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Robust Optimization: Concise Review of Current Status and Issues

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ABSTRACT

This paper gives a brief review of the theory and applications of the robust optimization. Major issues including tractability, relations with relevant optimization frameworks, and dynamic robust optimization are addressed and future research directions are presented.

Keywords: Robust Optimization, Stochastic Optimization, Survey

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1. INTRODUCTION

Since the late 1990s, robust optimization has established itself as a powerful tool to solve optimization problems under uncertainty. Its success is mainly due to the tractability of the robust optimization counterparts of widely-used optimization models such as linear programming. Especially, the development of efficient (interior point) solution methods for convex optimization problems such as conic optimizations (for example see Boyd and Vandenberghe, 1996 and Ben-Tal and Nemirovski, 2001) built a solid foundation of the vast array of successful applications of the robust optimization.

Early works such as Ben-Tal and Nemirovski (1998, 1999, and 2000) and El Ghaoui *et al*. (1997, 1998) paved the way for theoretical developments and practical applications of the robust optimization. Bertsimas and Sim (2004) extended its applicability to the discrete optimization problems including mixed integer programming models. Ben-Tal *et al*. (2003) presented a framework to deal with robust dynamic (multi-stage) optimization problems.

Applications of the robust optimization approach are diverse. Major application areas include portfolio optimization (Goldfarb and Iyengar, 2003; Tütüncü and Koenig, 2004; Fabozzi *et al*., 2010), statistics and learning (Xu *et al*., 2010; Ben-Tal *et al*., 2011), supply chain management (Bertsimas and Thiele, 2006; Ben-Tal *et al*., 2005), revenue management (Gao *et al*., 2009; Rusmevichientong and Topaloglu, 2012), facility location (Baron and Milner, 2011), antenna design (Lorenz and Boyd, 2005), and unit commitment problem (Jiang *et al*., 2011). For a comprehensive survey on the applications of the robust optimization approach, see Bertsimas *et al*. (2011) and Gabrel *et al*. (2014).

The purpose of this paper is to give a concise overview on the current status of robust optimization and future research directions to help in grasping important ideas of the robust optimization. After presenting some basic materials in this section, we explore important aspects of the robust optimization such as tractability, relations with other optimization frameworks, choice of the uncertainty set, and robust multi-stage optimization. Then directions of the future research are given as concluding remarks.

1.1 Robust Optimization Problem: Definition

Suppose we want to determine the value of decision variables $\mathbf{x} \in R^n$ to minimize an objective function $f(\mathbf{x}, \mathbf{u})$. The vector $\mathbf{u} \in R^L$ is a problem data whose value is not known precisely at the time of decision making. Rather we only know that $\mathbf{u}$ takes its value in a set $U \subset R^L$, called an *uncertainty set*. Then the problem of minimizing $f(\mathbf{x}, \mathbf{u})$ becomes problematic, since it is not a single function but a family of functions corresponding to each value $\mathbf{u} \in U$. Suppose we are interested in finding a solution which has the best performance guarantee independently of the realizations of the uncertain data. Then the problem can be cast as $\min_{\mathbf{x}\in X} \max_{\mathbf{u}\in U} f(\mathbf{x}, \mathbf{u})$, where $X \subset R^n$ is a set of feasible solutions. The resulting optimal solution is said to be robust since its guaranteed (worst-case) value does not depend on the possible realizations of the uncertain data. In a similar manner, to be robust, if there is uncertainty in the data of a constraint (for instance, $g(\mathbf{x}, \mathbf{u}) \le 0$) and the value of the uncertain data belongs to a set, the solution should be required to be feasible *for all possible realizations of the data*. Such a solution is called *robust feasible*.

Now we can define the robust optimization problem as follows: *Find a robust feasible solution which has the best guaranteed objective value independently of the realizations of the uncertain data*. Mathematically, a general form of robust optimization problems can be stated as follows.

$$\min_{\mathbf{x}\in X}\{f(\mathbf{x}): f_i(\mathbf{x}, \mathbf{u_i}) \le 0, \text{ for all } \mathbf{u_i} \in U_i,\ i = 1, \cdots, m\}. \qquad (1)$$

In the above formulation, the objective function is assumed to be certain. This is without loss of generality since if there is uncertainty in the objective function, it can be moved into the set of constraints by introducing an additional variable. Note that the uncertainty set (U_i) is defined constraint-wise.

