ON THE $m$-POTENT RANKS OF CERTAIN SEMIGROUPS OF ORIENTATION PRESERVING TRANSFORMATIONS

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ABSTRACT. It is known that the ranks of the semigroups $SOP_n, SPOP_n$ and $SSPOP_n$ (the semigroups of orientation preserving singular self-maps, partial and strictly partial transformations on $X_n = \{1, 2, \ldots, n\}$, respectively) are $n$, $2n$ and $n + 1$, respectively. The idempotent rank, defined as the smallest number of idempotent generating set, of $SOP_n$ and $SSPOP_n$ are the same value as the rank, respectively. Idempotent can be seen as a special case (with $m = 1$) of $m$-potent. In this paper, we investigate the $m$-potent ranks, defined as the smallest number of $m$-potent generating set, of the semigroups $SOP_n, SPOP_n$ and $SSPOP_n$.

Firstly, we characterize the structure of the minimal generating sets of $SOP_n$. As applications, we obtain that the number of distinct minimal generating sets is $(n - 1)^n!$. Secondly, we show that, for $1 \leq m \leq n - 1$, the $m$-potent ranks of the semigroups $SOP_n$ and $SPOP_n$ are also $n$ and $2n$, respectively. Finally, we find that the 2-potent rank of $SSPOP_n$ is $n + 1$.

1. Introduction and preliminaries

As usual we denote by $\mathcal{PT}_n$ the monoid of all partial transformations of a finite set $X_n$ with $n$ elements (under composition), by $\mathcal{T}_n$ the submonoid of $\mathcal{PT}_n$ of all full transformations of $X_n$ and by $\mathcal{S}_n$ the symmetric group on $X_n$, i.e., the subgroup of $\mathcal{PT}_n$ of all injective full transformations (permutations) of $X_n$. Denote by $\mathcal{SPT}_n$ the semigroup of $\mathcal{PT}_n \setminus \mathcal{S}_n$ of all singular partial transformations and by $\text{Sing}_n$ the semigroup $\mathcal{T}_n \setminus \mathcal{S}_n$ of all singular transformations of $X_n$.

Let now $X_n$ be a chain with $n$ elements, say $X_n = \{1 < 2 < \cdots < n\}$. Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of $t$ ($t \geq 0$) elements from the chain $X_n$. We say that $c$ is cyclic if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$, where $c_{t+1}$ denotes $c_1$. Let $\alpha \in \mathcal{PT}_n$ and suppose that

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The semigroup \( S_{POP} \) consists of the empty mapping and \(< 1\) orientation-preserving if the sequence of its image \((a_1a, \ldots, a_t)\) is cyclic. Denote by \( SPOP_n \) the subsemigroup of \( SPT_n \) of all partial orientation preserving singular transformations, by \( SOP_n \) the subsemigroup \( SPOP_n \cap \text{Sing}_n \) of all orientation preserving singular transformations and by \( SSSPOP_n \) the subsemigroup \( SPOP_n \backslash SOP_n \) of all orientation preserving strictly partial transformations.

Remark 1. In this paper it will always be clear from context when additions are modular.

An element \( a \) of any given semigroup is called \( m\)-potent if \( a^{m+1} = a^m \) and \( a, a^2, \ldots, a^m \) are distinct. In particular, we refer to idempotent as 1-potent. This concept was mentioned in [1, 6, 7].

As usual, the rank of a finite semigroup \( S \) is defined by rank \( S = \min \{|A| : A \subseteq S, \langle A \rangle = S \} \). If \( S \) is generated by its set \( E \) of idempotents, then the idempotent rank of \( S \) is defined by idrank \( S = \min \{|A| : A \subseteq E, \langle A \rangle = S \} \). If \( S \) is generated by its set \( E_m \) of \( m \)-potents, then the \( m \)-potent rank of \( S \) is defined by rank\( m \) \( S = \min \{|A| : A \subseteq E_m, \langle A \rangle = S \} \).

