Bull. Korean Math. Soc. **51** (2014), No. 6, pp. 1841–1850
http://dx.doi.org/10.4134/BKMS.2014.51.6.1841

ON THE m-POTENT RANKS OF CERTAIN SEMIGROUPS OF ORIENTATION PRESERVING TRANSFORMATIONS

Ping Zhao, Taijie You, and Huabi Hu

Abstract. It is known that the ranks of the semigroups $\mathcal{SOP}_n$, $\mathcal{SPOP}_n$ and $\mathcal{SSPOP}_n$ (the semigroups of orientation preserving singular self-maps, partial and strictly partial transformations on $X_n = \{1, 2, \ldots, n\}$, respectively) are n, $2n$ and $n+1$, respectively. The *idempotent rank*, defined as the smallest number of idempotent generating set, of $\mathcal{SOP}_n$ and $\mathcal{SSPOP}_n$ are the same value as the rank, respectively. Idempotent can be seen as a special case (with $m = 1$) of m-potent. In this paper, we investigate the m-potent ranks, defined as the smallest number of m-potent generating set, of the semigroups $\mathcal{SOP}_n$, $\mathcal{SPOP}_n$ and $\mathcal{SSPOP}_n$. Firstly, we characterize the structure of the minimal generating sets of $\mathcal{SOP}_n$. As applications, we obtain that the number of distinct minimal generating sets is $(n-1)^n n!$. Secondly, we show that, for $1 \leq m \leq n-1$, the m-potent ranks of the semigroups $\mathcal{SOP}_n$ and $\mathcal{SPOP}_n$ are also n and $2n$, respectively. Finally, we find that the 2-potent rank of $\mathcal{SSPOP}_n$ is $n+1$.

1. Introduction and preliminaries

As usual we denote by $\mathcal{PT}_n$ the monoid of all partial transformations of a finite set X_n with n elements (under composition), by $\mathcal{T}_n$ the submonoid of $\mathcal{PT}_n$ of all full transformations of X_n and by $\mathcal{S}_n$ the symmetric group on X_n, i.e., the subgroup of $\mathcal{PT}_n$ of all injective full transformations (permutations) of X_n. Denote by $\mathcal{SPT}_n$ the semigroup of $\mathcal{PT}_n \setminus \mathcal{S}_n$ of all singular partial transformations and by $\mathcal{S}ing_n$ the semigroup $\mathcal{T}_n \setminus \mathcal{S}_n$ of all singular transformations of X_n.

Let now X_n be a chain with n elements, say $X_n = \{1 < 2 < \cdots < n\}$. Let $c = (c_1, c_2, \ldots, c_t)$ be a sequence of t $(t \geq 0)$ elements from the chain X_n. We say that c is cyclic if there exists no more than one index $i \in \{1, \ldots, t\}$ such that $c_i > c_{i+1}$, where c_{t+1} denotes c_1. Let $\alpha \in \mathcal{PT}_n$ and suppose that

Received November 28, 2013; Revised February 23, 2014.
2010 *Mathematics Subject Classification.* 20M20, 20M10.
Key words and phrases. transformation, orientation-preserving, rank, idempotent rank, m-potent rank.
This work is supported by Natural Science Foundation of Guizhou (No.[2013]2225) and National Natural Science Foundation of China (No. 11461014).

$\mathrm{dom}(\alpha) = \{a_1, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < \cdots < a_t$. We say that α is orientation-preserving if the sequence of its image $(a_1\alpha, \ldots, a_t\alpha)$ is cyclic. Denote by $\mathcal{SPOP}_n$ the subsemigroup of $\mathcal{SPT}_n$ of all partial orientation preserving singular transformations, by $\mathcal{SOP}_n$ the subsemigroup $\mathcal{SPOP}_n \cap \mathcal{S}ing_n$ of all orientation preserving singular transformations and by $\mathcal{SSPOP}_n$ the subsemigroup $\mathcal{SPOP}_n \backslash \mathcal{SOP}_n$ of all orientation preserving strictly partial transformations.

Remark 1. In this paper it will always be clear from context when additions are modular.

An element a of any given semigroup is called *m-potent* if $a^{m+1} = a^m$ and $a, a^2, \ldots, a^m$ are distinct. In particular, we refer to idempotent as 1-potent. This concept was mentioned in [1, 6, 7].

As usual, the rank of a finite semigroup S is defined by $\operatorname{rank} S = \min\{|A| \mid A \subseteq S, \langle A \rangle = S\}$. If S is generated by its set E of idempotents, then the idempotent rank of S is defined by $\operatorname{idrank} S = \min\{|A| \mid A \subseteq E, \langle A \rangle = S\}$. If S is generated by its set E_m of m-potents, then the m-potent rank of S is defined by $\operatorname{rank}_m S = \min\{|A| \mid A \subseteq E_m, \langle A \rangle = S\}$.