1.2 Motivation of the Robust Optimization Approach

The robust optimization approach has been proposed to handle uncertainty inherent in many decision making problems. Usually, the uncertainty results from various types of errors in specifying the problem data. For example, if the data represents future state such as market demand or return of an asset, their value should be estimated, which results in prediction error. Even when the value of the data (for example, physical dimensions of technological devices) can be measured, measurement error can occur. Moreover, there frequently is an implementation error in the implementation phase, which means that the solution cannot be implemented exactly as specified. Note that the implementation error can be equivalently viewed as an error in the problem data. Hence if we make decisions based on the nominal (usually average) value, the resulting solution can be suboptimal and even worse, infeasible. So when the uncertainty is relevant, it is crucial to immunize the solution to prevent devastating outcomes, which calls for robustness of solutions.

Stochastic optimization has been used as a main vehicle to incorporate uncertainty in decision making problem. In this setting, the uncertainty is represented as a random variable whose distribution function is known. Though versatile in its applicability, stochastic optimization can be applied only when the relevant distribution function can be estimated very accurately, which may not be the case in many practical problems. Another drawback is that the resulting optimization problem usually is intractable. Also note that in some case, the nature of uncertainty is not stochastic. For instance, the uncertainty due to the measurement error is not stochastic (random) in general.

Besides minimal requirement for the accurate data, in many problems of interest, the robust optimization approach results in tractable robust counterpart (to be discussed in the next section). However, the approach can be very conservative in its nature. Hence it should not replace the stochastic optimization approach but should be used in a complementary way.

1.3 Main Issues in Robust Optimization

As in other optimization approach, there are many theoretical and practical issues in the robust optimization. The following is a summary of the main issues, each of which will be discussed in some detail in the following sections.

Tractability of the Robust Counterpart The tractability of the robust counterpart (robust optimization problem version of a nominal deterministic problem) depends both on the structure of the nominal mathematical programming model and on the structure of the uncertainty set. For linear optimization, robust counterpart with usually used uncertainty set is almost always tractable. However, this is not the case with other well-known mathematical programming models, such as quadratic optimization and conic optimization. The tractability issues will be discussed in Sections 2 and 3.

Relations with the Stochastic Optimization As mentioned earlier, the stochastic optimization has been served as a main tool to deal with uncertainty relevant in decision making problem. Robust optimization can be used to construct a tractable approximation of stochastic optimization model which in itself is usually intractable. In particular, robust optimization approach can result in a tractable approximation for chance constraints (probabilistic constraint) used in stochastic optimizations. This issue will be treated in Section 4. In addition, distribution robust optimization framework and risk-averse optimization will also be discussed briefly there, to give better understanding on the nature of robust optimization approaches.

Construction of the Uncertainty Set The choice of uncertainty set is naturally problem-dependent. However, general guidelines for constructing uncertainty set are desirable to systematically apply the robust optimization framework to diverse applications. Usual choices of the uncertainty set are discussed in Section 2 with tractability results. General guideline using the concept of coherent risk measure is considered in Section 4.

Dynamic Robust Optimization Many decision problems can be posed in a multi-stage setting, where some decisions can be tailored to the information available at the time of decision-making. Unlike the static (single-stage) case, general dynamic robust optimization problem is intractable even for LP. To get a tractable robust counterpart, some restriction on the form of future-stage decisions is required, which is discussed in Section 5.

2. UNCERTAINTY SETS AND ROBUST LINEAR OPTIMIZATION

In this section, we consider commonly used uncertainty sets and robust linear optimization. Tractability results are given for the robust linear optimization problem with those uncertainty sets.

The uncertainty sets commonly used in practice can be classified into three classes: finite sets of scenarios, norm-based sets (Bertsimas *et al*., 2004) including ellipsoidal uncertainty set (Ben-Tal and Nemirovski, 1999; El Ghaoui *et al*., 1997), and explicitly described sets such as polyhedral uncertainty set. In the following, we will consider each type of uncertainty sets together with robust linear optimization problem incorporated with it. For simplicity of presentation, we assume that a linear constraint $\mathbf{a}^T\mathbf{x} \le b$ is given. The coefficient vector $\mathbf{a}$ is uncertain but the right-hand side scalar b is assumed to be known exactly. Note this is without loss of generality since when b also is uncertain, it can be moved into the left-hand side with an auxiliary variable whose value is fixed at 1.