It is known that the ranks of the semigroups \( SOP_n \), \( SPOP_n \) and \( SSSPOP_n \) are \( n, 2n \) and \( n + 1 \), respectively. The idempotent rank of \( SOP_n \) and \( SPOP_n \) are the same value as the rank, respectively (see [8]). In this paper, we investigate the \( m \)-potent ranks of the semigroup \( SOP_n \), \( SPOP_n \) and \( SSSPOP_n \). In Section 2 we characterize the structure of the minimal generating sets of \( SOP_n \). As applications, we prove that the number of distinct minimal generating sets is \((n-1)^{n+1} n!\). Moreover, we show that, for \( 1 \leq m \leq n-1 \), the \( m \)-potent ranks of the semigroups \( SOP_n \) and \( SPOP_n \) are \( n \) and \( 2n \), respectively. In Section 3 we find that the 2-potent rank of \( SSSPOP_n \) is \( n + 1 \).

Let \( \alpha \in PT_n \). As usual, we write \( \text{im}(\alpha) \) and \( \text{rank}(\alpha) \) for the image of \( \alpha \) and the rank of \( \alpha \), respectively. (The rank of a transformation is defined to be the size of its image.) The kernel of \( \alpha \) is the equivalence \( \ker(\alpha) = \{(x, y) \in X_n \times X_n : \alpha x = y\} \). From Fernandes, Gomes, and Jesus [3], Green’s relations on \( SPOP_n \) are characterized by

\[
\alpha \mathcal{L} \beta \quad \text{if and only if} \quad \text{im}(\alpha) = \text{im}(\beta),
\]

\[
\alpha \mathcal{R} \beta \quad \text{if and only if} \quad \ker(\alpha) = \ker(\beta),
\]

\[
\alpha \mathcal{J} \beta \quad \text{if and only if} \quad \text{rank}(\alpha) = \text{rank}(\beta).
\]

The semigroup \( SPOP_n \) has \( n \) \( \mathcal{J} \)-classes, namely \( J_0, J_1, \ldots, J_{n-1} \), where \( J_0 \) consists of the empty mapping and \( J_r = \{ \alpha \in SPOP_n : \text{rank}(\alpha) = r \} \) for \( 1 \leq r \leq n - 1 \). For \( 0 \leq r \leq s \leq n \), let

\[
[s, r] = \{ \alpha \in SPOP_n : |\text{dom}(\alpha)| = s, \text{rank}(\alpha) = r \}.
\]

Then \( J_r = \bigcup^n_{i=r} [i, r] \) and \( J_{n-1} = [n, n-1] \cup [n-1, n-1] \). We draw attention to the top \( \mathcal{J} \)-class \( J_{n-1} \). As in [8] we use the notation

\[
L_k = \{ \alpha \in J_{n-1} : \text{im}(\alpha) = X_n \setminus \{k\} \}.
\]
$R_{(k,k+1)} = \{ \alpha \in [n,n-1] : \text{the unique non-singleton class of } \ker(\alpha) \text{ is } \{k,k+1\} \}$
and
$R_k = \{ \alpha \in [n-1,n-1] : \text{dom}(\alpha) = X_n \setminus \{k\} \}, \; k \in X_n$
for $\mathcal{L}$-classes and $\mathcal{R}$-classes in $J_{n-1}$. Hence $J_{n-1}$ has $n$ $\mathcal{L}$-classes $L_1, L_2, \ldots, L_n$ and $2n$ $\mathcal{R}$-classes $R_{(1,2)}, R_{(2,3)}, \ldots, R_{(n,1)}, R_1, R_2, \ldots, R_n$.

Gomes and Howie [4] used the notation $[i \rightarrow i - 1]$ for the idempotent $e$ defined by $i e = i - 1, x e = x (x \neq i)$ and the notation $[i \rightarrow i + 1]$ for the idempotent $f$ defined by $i f = i + 1, x f = x (x \neq i)$. They also used the notation $\delta_k, k = 1, 2, \ldots, n$, for the identity mapping on $X_n \setminus \{k\}$.

Let $S$ be a subset of $S^{\text{POP}}_n$. As usual, we denote by $E(S)$ the set of all idempotents of $S$. Employing the above notation, the set $E(J_{n-1})$ consists of $n$ decreasing idempotents $[i \rightarrow i - 1] (i \in X_n)$, $n$ increasing idempotents $[i \rightarrow i + 1] (i \in X_n)$ and $n$ idempotents $\delta_k$. Notice that $[1 \rightarrow 0] = [1 \rightarrow n]$, $[n \rightarrow n + 1] = [n \rightarrow 1]$, etc., by Remark 1. Let $E_{n-1}^+ = \{ [i \rightarrow i + 1] : i \in X_n \}$
and $E_{n-1}^- = \{ [i + 1 \rightarrow i] : i \in X_n \}$ be the increasing and decreasing idempotent set of $[n,n-1]$, respectively, and let $F_{n-1} = \{ \delta_1, \ldots, \delta_n \}$. Then $E(J_{n-1}) = E_{n-1}^+ \cup E_{n-1}^- \cup F_{n-1}$.