It is known that the ranks of the semigroups $\mathcal{SOP}_n$, $\mathcal{SPOP}_n$ and $\mathcal{SSPOP}_n$ are n, $2n$ and $n+1$, respectively. The idempotent rank of $\mathcal{SOP}_n$ and $\mathcal{SPOP}_n$ are the same value as the rank, respectively (see [8]). In this paper, we investigate the m-potent ranks of the semigroup $\mathcal{SOP}_n$, $\mathcal{SPOP}_n$ and $\mathcal{SSPOP}_n$. In Section 2 we characterize the structure of the minimal generating sets of $\mathcal{SOP}_n$. As applications, we prove that the number of distinct minimal generating sets is $(n-1)^n n!$. Moreover, we show that, for $1 \leq m \leq n-1$, the m-potent ranks of the semigroups $\mathcal{SOP}_n$ and $\mathcal{SPOP}_n$ are n and $2n$, respectively. In Section 3 we find that the 2-potent rank of $\mathcal{SSPOP}_n$ is $n+1$.

Let $\alpha \in \mathcal{PT}_n$. As usual, we write $\mathrm{im}(\alpha)$ and $\mathrm{rank}(\alpha)$ for the image of α and the rank of α, respectively. (The rank of a transformation is defined to be the size of its image.) The *kernel* of α is the equivalence $\ker(\alpha) = \{(x, y) \in X_n \times X_n : x\alpha = y\alpha\}$. From Fernandes, Gomes, and Jesus [3], Green's relations on $\mathcal{SPOP}_n$ are characterized by

$$\begin{aligned} \alpha\mathcal{L}\beta &\text{ if and only if } \mathrm{im}(\alpha) = \mathrm{im}(\beta), \\ \alpha\mathcal{R}\beta &\text{ if and only if } \ker(\alpha) = \ker(\beta), \\ \alpha\mathcal{J}\beta &\text{ if and only if } \mathrm{rank}(\alpha) = \mathrm{rank}(\beta). \end{aligned}$$

The semigroup $\mathcal{SPOP}_n$ has n $\mathcal{J}$-classes, namely $J_0, J_1, \ldots, J_{n-1}$, where J_0 consists of the empty mapping and $J_r = \{\alpha \in \mathcal{SPOP}_n : \mathrm{rank}(\alpha) = r\}$ for $1 \leq r \leq n-1$. For $0 \leq r \leq s \leq n$, let

$$[s, r] = \{\alpha \in \mathcal{SPOP}_n : |\operatorname{dom}(\alpha)| = s, \mathrm{rank}(\alpha) = r\}.$$

Then $J_r = \cup_{i=r}^{n} [i, r]$ and $J_{n-1} = [n, n-1] \cup [n-1, n-1]$. We draw attention to the top $\mathcal{J}$-class J_{n-1}. As in [8] we use the notation

$$L_k = \{\alpha \in J_{n-1} : \mathrm{im}(\alpha) = X_n \setminus \{k\}\},$$

$R_{(k,k+1)} = \{\alpha \in [n, n-1] : \text{the unique non-singleton class of } \ker(\alpha) \text{ is } \{k, k+1\}\}$ and

$$R_k = \{\alpha \in [n-1, n-1] : \mathrm{dom}(\alpha) = X_n \backslash \{k\}\},\ k \in X_n$$

for $\mathcal{L}$-classes and $\mathcal{R}$-classes in J_{n-1}. Hence J_{n-1} has n $\mathcal{L}$-classes $L_1, L_2, \ldots, L_n$ and $2n$ $\mathcal{R}$-classes $R_{(1,2)}$, $R_{(2,3)}, \ldots, R_{(n,1)}$, $R_1, R_2, \ldots, R_n$.

Gomes and Howie [4] used the notation $[i \to i-1]$ for the idempotent e defined by $ie = i-1$, $xe = x$ $(x \neq i)$ and the notation $[i \to i+1]$ for the idempotent f defined by $if = i+1$, $xf = x$ $(x \neq i)$. They also used the notation δ_k, $k = 1, 2, \ldots, n$, for the identity mapping on $X_n \backslash \{k\}$.