2.1 Scenario

The simplest way of specifying the uncertainty set is to enumerate all possible realizations of the vector $\mathbf{a}$, that is, $U = \{\mathbf{a}^1, \cdots, \mathbf{a}^L\}$. In this case, the robust constraint $\mathbf{a}^T\mathbf{x} \le b, \forall \mathbf{a} \in U$ is just an array of L linear inequalities each of which corresponds to a realized value of $\mathbf{a}$. Hence the robust optimization counterpart results in another linear programming (LP) problem with its size proportional to the number of scenarios.

2.2 Norm-based Uncertainty Set

Let $\mathbf{a} = \bar{\mathbf{a}} + \mathbf{u}$, where $\bar{\mathbf{a}}$ is a nominal (deterministic) value and $\mathbf{u}$ is a disturbance (uncertain element). A general form of the norm-based uncertainty set is the following:

$$U = \{\mathbf{a} = \bar{\mathbf{a}} + \mathbf{u} : \| B\mathbf{u} \| \le r\}, \tag{2}$$

where B is an invertible matrix and $\|\cdot\|$ is a norm on R^n. The meaning is clear when $B = I$ (identity matrix); the value of $\mathbf{a}$ is within the distance r from the given center $\bar{\mathbf{a}}$. This type of uncertainty set is originated either from statistical inferences on the random data or tolerance used in engineering designs. The matrix B is introduced to reflect dependencies among the uncertain elements. The value of r is used to control the robustness of the solution and is usually called *budget of uncertainty*. Larger value of r results in more robust but more conservative solution. The particular form of the norm functional $\|\cdot\|$ describes the shape of the uncertainty set.

The (semi-infinite, that is, infinite number of constraints with a finite number of variables) constraints $\mathbf{a}^T\mathbf{x} \le b, \forall \mathbf{a} \in U$ are equivalent to a single constraint $\max_{\mathbf{a}\in U} \mathbf{a}^T\mathbf{x} \le b$. Hence to reformulate the problem, we need to consider the following optimization problem:

$$\max_{\mathbf{u}, \|B\mathbf{u}\| \le r} \mathbf{u}^T\mathbf{x} . \tag{3}$$

Note the variable in the above problem is not the decision variable $\mathbf{x}$ but the disturbance vector $\mathbf{u}$. It is known that the optimal value of the above problem is given by $r\|(B^T)^{-1}\mathbf{x}\|_*$, where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. Using these results, the constraints $\mathbf{a}^T\mathbf{x} \le b, \forall \mathbf{a} \in U$ can be replaced with a single constraint

$$\bar{\mathbf{a}}^T\mathbf{x} + r\|(B^T)^{-1}\mathbf{x}\|_* \le b. \tag{4}$$

Commonly used norms are l_p-norms $\|\cdot\|_p$ with $p \ge 1$, where $\|\mathbf{x}\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. When $p = \infty$, l_∞-norm is defined as a limit and so $\|\mathbf{x}\|_\infty = \max_i |x_i|$. The dual norm of $\|\cdot\|_p$ is $\|\cdot\|_q$, where q satisfies $1/p + 1/q = 1$ (when $p = 1$, $q = \infty$).

If l_1-norm or l_∞-norm is used, (4) becomes a set of linear constraints (after an appropriate transform of the variables). Hence in these cases, the robust linear optimization problem results in an ordinary linear programming problem with its size polynomial in the size of the nominal problem. When l_2-norm is used, the problem reduces to a second-order cone programming problem. Both cases can be solved in an efficient way.

Remark 1: The l_2-norm case includes widely-used ellipsoidal uncertainty sets. In this case, the matrix B is a symmetric positive semi-definite matrix.

2.3 Explicitly Described Uncertainty Set

In this case, the uncertainty set is described explicitly by a set of constraints, that is,

$$U = \{\mathbf{a} = \bar{\mathbf{a}} + \mathbf{u} : g_k(\mathbf{u}) \le 0, k = 1, \cdots, K\}. \tag{5}$$

Each of the constraints in the uncertainty set represents a certain relation on the elements of the disturbance, reflecting a priori knowledge on the problem data.