Given a subset $U$ of a semigroup $S$ and $s \in S$, we denote by $L_s, R_s$ and $H_s$
the $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class of $s$, respectively. For general background on Semigroup Theory, we refer the reader to Howie’s book [5].

2. The $m$-potent ranks of $SOP_n$ and $S^{\text{POP}}_n$

In this section we characterize the structure of the minimal generating sets of $SOP_n$. As applications, we prove that the number of distinct minimal generating sets is $(n-1)^n!$. Moreover, we show that, for $1 \leq m \leq n-1$, the $m$-potent ranks of the semigroups $SOP_n$ and $S^{\text{POP}}_n$ are $n$ and $2n$, respectively.

We begin by recalling that Zhao, Xu and Yang [9, Theorem 2.1] proved:

Lemma 2.1. Let $n \geq 3$. Let $G \subseteq E(SOP_n)$. Then
\[(G) = SOP_n \text{ if and only if } E_{n-1}^+ \subseteq G \text{ or } E_{n-1}^- \subseteq G.\]

For any $i, j \in X_n$, let
$H_{(i,i+1)}^{[j]} = R_{(i,i+1)} \cap L_j$ and $H_{(i)}^{[j]} = R_i \cap L_j$.

Notice that each $\mathcal{H}$-classes contained in $[n,n-1]$ has the form $H_{(i,i+1)}^{[j]}$, for some $i, j \in X_n$, and each $\mathcal{H}$-classes contained in $[n-1,n-1]$ has the form $H_{(i)}^{[j]}$, for some $i, j \in X_n$. With above notation, we have the following simple observation:

Lemma 2.2. Let $n \geq 3$. Then
$H_{(i,i+1)}^{[k]} H_{(j,j+1)}^{[l]} = H_{(i,i+1)}^{[l]}$ if $k = j$ or $k = j + 1$,
$H_{(i)}^{[k]} H_{(j,j+1)}^{[l]} = H_{(i)}^{[l]}$ if $k = j$ or $k = j + 1$,.
Notice that \( SOP_n \cap J_{n-1} = [n, n-1] \) and \([n, n-1]\) contains \( n \) \( \mathcal{R} \)-classes and \( n \) \( \mathcal{L} \)-classes of \( SOP_n \). Thus the number of \( \mathcal{R} \)-classes and \( \mathcal{L} \)-classes of \( SOP_n \) contained in \([n, n-1]\) are both equal to \( n \).

**Lemma 2.3.** Let \( n \geq 3 \), and let \( A \) be a nonempty subset of \( SOP_n \) with \( n \) elements. If \( A \) contains exactly one element from each \( \mathcal{R} \)-class and from each \( \mathcal{L} \)-class of \( SOP_n \) contained in \([n, n-1]\), then \( SOP_n = \langle A \rangle \).

**Proof.** By Lemma 2.1, we have \( SOP_n = \langle E_{n-1}^+ \rangle \). Notice that \( A \subseteq [n, n-1] \subseteq SOP_n \). We shall show that \( E_{n-1}^+ \subseteq \langle A \rangle \) and so \( SOP_n = \langle A \rangle \).

Let \( E_{n-1}^+ (A) = \{ i \to i + 1 \in E_{n-1}^+ : \) there exists \( \alpha \in A \) such that \( \alpha \mathcal{H}[i \to i + 1] \} \) and \( X_n(A) = \{ i \in X_n : i \to i + 1 \notin E_{n-1}^+ (A) \} \).

Let \( [i \to i + 1] \in E_{n-1}^+ \). If \( [i \to i + 1] \in E_{n-1}^+ (A) \), then there exists \( \alpha \in A \) such that \( \alpha \mathcal{H}[i \to i + 1] \). Since \( [i \to i + 1] \) is an idempotent, it follows that \( [i \to i + 1] = \alpha^\omega \) for some \( \omega \in \mathbb{N} \), and so \( [i \to i + 1] \in \langle A \rangle \). If \( [i \to i + 1] \in E_{n-1}^+ \setminus E_{n-1}^+ (A) \), then \( i \in X_n(A) \) and, by the property of \( A \), there exists \( i_1 \in X_n(A) \setminus \{ i \} \) such that \( \alpha_{i_1} \in A \) and \( [i \to i + 1] \mathcal{L} \alpha_{i_1} \mathcal{R} [i_1 \to i + 1] \).