Let S be a subset of $\mathcal{SPOP}_n$. As usual, we denote by $E(S)$ the set of all idempotents of S. Employing the above notation, the set $E(J_{n-1})$ consists of n decreasing idempotents $[i \to i-1]$ $(i \in X_n)$, n increasing idempotents $[i \to i+1]$ $(i \in X_n)$ and n idempotents δ_k. Notice that $[1 \to 0] = [1 \to n]$, $[n \to n+1] = [n \to 1]$, etc., by Remark 1. Let $E^+_{n-1} = \{[i \to i+1] : i \in X_n\}$ and $E^-_{n-1}\{[i+1 \to i] : i \in X_n\}$ be the increasing and decreasing idempotent set of $[n, n-1]$, respectively, and let $F_{n-1} = \{\delta_1, \ldots, \delta_n\}$. Then $E(J_{n-1}) = E^+_{n-1} \cup E^-_{n-1} \cup F_{n-1}$.

Given a subset U of a semigroup S and $s \in S$, we denote by L_s , R_s and H_s the $\mathcal{L}$-class, $\mathcal{R}$-class and $\mathcal{H}$-class of s, respectively. For general background on Semigroup Theory, we refer the reader to Howie's book [5].

2. The m-potent ranks of $\mathcal{SOP}_n$ and $\mathcal{SPOP}_n$

In this section we characterize the structure of the minimal generating sets of $\mathcal{SOP}_n$. As applications, we prove that the number of distinct minimal generating sets is $(n-1)^n n!$. Moreover, we show that, for $1 \leq m \leq n-1$, the m-potent ranks of the semigroups $\mathcal{SOP}_n$ and $\mathcal{SPOP}_n$ are n and $2n$, respectively.

We begin by recalling that Zhao, Xu and Yang [9, Theorem 2.1] proved:

Lemma 2.1. *Let $n \geq 3$. Let $G \subseteq E(\mathcal{SOP}_n)$. Then*

$$\langle G \rangle = \mathcal{SOP}_n \text{ if and only if } E^+_{n-1} \subseteq G \text{ or } E^-_{n-1} \subseteq G.$$

For any $i, j \in X_n$, let

$$H^{[j]}_{(i,i+1)} = R_{(i,i+1)} \cap L_j \text{ and } H^{[j]}_{(i)} = R_i \cap L_j.$$

Notice that each $\mathcal{H}$-classes contained in $[n, n-1]$ has the form $H^{[j]}_{(i,i+1)}$, for some $i, j \in X_n$, and each $\mathcal{H}$-classes contained in $[n-1, n-1]$ has the form $H^{[j]}_{(i)}$, for some $i, j \in X_n$. With above notation, we have the following simple observation:

Lemma 2.2. *Let $n \geq 3$. Then*

$$H^{[k]}_{(i,i+1)} H^{[l]}_{(j,j+1)} = H^{[l]}_{(i,i+1)},\ \text{if } k = j \text{ or } k = j+1,$$
$$H^{[k]}_{(i)} H^{[l]}_{(j,j+1)} = H^{[l]}_{(i)},\ \text{if } k = j \text{ or } k = j+1,$$

$$H_{(k)}^{[i]}H_{(l)}^{[j]} = H_{(k)}^{[j]}, \text{ if } i = l.$$

Notice that $\mathcal{SOP}_n \cap J_{n-1} = [n, n-1]$ and $[n, n-1]$ contains n $\mathcal{R}$-classes and n $\mathcal{L}$-classes of $\mathcal{SOP}_n$. Thus the number of $\mathcal{R}$-classes and $\mathcal{L}$-classes of $\mathcal{SOP}_n$ contained in $[n, n-1]$ are both equal to n.

Lemma 2.3. *Let $n \geq 3$, and let A be a nonempty subset of $\mathcal{SOP}_n$ with n elements. If A contains exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{SOP}_n$ contained in $[n, n-1]$, then $\mathcal{SOP}_n = \langle A \rangle$.*

Proof. By Lemma 2.1, we have $\mathcal{SOP}_n = \langle E_{n-1}^+ \rangle$. Notice that $A \subseteq [n, n-1] \subseteq \mathcal{SOP}_n$. We shall show that $E_{n-1}^+ \subseteq \langle A \rangle$ and so $\mathcal{SOP}_n = \langle A \rangle$. Let

$$E_{n-1}^+(A) = \{[i \to i+1] \in E_{n-1}^+ : \text{ there exists } \alpha \in A \text{ such that } \alpha\mathcal{H}[i \to i+1]\}$$

and

$$X_n(A) = \{i \in X_n : [i \to i+1] \notin E_{n-1}^+(A)\}.$$

Let $[i \to i+1] \in E_{n-1}^+$. If $[i \to i+1] \in E_{n-1}^+(A)$, then there exists $\alpha \in A$ such that $\alpha\mathcal{H}[i \to i+1]$. Since $[i \to i+1]$ is an idempotent, it follows that $[i \to i+1] = \alpha^\omega$ for some $\omega \in \mathbb{N}$, and so $[i \to i+1] \in \langle A \rangle$. If $[i \to i+1] \in E_{n-1}^+ \backslash E_{n-1}^+(A)$, then $i \in X_n(A)$ and, by the property of A, there exists $i_1 \in X_n(A) \backslash \{i\}$ such that

$$\alpha_{i_1} \in A \text{ and } [i \to i+1]\mathcal{L}\alpha_{i_1}\mathcal{R}[i_1 \to i_1+1].$$