The polyhedral uncertainty set is a usual choice, which can be written as

$$U = \{\mathbf{a} = \bar{\mathbf{a}} + \mathbf{u} : \mathbf{Bu} \le \mathbf{d}\}. \quad (6)$$

To derive an alternative representation of the constraints $\mathbf{a}^T\mathbf{x} \le b,\ \forall \mathbf{a} \in U$, we should focus on the problem $\max\{\mathbf{u}^T\mathbf{x} \mid \mathbf{Bu} \le \mathbf{d}\}$, which is a LP. By the strong duality of the LP, we have $\max\{\mathbf{u}^T\mathbf{x} : \mathbf{Bu} \le \mathbf{d}\} = \min\{\mathbf{v}^T\mathbf{d} : \mathbf{v}^T\mathbf{B} = \mathbf{x},\ \mathbf{v} \ge 0\}$. So $\mathbf{a}^T\mathbf{x} \le b, \forall \mathbf{a} \in U$ is equivalent to a set of const-raints

$$\bar{\mathbf{a}}^T\mathbf{x} + \mathbf{v}^T\mathbf{d} \le b, \quad (7)$$
$$\mathbf{v}^T\mathbf{B} = \mathbf{x},\ \mathbf{v} \ge 0.$$

Note that the robust linear counterpart is another LP with additional variables and constraints whose number is polynomial in the size of the description of the uncertainty set.

An interesting special case is a cardinality-constr-ained uncertainty set (Bertsimas and Sim, 2004). In this setting, each of the coefficients a_j is assumed to take its value in an interval $[\bar{a}_j - \hat{a}_j,\ \bar{a}_j + \hat{a}_j]$, where $\bar{a}_j$ is its nominal value and $\hat{a}_j$ is its maximum possible deviation from the nominal value. The uncertainty set is defined as the set of realizations whose number of maximum deviations is at most Γ. The parameter Γ plays the role of uncertainty budget as in r in (2). The problem to be considered is $\max\{\sum_{j=1}^n \hat{a}_j \mid x_j \mid y_j : \sum_{j=1}^n y_j \le \Gamma,\ y_j \in \{0, 1\},\ \forall j\}$. One can easily see that the integer restriction can be relaxed and so the problem reduces to a LP.

Remark 2: Tractability results of the robust linear optimization with a polyhedral uncertainty set can be generalized to the case of the uncertainty set described by cone constraints. Suppose the uncertainty set is given by $U = \{\mathbf{a} = \bar{\mathbf{a}} + \mathbf{u} : \mathbf{d} - \mathbf{Bu} \in K\}$, where K is a closed convex cone. Then the relevant maximization problem is $\max\{\mathbf{u}^T\mathbf{x} : \mathbf{d} - \mathbf{Bu} \in K\}$. If the uncertainty set is strictly feasible (that is, $\exists \mathbf{u},\ \mathbf{d} - \mathbf{Bu} \in \text{int}(K)$), then cone duality (see Ben-Tal and Nemirovski, 2001) states that

$$\max\{\mathbf{u}^T\mathbf{x} : \mathbf{d} - \mathbf{Bu} \in K\} = \min\{\mathbf{v}^T\mathbf{d} : \mathbf{v}^T\mathbf{B} = \mathbf{x},\ \mathbf{v} \in K^*\},$$

where K^* is the dual cone of K. Since widely used cones (nonnegative orthant, Lorentz cone, and semi-definite cone) are self-dual $(K^* = K)$, the corresponding robust counterpart is a cone program with the same cone used in describing the uncertainty set.

3. TRACTABILITY OF GENERAL ROBUST OPTIMIZATION PROBLEMS

In the previous section, we see that robust linear optimization problem is tractable for all commonly-used uncertainty sets. Unfortunately, this is not the case for other optimization models such as general cone programming. In this section, we first consider tractability issues of general robust optimization problems and then robust cone programming models are discussed with their tractability results.