Since \( i_1 \in X_n(A) \setminus \{ i \} \), then, by the property of \( A \), there exists \( i_2 \in X_n(A) \setminus \{ i_1 \} \) such that \( \alpha_{i_2} \in A \) and \( [i_1 \to i_1 + 1] \mathcal{L} \alpha_{i_2} \mathcal{R} [i_2 \to i_2 + 1] \).

If \( i_2 = i \), then, by Lemma 2.2,

\[
\alpha_{i_2} = \alpha_{i_2} \alpha_{i_1} \in H_{(i_2, i_2 + 1)}^{[i_1]} H_{(i_1, i_1 + 1)}^{[i]} = H_{(i_2, i_2 + 1)}^{[i]} = H_{(i, i + 1)}^{[i]},
\]

Since \( [i \to i + 1] \) is an idempotent of the group \( H_{(i, i + 1)}^{[i]} \), it follows that \( [i \to i + 1] = (\alpha_{i_2}^\omega) \) for some \( \omega \in \mathbb{N} \), and so \( [i \to i + 1] \in \langle A \rangle \). Notice that \( i_2 \in X_n(A) \setminus \{ i_1 \} \). If \( i_2 \neq i \), then, by the property of \( A \), there exists \( i_3 \in X_n(A) \setminus \{ i_1, i_2 \} \) such that \( \alpha_{i_3} \in A \) and \( [i_2 \to i_2 + 1] \mathcal{L} \alpha_{i_3} \mathcal{R} [i_3 \to i_3 + 1] \).

Notice that \( i \in X_n(A) \). Continuing this demonstration, by the property of \( A \), there exist distinct \( i_1, i_2, \ldots, i_m-1, i_m \in X_n(A) \) (\( m \leq |X_n(A)| \)) such that \( i_m = i \) and

\[
\alpha_{i_1} \in A \] and \( [i \to i + 1] \mathcal{L} \alpha_{i_1} \mathcal{R} [i_1 \to i_1 + 1] \),
\[
\alpha_{i_k} \in A \] and \( [i_{k-1} \to i_k - 1 + 1] \mathcal{L} \alpha_{i_k} \mathcal{R} [i_k \to i_k + 1] \),

for \( k \in \{2, \ldots, m\} \). Then, by Lemma 2.2,

\[
\alpha_{i_m} = \alpha_{i_m} \alpha_{i_{m-1}} \cdots \alpha_{i_1} 
\in H_{(i_m, i_m + 1)}^{[i_{m-1}]} H_{(i_{m-1}, i_{m-1} + 1)}^{[i_{m-2}]} \cdots H_{(i_2, i_3 + 1)}^{[i_1]} H_{(i_2, i_2 + 1)}^{[i_1]} H_{(i_1, i_1 + 1)}^{[i]}
\]
\[ \begin{align*}
&= H^{[i]}_{\{m, \ldots, m+1\}} = H^{[i]}_{\{i, i+1\}}.
\end{align*} \]

Since \([i \to i + 1]\) is an idempotent of the group \(H^{[i]}_{\{i, i+1\}}\), it follows that \([i \to i + 1] = (\alpha^*_m)^\omega\) for some \(\omega \in \mathbb{N}\), and so \([i \to i + 1] \in (A)\). \qed

Since \(\text{SOP}_n\) has rank \(n\) (see \([8, \text{Theorem 2.2}]\)), a generating set of \(\text{SOP}_n\) with \(n\) elements is a minimal generating set. Moreover, if \(\alpha\) is an element of \(\text{SOP}_n\) of rank \(n - 1\) and \(\beta\) and \(\gamma\) are two elements of \(\text{SOP}_n\) such that \(\alpha = \beta \gamma\), then \(\ker(\alpha) = \ker(\beta)\) and \(\im(\alpha) = \im(\gamma)\). Then any generating set of \(\text{SOP}_n\) with \(n\) elements must be the subset having exactly one element from each \(\mathcal{R}\)-class and from each \(\mathcal{L}\)-class of \(\text{SOP}_n\) contained in \([n, n - 1]\). These observations, together with the Lemma 2.3, prove the following result:

**Theorem 2.4.** Let \(n \geq 3\), and let \(M\) be a nonempty subset of \(\text{SOP}_n\) with \(n\) elements. Then \(M\) is a minimal generating set of \(\text{SOP}_n\) if and only if \(M\) is the subset having exactly one element from each \(\mathcal{R}\)-class and from each \(\mathcal{L}\)-class of \(\text{SOP}_n\) contained in \([n, n - 1]\).