Since $i_1 \in X_n(A) \backslash \{i\}$, then, by the property of A, there exists $i_2 \in X_n(A) \backslash \{i_1\}$ such that

$$\alpha_{i_2} \in A \text{ and } [i_1 \to i_1+1]\mathcal{L}\alpha_{i_2}\mathcal{R}[i_2 \to i_2+1].$$

If $i_2 = i$, then, by Lemma 2.2,

$$\alpha_{i_2}^* = \alpha_{i_2}\alpha_{i_1} \in H_{(i_2,i_2+1)}^{[i_1]}H_{(i_1,i_1+1)}^{[i]} = H_{(i_2,i_2+1)}^{[i]} = H_{(i,i+1)}^{[i]}.$$

Since $[i \to i+1]$ is an idempotent of the group $H_{(i,i+1)}^{[i]}$, it follows that $[i \to i+1] = (\alpha_{i_2}^*)^\omega$ for some $\omega \in \mathbb{N}$, and so $[i \to i+1] \in \langle A \rangle$. Notice that $i_2 \in X_n(A) \backslash \{i_1\}$. If $i_2 \neq i$, then, by the property of A, there exists $i_3 \in X_n(A) \backslash \{i_1, i_2\}$ such that

$$\alpha_{i_3} \in A \text{ and } [i_2 \to i_2+1]\mathcal{L}\alpha_{i_3}\mathcal{R}[i_3 \to i_3+1].$$

Notice that $i \in X_n(A)$. Continuing this demonstration, by the property of A, there exist distinct $i_1, i_2, \ldots, i_{m-1}, i_m \in X_n(A)$ $(m \leq |X_n(A)|)$ such that $i_m = i$ and

$$\alpha_{i_1} \in A \text{ and } [i \to i+1]\mathcal{L}\alpha_{i_1}\mathcal{R}[i_1 \to i_1+1],$$

$$\alpha_{i_k} \in A \text{ and } [i_{k-1} \to i_{k-1}+1]\mathcal{L}\alpha_{i_k}\mathcal{R}[i_k \to i_k+1],$$

for $k \in \{2, \ldots, m\}$. Then, by Lemma 2.2,

$$\begin{aligned}\alpha_{i_m}^* &= \alpha_{i_m}\alpha_{i_{m-1}}\cdots\alpha_{i_1} \\ &\in H_{(i_m,i_m+1)}^{[i_{m-1}]}H_{(i_{m-1},i_{m-1}+1)}^{[i_{m-2}]}\cdots H_{(i_3,i_3+1)}^{[i_2]}H_{(i_2,i_2+1)}^{[i_1]}H_{(i_1,i_1+1)}^{[i]}\end{aligned}$$

$$= H^{[i]}_{(i_m,i_m+1)} = H^{[i]}_{(i,i+1)}.$$

Since $[i \to i+1]$ is an idempotent of the group $H^{[i]}_{(i,i+1)}$, it follows that $[i \to i+1] = (\alpha^*_{i_m})^\omega$ for some $\omega \in \mathbb{N}$, and so $[i \to i+1] \in \langle A \rangle$. $\square$

Since $\mathcal{SOP}_n$ has rank n (see [8, Theorem 2.2]), a generating set of $\mathcal{SOP}_n$ with n elements is a minimal generating set. Moreover, if α is an element of $\mathcal{SOP}_n$ of rank $n-1$ and β and γ are two elements of $\mathcal{SOP}_n$ such that $\alpha = \beta\gamma$, then $\ker(\alpha) = \ker(\beta)$ and $\mathrm{im}(\alpha) = \mathrm{im}(\gamma)$. Then any generating set of $\mathcal{SOP}_n$ with n elements must be the subset having exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{SOP}_n$ contained in $[n, n-1]$. These observations, together with the Lemma 2.3, prove the following result:

Theorem 2.4. *Let $n \geq 3$, and let M be a nonempty subset of $\mathcal{SOP}_n$ with n elements. Then M is a minimal generating set of $\mathcal{SOP}_n$ if and only if M be the subset having exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{SOP}_n$ contained in $[n, n-1]$.*

Notice also that each $\mathcal{H}$-class of $\mathcal{SOP}_n$ contained in $[n, n-1]$ has $n-1$ elements (see [2, Corollary 3.6]). Thus we have the following corollary from Theorem 2.4:

