mathbf{K}_M' \) and \( e^{k|t|} \) in the proof of Lemma 5.9 in [3, pp. 1052–1053] by \( \mathbf{K}_M' \) and \( e^{M(|t|)} \), respectively, we have that

\[
\lim_{y \rightarrow 0} e^{-2\pi i(y,t)}V = V, \quad y \in \mathbb{R}^n,
\]

in the weak topology of \( \mathbf{K}_M' \). Since \( \mathbf{K}_M \) is a Montel space by Lemma 7, we also have the convergence (73) in the strong topology of \( \mathbf{K}_M' \). Since the Fourier transform is a topological isomorphism of \( \mathbf{K}_M' \) onto \( \mathbf{K}_M' \) by Lemma 10, \( f(z) \rightarrow \mathcal{F}[V] \in \mathbf{K}_M \) in the strong and weak topology of \( \mathbf{K}_M' \) as \( y = \text{Im}(z) \rightarrow 0 \), \( y \in C' \subset C \). Since \( V \) is independent of how \( y \rightarrow 0 \) in \( C' \subset C \), the boundary value
$F[V]$ is obtained independently of how $y \to 0$ in $C' \subset C$. This completes the proof of (VII).

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