Notice also that each \(\mathcal{R}\)-class of \(\text{SOP}_n\) contained in \([n, n - 1]\) has \(n - 1\) elements (see \([2, \text{Corollary 3.6}]\)). Thus we have the following corollary from Theorem 2.4:

**Corollary 2.5.** Let \(n \geq 3\), and let \(M\) be a minimal generating set of \(\text{SOP}_n\). Then the number of distinct sets \(M\) is \((n - 1)^n n!\).

Now, consider the permutation \((n\text{-cycle})\) \(g = \begin{pmatrix} 1 & 2 & \cdots & n - 1 & n \\ 2 & 3 & \cdots & s + 1 & s + 2 & \cdots & n \end{pmatrix}\) of \(X_n\). For \(1 \leq s \leq n - 1\) and \(1 \leq i \leq n\), let

\[ \alpha^{[s+1]}_{(1,2)} = \begin{pmatrix} 1 & 2 & \cdots & s + 1 & s + 2 & \cdots & n \end{pmatrix} \]

and

\[ \alpha^{[s+1]}_{(i+1,i+2)} = g^{-i} \alpha^{[s+1]}_{(1,2)} g^i. \]

Then clearly \(\alpha^{[s+1]}_{(1,2)} \in H^{[s+1]}_{(1,2)}\). Notice also that \(\alpha^{[s+1]}_{(n+1,n+2)} = \alpha^{[s+1]}_{(1,2)}\).

**Lemma 2.6.** Let \(1 \leq s \leq n - 1\) and \(1 \leq i \leq n\). Then \(\alpha^{[s+1]}_{(i+1,i+2)} \in H^{[s+1]}_{(i+1,i+2)}\) and \(\alpha^{[s+1]}_{(1,2)}\) is \(s\)-potent.

**Proof.** Since \(g\) is a permutation \((n\text{-cycle})\) of \(X_n\) and \([1 \to 2]\mathcal{R}_n^{[s+1]}\mathcal{L}[s + 1 \to s + 2]\), we have

\[ g^{-i}[1 \to 2]\mathcal{R}g^{-i} \alpha^{[s+1]}_{(1,2)} \quad \text{and} \quad \alpha^{[s+1]}_{(1,2)} g^i \mathcal{L}[s + 1 \to s + 2]g^i. \]

Notice that

\[ g^{-i}[1 \to 2]g^i = [i + 1 \to i + 2], \quad g^{-i}[s + 1 \to s + 2]g^i = [i + s + 1 \to i + s + 2] \]
and $\mathcal{R}$ ($\mathcal{L}$) is a right (left) congruence. Then
\[
a_{[i+1]}^{[i+1]}(i+1,i+2) = g^{-1}\alpha_{[1]}^{[i+1]}g'\mathcal{L}g^{-1}[s+1 \to s+2]g^i = [i+1 \to i+2]
\]
and
\[
a_{[i+1]}^{[i+1]}(i+1,i+2) = g^{-1}\alpha_{[1]}^{[i+1]}g'\mathcal{L}g^{-1}[s+1 \to s+2]g^i = [i+1 \to i+2].
\]
Thence $\alpha_{[i]}^{[i]}(i+1,i+2) \in H^{[i+1]}(i+1,i+2)$.