Corollary 2.5. *Let $n \geq 3$, and let M be a minimal generating set of $\mathcal{SOP}_n$. Then the number of distinct sets M is $(n-1)^n n!$.*

Now, consider the permutation (n-cycle) $g = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix}$ of X_n. For $1 \leq s \leq n-1$ and $1 \leq i \leq n$, let

$$\alpha^{[s+1]}_{(1,2)} = \begin{pmatrix} \{1,2\} & 3 & \cdots & s+1 & s+2 & \cdots & n \\ 1 & 2 & \cdots & s & s+2 & \cdots & n \end{pmatrix}.$$

and

$$\alpha^{[s+1]}_{(i+1,i+2)} = g^{-i}\alpha^{[s+1]}_{(1,2)}g^i.$$

Then clearly $\alpha^{[s+1]}_{(1,2)} \in H^{[s+1]}_{(1,2)}$. Notice also that $\alpha^{[s+1]}_{(n+1,n+2)} = \alpha^{[s+1]}_{(1,2)}$.

Lemma 2.6. *Let $1 \leq s \leq n-1$ and $1 \leq i \leq n$. Then $\alpha^{[s+1]}_{(i+1,i+2)} \in H^{[i+s+1]}_{(i+1,i+2)}$ and $\alpha^{[s+1]}_{(i+1,i+2)}$ is s-potent.*

Proof. Since g is a permutation (n-cycle) of X_n and $[1 \to 2]\mathcal{R}\alpha^{[s+1]}_{(1,2)}\mathcal{L}[s+1 \to s+2]$, we have

$$g^{-i}[1 \to 2]\mathcal{R}g^{-i}\alpha^{[s+1]}_{(1,2)} \quad \text{and} \quad \alpha^{[s+1]}_{(1,2)}g^i\mathcal{L}[s+1 \to s+2]g^i.$$

Notice that

$$g^{-i}[1 \to 2]g^i = [i+1 \to i+2],\ g^{-i}[s+1 \to s+2]g^i = [i+s+1 \to i+s+2]$$

and $\mathcal{R}$ ($\mathcal{L}$) is a right (left) congruence. Then

$$\alpha^{[s+1]}_{(i+1,i+2)} = g^{-i}\alpha^{[s+1]}_{(1,2)}g^i\mathcal{R}g^{-i}[1 \to 2]g^i = [i+1 \to i+2]$$

and

$$\alpha^{[s+1]}_{(i+1,i+2)} = g^{-i}\alpha^{[s+1]}_{(1,2)}g^i\mathcal{L}g^{-i}[s+1 \to s+2]g^i = [i+s+1 \to i+s+2].$$

Thence $\alpha^{[s+1]}_{(i+1,i+2)} \in H^{[i+s+1]}_{(i+1,i+2)}$.

Notice that $\alpha^{[2]}_{(1,2)}$ is an idempotent and $g^ig^{-i} = 1_{X_n}$. Then

$$(\alpha^{[2]}_{(i+1,i+2)})^2 = g^{-i}(\alpha^{[2]}_{(1,2)})^2g^i = g^{-i}\alpha^{[2]}_{(1,2)}g^i = \alpha^{[2]}_{(i+1,i+2)}$$

and so $\alpha^{[2]}_{(i+1,i+2)}$ is 1-potent. Notice also that, for $2 \le s \le n-1$,

$$\begin{aligned}
&(\alpha^{[s+1]}_{(i+1,i+2)})^k \\
&= g^{-i}(\alpha^{[s+1]}_{(1,2)})^kg^i \\
&= g^{-i}\begin{pmatrix} \{1,2,\cdots,k+1\} & k+2 & \cdots & s+1 & s+2 & \cdots & n \\ 1 & 2 & \cdots & s+1-k & s+2 & \cdots & n \end{pmatrix} g^i, \\
&\quad 1 \le k \le s-1
\end{aligned}$$

and

$$(\alpha^{[s+1]}_{(i+1,i+2)})^s = g^{-i}\begin{pmatrix} \{1,2,\cdots,s+1\} & s+2 & \cdots & n \\ 1 & s+2 & \cdots & n \end{pmatrix} g^i.$$