Notice that $\alpha_{[1]}^{[2]}(i+1,i+2)$ is an idempotent and $g^ig^{-1} = 1_X$. Then
\[
(\alpha_{[i]}^{[2]}(i+1,i+2))^2 = g^{-1}(\alpha_{[i]}^{[2]}(i+1,i+2))^2g^i = g^{-1}\alpha_{[i]}^{[2]}(i+1,i+2)g^i = \alpha_{[i]}^{[2]}(i+1,i+2)
\]
and so $\alpha_{[i]}^{[3]}(i+1,i+2)$ is 1-potent. Notice also that, for $2 \leq s \leq n-1$,
\[
(\alpha_{[i]}^{[1]}(i+1,i+2))^k = g^{-1}(\alpha_{[i]}^{[1]}(i+1,i+2))^kg^i = g^{-1}\left(\begin{array}{cccc}
1,2,\ldots,k+1 & k+2 & \cdots & s+1 \\
1 & 2 & \cdots & s+1-k \\
1 & 2 & \cdots & s+2 \\
1 & 2 & \cdots & n
\end{array}\right)g^i,
\]
with $1 \leq k \leq s-1$ and
\[
(\alpha_{[i]}^{[1]}(i+1,i+2))^s = g^{-1}\left(\begin{array}{cccc}
1,2,\ldots,s+1 & s+2 & \cdots & n \\
1 & s+1 & \cdots & n
\end{array}\right)g^i.
\]
Then $(\alpha_{[i]}^{[1]}(i+1,i+2))^{[s+1]} = (\alpha_{[i]}^{[1]}(i+1,i+2))^s$ and
\[
(\alpha_{[i]}^{[1]}(i+1,i+2))^{[s+1]}, (\alpha_{[i]}^{[1]}(i+1,i+2))^{[s+1]}, \ldots, (\alpha_{[i]}^{[1]}(i+1,i+2))^{[s+1]}
\]
are distinct. Hence $\alpha_{[i]}^{[s+1]}(i+1,i+2)$ is $s$-potent. □

Now, for $1 \leq s \leq n-1$, let
\[
G(s) = \{\alpha_{[i]}^{[s+1]}(i+1,i+2) : 1 \leq i \leq n\}.
\]
Then, by Lemma 2.6, the set $G(s)$ contains exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $SOP_n$ contained in $[n,n-1]$. Thus, by Theorem 2.4, $SOP_n = (G(s))$. Notice that $|G(s)| = n$ and the $s$-potent rank of $SOP_n$ is at least as large as the rank of $SOP_n$. Recall that the rank and the idempotent rank of $SOP_n$ are both equal to $n$ (see [8, Theorem 2.2]). These observations, together with Lemma 2.6, prove the following result:

**Theorem 2.7.** Let $1 \leq m \leq n-1$. Then $\operatorname{rank}_m SOP_n = n$.

Now, recall that Zhao [8, Theorem 2.3] proved:

**Lemma 2.8.** Let $n \geq 3$. Let $G \subseteq E(SPOP_n)$. Then
\[
(G) = SPOP_n \text{ if and only if } E^1_{n-1} \cup F_{n-1} \subseteq G \text{ or } E^0_{n-1} \cup F_{n-1} \subseteq G.
\]

Using Lemma 2.8, it is easy to prove the following result:
Lemma 2.9. Let \( n \geq 3 \). Let \( A \) be a subset of \([n-1, n-1]\). If \( A \) contains one element from each \( R \)-class of \( SPOP_n \) contained in \([n-1, n-1]\), then \( SPOP_n = (E_{n-1}^+ \cup A) \).

Proof. By Lemma 2.8, we have \( SPOP_n = \langle E_{n-1}^+ \cup F_{n-1} \rangle \). We shall show that \( F_{n-1} \subseteq \langle E_{n-1}^+ \cup A \rangle \) and so \( SPOP_n = \langle E_{n-1}^+ \cup A \rangle \). Let \( \delta_i \in F_{n-1} \), then there exists \( \alpha \in A \) such that \( \alpha \delta_i \). Then \( \alpha \in R_i \). Suppose that \( \alpha \in H_{(i)} \) for some \( k \in \{1, 2, \ldots, n\} \). Let \( \beta \in H_{(i)}^{[k]} \). Then, by Lemmas 2.1 and 2.2, \( \beta \in SPOP_n = \langle E_{n-1}^+ \rangle \) and \( \alpha \beta \in H_{(i)}^{[k]} H_{(i)}^{[k+i]} = H_{(i)}^{[i]} \). Since \( \delta_i \) is an idempotent of the group \( H_{(i)}^{[i]} \), it follows that \( \delta_i = (\alpha \beta)\omega \) for some \( \omega \in N \), whence \( \delta_i \in (E_{n-1}^+ \cup A) \). \( \square \)

For \( 1 \leq s \leq n-1 \) and \( 1 \leq i \leq n \), let
\[
\beta_{[1,s+1]} = \begin{pmatrix} 2 & 3 & \cdots & s+1 & s+2 & \cdots & n \\ 1 & 2 & \cdots & s & s+2 & \cdots & n \end{pmatrix}
\]
and
\[
\beta_{[i+1,s+1]} = g^{-i} \beta_{[1,s+1]} g^i.
\]
Then clearly \( \beta_{[1,s+1]} \in H_{(1)}^{[s+1]} \). Notice also that \( \beta_{[n+1,s+1]} = \beta_{[1,s+1]} \).

Lemma 2.10. Let \( 1 \leq s \leq n-1 \) and \( 1 \leq i \leq n \). Then \( \beta_{[i+1,s+1]} \in H_{(i+1)}^{[s+1]} \) and \( \beta_{[i+1,s+1]} \) is \((s+1)\)-potent.

Proof. Since \( g \) is a permutation \((n\)-cycle\) of \( X_n \) and \( \delta_i R \beta_{[1,s+1]} \mathcal{L} \delta_{s+1} \), we have
\[
g^{-i} \delta_i R g^{-i} \beta_{[1,s+1]} \text{ and } \beta_{[1,s+1]} g^s \mathcal{L} \delta_{s+1} g^i.
\]
Notice that
\[
g^{-i} \delta_i g^i = \delta_{i+1}, \quad g^{-i} \delta_{s+1} g^i = \delta_{i+s+1}
\]
and \( R \) \((\mathcal{L})\) is a right \((\text{left})\) congruence. We mean \( \delta_{n+j} = \delta_j \) for \( j = 1, 2, \ldots, n \) by Remark 1. Then
\[
\beta_{[i+1,s+1]} = g^{-i} \beta_{[1,s+1]} g^i \mathcal{L} \delta_i g^i = \delta_{i+1},
\]
\[
\beta_{[i+1,s+1]} = g^{-i} \beta_{[1,s+1]} g^i \mathcal{L} g^{-i} \delta_{s+1} g^i = \delta_{i+s+1}.
\]
Hence \( \beta_{[i+1,s+1]} \in H_{(i+1)}^{[s+1]} \).

Notice that \( \beta_{[1,2]} = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 3 & \cdots & n \end{pmatrix} \) and \( g^n g^{-1} = 1_{X_n} \). Then
\[
(\beta_{[1,2]})^2 = g^{-i} (\beta_{[1,2]})^2 g^i = g^{-i} \begin{pmatrix} 3 & \cdots & n \\ 3 & \cdots & n \end{pmatrix} g^i.
\]
Then \( (\beta_{[1,2]})^3 = (\beta_{[1,2]})^2 \) and \( (\beta_{[1,2]})^4 \), \( (\beta_{[1,2]})^2 \) are distinct. Hence \( \beta_{[1,2]} \) is \(2\)-potent. Notice also that, for \( 2 \leq s \leq n-1 \),
\[
(\beta_{[i+1,s+1]})^k
\]
\[
= g^{-i} (\beta_{k+1,k+1}^{i+1} g^k)
= g^{-i} \left( \begin{array}{cccc}
  k+1 & k+2 & \cdots & s+1 \\
  1 & 2 & \cdots & s+1-k \\
\end{array} \right) g^i, \quad 1 \leq k \leq s
\]

and
\[
= (\beta_{k+1,k+1}^{i+1})^{s+1} = g^{-i} \left( \begin{array}{cccc}
  s+2 & \cdots & n \\
  s+2 & \cdots & n \\
\end{array} \right) g^i.
\]

Then \((\beta_{[i+1,s+1]}^{i+1})^{s+2} = (\beta_{[i+1,s+1]}^{i+1})^{s+1} and
\[(\beta_{[i+1,s+1]}^{i+1})^1, (\beta_{[i+1,s+1]}^{i+1})^2, \ldots, (\beta_{[i+1,s+1]}^{i+1})^{s+1}
\]
are distinct. Hence \(\beta_{[i+1,s+1]}^{i+1}\) is \((s+1)\)-potent. \(\square\)

For \(1 \leq s \leq n-1\), let
\[
F(s) = \{\beta_{[i+1,s+1]} : 1 \leq i \leq n\}.
\]