Then $(\alpha^{[s+1]}_{(i+1,i+2)})^{s+1} = (\alpha^{[s+1]}_{(i+1,i+2)})^s$ and

$$(\alpha^{[s+1]}_{(i+1,i+2)})^1, (\alpha^{[s+1]}_{(i+1,i+2)})^2, \ldots, (\alpha^{[s+1]}_{(i+1,i+2)})^s$$

are distinct. Hence $\alpha^{[s+1]}_{(i+1,i+2)}$ is s-potent. □

Now, for $1 \le s \le n-1$, let

$$G(s) = \{\alpha^{[s+1]}_{(i+1,i+2)} : 1 \le i \le n\}.$$

Then, by Lemma 2.6, the set $G(s)$ contains exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{SOP}_n$ contained in $[n, n-1]$. Thus, by Theorem 2.4, $\mathcal{SOP}_n = \langle G(s)\rangle$. Notice that $|G(s)| = n$ and the s-potent rank of $\mathcal{SOP}_n$ is at least as large as the rank of $\mathcal{SOP}_n$. Recall that the rank and the idempotent rank of $\mathcal{SOP}_n$ are both equal to n (see [8, Theorem 2.2]). These observations, together with Lemma 2.6, prove the following result:

Theorem 2.7. *Let $1 \le m \le n-1$. Then* $\operatorname{rank}_m \mathcal{SOP}_n = n$.

Now, recall that Zhao [8, Theorem 2.3] proved:

Lemma 2.8. *Let $n \ge 3$. Let $G \subseteq E(\mathcal{SPOP}_n)$. Then*

$$\langle G\rangle = \mathcal{SPOP}_n \text{ if and only if } E^+_{n-1} \cup F_{n-1} \subseteq G \text{ or } E^-_{n-1} \cup F_{n-1} \subseteq G.$$

Using Lemma 2.8, it is easy to prove the following result:

Lemma 2.9. *Let $n \geq 3$. Let A be a subset of $[n-1, n-1]$. If A contains one element from each $\mathcal{R}$-class of $\mathcal{SPOP}_n$ contained in $[n-1, n-1]$, then $\mathcal{SPOP}_n = \langle E_{n-1}^+ \cup A \rangle$.*

Proof. By Lemma 2.8, we have $\mathcal{SPOP}_n = \langle E_{n-1}^+ \cup F_{n-1} \rangle$. We shall show that $F_{n-1} \subseteq \langle E_{n-1}^+ \cup A \rangle$ and so $\mathcal{SPOP}_n = \langle E_{n-1}^+ \cup A \rangle$. Let $\delta_i \in F_{n-1}$, then there exists $\alpha \in A$ such that $\alpha \mathcal{R} \delta_i$. Then $\alpha \in R_i$. Suppose that $\alpha \in H_{(i)}^{[k]}$ for some $k \in \{1, 2, \ldots, n\}$. Let $\beta \in H_{(k,k+1)}^{[i]}$. Then, by Lemmas 2.1 and 2.2, $\beta \in \mathcal{SOP}_n = \langle E_{n-1}^+ \rangle$ and $\alpha\beta \in H_{(i)}^{[k]} H_{(k,k+1)}^{[i]} = H_{(i)}^{[i]}$. Since δ_i is an idempotent of the group $H_{(i)}^{[i]}$, it follows that $\delta_i = (\alpha\beta)^\omega$ for some $\omega \in \mathbb{N}$, whence $\delta_i \in \langle E_{n-1}^+ \cup A \rangle$. $\square$

For $1 \leq s \leq n-1$ and $1 \leq i \leq n$, let

$$\beta_{[1,s+1]} = \begin{pmatrix} 2 & 3 & \cdots & s+1 & s+2 & \cdots & n \\ 1 & 2 & \cdots & s & s+2 & \cdots & n \end{pmatrix}$$

and

$$\beta_{[i+1,s+1]} = g^{-i} \beta_{[1,s+1]} g^i.$$

Then clearly $\beta_{[1,s+1]} \in H_{(1)}^{[s+1]}$. Notice also that $\beta_{[n+1,s+1]} = \beta_{[1,s+1]}$.

Lemma 2.10. *Let $1 \leq s \leq n-1$ and $1 \leq i \leq n$. Then $\beta_{[i+1,s+1]} \in H_{(i+1)}^{[i+s+1]}$ and $\beta_{[i+1,s+1]}$ is $(s+1)$-potent.*

Proof. Since g is a permutation (n-cycle) of X_n and $\delta_1 \mathcal{R} \beta_{[1,s+1]} \mathcal{L} \delta_{s+1}$, we have

$$g^{-i} \delta_1 \mathcal{R} g^{-i} \beta_{[1,s+1]} \quad \text{and} \quad \beta_{[1,s+1]} g^i \mathcal{L} \delta_{s+1} g^i.$$

Notice that

$$g^{-i} \delta_1 g^i = \delta_{i+1}, \ g^{-i} \delta_{s+1} g^i = \delta_{i+s+1}$$

and $\mathcal{R}$ ($\mathcal{L}$) is a right (left) congruence. We mean $\delta_{n+j} = \delta_j$ for $j = 1, 2, \ldots, n$ by Remark 1. Then

$$\beta_{[i+1,s+1]} = g^{-i} \beta_{[1,s+1]} g^i \mathcal{R} g^{-i} \delta_1 g^i = \delta_{i+1},$$
$$\beta_{[i+1,s+1]} = g^{-i} \beta_{[1,s+1]} g^i \mathcal{L} g^{-i} \delta_{s+1} g^i = \delta_{i+s+1}.$$

Thence $\beta_{[i+1,s+1]} \in H_{(i+1)}^{[i+s+1]}$.