Then, by Lemma 2.10, the set \(F(s)\) contains exactly one element from each \(R\)-class and from each \(L\)-class of \(SPOP_n\) contained in \([n-1,n-1]\). Thus, by

**Theorem 2.4** and Lemma 2.9, \(SPOP_n = \langle G(s+1) \cup F(s) \rangle\) for \(1 \leq s \leq n-2\). Notice that \(|G(s+1) \cup F(s)| = 2n\) and the \((s+1)\)-potent rank of \(SPOP_n\) is at least as large as the rank of \(SPOP_n\). Recall that the rank and the idempotent rank of \(SPOP_n\) are both equal to \(2n\) (see [8, Theorem 2.9]). These observations, together with Lemmas 2.6 and 2.10, prove the following result:

**Theorem 2.11.** Let \(1 \leq m \leq n-1\). Then \(\text{rank}_m SPOP_n = 2n\).

### 3. The 2-potent rank of \(SSPOP_n\)

The subset \(SSPOP_n = SPOP_n \setminus SOP_n\) of strictly partial orientation preserving mappings on \(X_n\) is a subsemigroup of \(SPOP_n\). Since a non-idempotent element in \([n-1,n-1]\) can not be expressed as a product of idempotents of \([n-1,n-1]\), the semigroup \(SSPOP_n\) is not idempotent-generated. Then \(SSPOP_n\) has no \(1\)-potent rank.

Let
\[
\beta = \left( \begin{array}{cccc}
  1 & 2 & \cdots & n-1 \\
  2 & 2 & \cdots & n-1 \\
\end{array} \right),
\]

\[
\alpha_1 = \left( \begin{array}{cccc}
  2 & 3 & \cdots & n-1 \\
  2 & 3 & \cdots & n-1 \\
  1 & 1 & \cdots & 1 \\
\end{array} \right) \in H^{[n]}_{(1)}
\]

and
\[
\alpha_i = \left( \begin{array}{cccc}
  1 & \cdots & i-2 & i-1 & i & i+1 & i+2 & \cdots & n \\
  1 & \cdots & i-2 & i & i+1 & i+2 & \cdots & n \\
\end{array} \right) \in H^{[n]}_{(i)}
\]

for \(2 \leq i \leq n\). Recall that Zhao [8, Lemma 3.3 and Lemma 3.8] proved:

**Lemma 3.1.** Let \(B = \{\alpha_i : 1 \leq i \leq n\}\) and \(G = B \cup \{\beta\}\). Then \([n-1,n-1] \subseteq \langle B \rangle\) and \(SSPOP_n = \langle G \rangle\).
Now, let
\[ \tilde{\beta} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 \\ n & 2 & 3 & \cdots & n-2 & n \end{pmatrix}. \]
Then \( \tilde{\beta} \) is 2-potent.

Lemma 3.2. Let \( B = \{ \alpha_i : 1 \leq i \leq n \} \) and \( \tilde{G} = B \cup \{ \tilde{\beta} \} \). Then \( \text{SSPOP}_n = \langle \tilde{G} \rangle \).

Proof. Notice that \( \alpha_i \in [n-1, n-1] \), for \( 1 \leq i \leq n \). Then, by Lemma 3.1, \( \text{SSPOP}_n = \langle [n-1, n-1] \cup \{ \beta \} \rangle \). Now, as
\[ \beta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 \\ n-1 & 1 & 2 & \cdots & n-3 & n-2 \end{pmatrix} \tilde{\beta} \begin{pmatrix} 2 & 3 & \cdots & n-2 & n-1 & n \\ 3 & 4 & \cdots & n-1 & n & 2 \end{pmatrix}, \]
it follows that \( \text{SSPOP}_n = \langle [n-1, n-1] \cup \{ \tilde{\beta} \} \rangle \). Then, again by Lemma 3.1, \( \text{SSPOP}_n = \langle \tilde{G} \rangle \).

Notice that \( |\tilde{G}| = n+1 \), each element of the set \( \tilde{G} \) is 2-potent and the 2-potent rank of \( \text{SSPOP}_n \) is at least as large as the rank of \( \text{SSPOP}_n \). Recall that the rank of \( \text{SSPOP}_n \) is equal to \( n+1 \) (see [8, Theorem 3.1]). These observations, together with Lemma 3.2, prove the following result:

Theorem 3.3. Let \( n \geq 3 \). Then \( \text{rank}_2 \text{SSPOP}_n = n+1 \).

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