Notice that $\beta_{[1,2]} = \begin{pmatrix} 2 & 3 & \cdots & n \\ 1 & 3 & \cdots & n \end{pmatrix}$ and $g^i g^{-i} = 1_{X_n}$. Then

$$(\beta_{[i+1,2]})^2 = g^{-i} (\beta_{[1,2]})^2 g^i = g^{-i} \begin{pmatrix} 3 & \cdots & n \\ 3 & \cdots & n \end{pmatrix} g^i.$$

Then $(\beta_{[i+1,2]})^3 = (\beta_{[i+1,2]})^2$ and $(\beta_{[i+1,2]})^1$, $(\beta_{[i+1,2]})^2$ are distinct. Hence $\beta_{[i+1,2]}$ is 2-potent. Notice also that, for $2 \leq s \leq n-1$,

$$(\beta_{[i+1,s+1]})^k$$

$$= g^{-i}[\beta_{[1,s+1]}]^k g^i$$
$$= g^{-i}\begin{pmatrix} k+1 & k+2 & \cdots & s+1 & s+2 & \cdots & n \\ 1 & 2 & \cdots & s+1-k & s+2 & \cdots & n \end{pmatrix} g^i,\ 1 \le k \le s$$

and

$$(\beta_{[i+1,s+1]})^{s+1} = g^{-i}\begin{pmatrix} s+2 & \cdots & n \\ s+2 & \cdots & n \end{pmatrix} g^i.$$

Then $(\beta_{[i+1,s+1]})^{s+2} = (\beta_{[i+1,s+1]})^{s+1}$ and

$$(\beta_{[i+1,s+1]})^1, (\beta_{[i+1,s+1]})^2, \ldots, (\beta_{[i+1,s+1]})^{s+1}$$

are distinct. Hence $\beta_{[i+1,s+1]}$ is $(s+1)$-potent. □

For $1 \le s \le n-1$, let

$$F(s) = \{\beta_{[i+1,s+1]} : 1 \le i \le n\}.$$

Then, by Lemma 2.10, the set $F(s)$ contains exactly one element from each $\mathcal{R}$-class and from each $\mathcal{L}$-class of $\mathcal{SPOP}_n$ contained in $[n-1, n-1]$. Thus, by Theorem 2.4 and Lemma 2.9, $\mathcal{SPOP}_n = \langle G(s+1) \cup F(s)\rangle$ for $1 \le s \le n-2$. Notice that $|G(s+1) \cup F(s)| = 2n$ and the $(s+1)$-potent rank of $\mathcal{SPOP}_n$ is at least as large as the rank of $\mathcal{SPOP}_n$. Recall that the rank and the idempotent rank of $\mathcal{SPOP}_n$ are both equal to $2n$ (see [8, Theorem 2.9]). These observations, together with Lemmas 2.6 and 2.10, prove the following result:

Theorem 2.11. *Let* $1 \le m \le n-1$. *Then* $\operatorname{rank}_m \mathcal{SPOP}_n = 2n$.

3. The 2-potent rank of $\mathcal{SSPOP}_n$

The subset $\mathcal{SSPOP}_n = \mathcal{SPOP}_n \backslash \mathcal{SOP}_n$ of strictly partial orientation preserving mappings on X_n is a subsemigroup of $\mathcal{SPOP}_n$. Since a non-idempotent element in $[n-1, n-1]$ can not be expressed as a product of idempotents of $[n-1, n-1]$, the semigroup $\mathcal{SSPOP}_n$ is not idempotent-generated. Then $\mathcal{SSPOP}_n$ has no 1-potent rank.

Let

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & n-1 \\ 2 & 2 & \cdots & n-1 \end{pmatrix},$$

$$\alpha_1 = \begin{pmatrix} 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 \end{pmatrix} \in H_{(1)}^{[n]}$$

and

$$\alpha_i = \begin{pmatrix} 1 & \cdots & i-2 & i-1 & i+1 & i+2 & \cdots & n \\ 1 & \cdots & i-2 & i & i+1 & i+2 & \cdots & n \end{pmatrix} \in H_{(i)}^{[i-1]}$$

for $2 \le i \le n$. Recall that Zhao [8, Lemma 3.3 and Lemma 3.8] proved:

Lemma 3.1. *Let* $B = \{\alpha_i : 1 \le i \le n\}$ *and* $G = B \cup \{\beta\}$. *Then* $[n-1, n-1] \subseteq \langle B\rangle$ *and* $\mathcal{SSPOP}_n = \langle G\rangle$.

Now, let

$$\widetilde{\beta} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 \\ n & 2 & 3 & \cdots & n-2 & n \end{pmatrix}.$$

Then $\widetilde{\beta}$ is 2-potent.

Lemma 3.2. *Let $B = \{\alpha_i : 1 \leq i \leq n\}$ and $\widetilde{G} = B \cup \{\widetilde{\beta}\}$. Then $\mathcal{SSPOP}_n = \langle \widetilde{G} \rangle$.*

Proof. Notice that $\alpha_i \in [n-1, n-1]$, for $1 \leq i \leq n$. Then, by Lemma 3.1, $\mathcal{SSPOP}_n = \langle [n-1, n-1] \cup \{\beta\} \rangle$. Now, as

$$\beta = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 \\ n-1 & 1 & 2 & \cdots & n-3 & n-2 \end{pmatrix} \widetilde{\beta} \begin{pmatrix} 2 & 3 & \cdots & n-2 & n-1 & n \\ 3 & 4 & \cdots & n-1 & n & 2 \end{pmatrix}$$
$$\in \langle [n-1, n-1] \cup \{\widetilde{\beta}\} \rangle,$$

it follows that $\mathcal{SSPOP}_n = \langle [n-1, n-1] \cup \{\widetilde{\beta}\} \rangle$. Then, again by Lemma 3.1, $\mathcal{SSPOP}_n = \langle \widetilde{G} \rangle$. □

Notice that $|\widetilde{G}| = n+1$, each element of the set $\widetilde{G}$ is 2-potent and the 2-potent rank of $\mathcal{SSPOP}_n$ is at least as large as the rank of $\mathcal{SSPOP}_n$. Recall that the rank of $\mathcal{SSPOP}_n$ is equal to $n+1$ (see [8, Theorem 3.1]). These observations, together with Lemma 3.2, prove the following result:

Theorem 3.3. *Let $n \geq 3$. Then* $\operatorname{rank}_2 \mathcal{SSPOP}_n = n+1$.

Acknowledgments. The authors would like to thank the anonymous referee for his/her helpful comments and suggestions that helped to improve this paper.

References

[1] G. Ayık, H. Ayık, Y. Ünlü, and J. M. Howie, *The structure of elements in finite full transformation semigroups*, Bull. Aust. Math. Soc. **71** (2005), no. 1, 69–74.

[2] P. M. Catarino and P. M. Higgins, *The monoid of orientation-preserving mappings on a chain*, Semigroup Forum **58** (1999), no. 2, 190–206.

[3] V. H. Fernandes, G. M. S. Gomes, and M. M. Jesus, *Congruences on monoids of transformations preserving the orientation on a finite chain*, J. Algebra **321** (2009), no. 3, 743–757.

[4] G. M. S. Gomes and J. M. Howie, *On the ranks of certain semigroups of order-preserving transformations*, Semigroup Forum **45** (1992), no. 3, 272–282.

[5] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford, Oxford University Press, 1995.

[6] O. Sönmez, *Combinatorics of elements in T_n and PT_n*, Int. J. Algebra **4** (2010), no. 1-4, 45–51.

[7] O. Sönmez and Y. Ünlü, *m-potent elements in order-preserving transformation semigroups and ordered trees*, Comm. Algebra **42** (2014), no. 1, 332–342.

[8] P. Zhao, *On the ranks of certain semigroups of orientation preserving transformations*, Comm. Algebra **39** (2011), no. 11, 4195–4205.

[9] P. Zhao, X. Bo, and Y. Mei, *Locally maximal idempotent-generated subsemigroups of singular orientation-preserving transformation semigroups*, Semigroup Forum **77** (2008), no. 2, 187–195.

Ping Zhao
School of Mathematics and Computer Science
Guizhou Normal University
Guiyang, Guizhou Province 550001, P. R. China
and
Mathematics Teaching & Research Section
Guiyang Medical College
Guiyang, Guizhou Province 550004, P. R. China
E-mail address: zhaoping731108@hotmail.com

Taijie You
School of Mathematics and Computer Science
Guizhou Normal University
Guiyang, Guizhou Province 550001, P. R. China
E-mail address: taijieyou@163.com

Huabi Hu
Department of Medicine Bioengineering
Guiyang Medical College
Guiyang, Guizhou Province 550004, P. R. China
E-mail address: huhuabi@hotmail.com