3.1 Tractability of General Robust Optimization Problems

The feasible set of the general robust optimization problem (1) can be equivalently described as

$$X(U) = \{\mathbf{x} \in X : F_i(\mathbf{x}) \le 0, i = 1, \cdots, m\}, \quad (8)$$

where we denote $F_i(\mathbf{x}) = \max\{f_i(\mathbf{x}, \mathbf{u}_i) : \mathbf{u}_i \in U_i\}$. Since the function $F_i(\mathbf{x})$ is defined as a supremum of a set of functions, it is convex if the functions $f_i(\mathbf{x}, \mathbf{u}_i)$ are convex in $\mathbf{x}$ for all fixed $\mathbf{u}_i \in U_i$. Hence in this case, the feasible set $X(U)$ is convex. However, the convexity of the feasible set in itself does not guarantee tractability of the problem. In general, the tractability of the problem are closely related to the tractability of separation problem for the feasible region; given $\mathbf{x} \in R^n$, either prove the point is feasible or find a hyper plane that separates the point from the feasible region. Hence to be tractable, the set $X(U)$ should be well structured. Of course, the structure of the feasible set depends on the structure of the uncertainty set as well as the form of the constraints in the nominal problem.

To get an insight on the tractability of general robust optimization problems, consider the following essentially linear constraint:

$$\| \mathbf{Ax} - \mathbf{b} \|_1 \le 1,\ \forall \mathbf{b} \in U = \{\mathbf{b} : \mathbf{b} = \mathbf{Du},\ \| \mathbf{u} \|_2 \le 1\}.$$

Suppose we want to check the feasibility of the point $\mathbf{x} = 0$. Since $\max_{\|\mathbf{u}\|_2 \le 1} \| D\mathbf{u} \|_1 = (\max_{\|\mathbf{v}\|_\infty \le 1} \mathbf{v}^T DD^T \mathbf{v})^{1/2}$, the feasibility checking problem is NP-hard (it can be seen as a generalization of the well-known NP-hard max-cut problem). Hence the corresponding robust optimization problem is computationally intractable.

Remark 3: After transforming the above problem into a linear programming problem, one can see that the uncertainty is not defined constraint-wise. In other words, the same uncertainty may affect the feasibility of more than one constraint. Hence in general, if the uncertainty is not constraint-wise, even the robust LP with a simple ellipsoidal uncertainty set may be intractable.

3.2 Robust Conic Optimization

A canonical form of cone programming problems is the following:

$$\min\{\mathbf{c}^T\mathbf{x} : \mathbf{A}_j\mathbf{x} + \mathbf{b}_j \in K_j,\ j = 1, \cdots, m\}, \quad (9)$$

where K_j is a simple convex cone, usually one of the following: nonnegative orthant, Lorentz cone, both defined on the vector space R^{m_j}, or semidefinite cone defined on the space of symmetric matrices (for a general background on the conic optimization, see Ben-Tal and Nemirovski, 2001). Uncertainty can be imposed on the data of the problem; $\mathbf{c}$, $\mathbf{A}_j$, $\mathbf{b}_j$, $j=1, \cdots, m$. The corresponding robust counterpart can be written as:

$$\min\{t : t - \mathbf{c}(\mathbf{u})^T \mathbf{x} \le 0, \mathbf{A}_j(\mathbf{u})\mathbf{x} + \mathbf{b}_j(\mathbf{u}) \in K_j, \quad \forall (\mathbf{c}(\mathbf{u}), (\mathbf{A}_j(\mathbf{u}), \mathbf{b}_j(\mathbf{u}))_{j=1,\cdots,m}) \in U\}. \tag{10}$$

Unfortunately, the robust cone programming problem turns out to be rarely tractable. One notable exception is the second-order cone programming problem (corresponding to the Lorentz cone) with an ellipsoidal uncertainty set (in this case, the robust counterpart is a semi-definite program). However, the problem becomes intractable even when the uncertainty set is polyhedral or an intersection of ellipsoids (Ben-Tal and Nemirovski, 1998, 1999; Ben-Tal *et al*., 2002). For the case of semidefinite program (SDP), the robust counterpart is NP-hard even under simple ellipsoidal uncertainty sets (Ben-Tal and Nemirovski, 1998).

In general, robust counterpart of conic optimization problems is computationally intractable. Hence the research on the robust conic optimization is focused either on finding tractable special cases (for an example of a tractable robust SDP, see Boyd *et al*., 1994) or on finding tight tractable approximations of the robust counterpart (see El Ghaoui *et al*., 1998; Ben-Tal and Nemirovski, 2000; Bertsimas and Sim, 2006).

3.3 Robust Discrete Optimization

Unlike robust linear programming problem, robust discrete optimization problems are usually NP-hard. As discussed in Kouvelis and Yu (1997), the problem of minimizing the maximum shortest path on a graph can be shown to be NP-hard even for the case of only two scenarios for the distance vector.

One notable exception is the result of Bertsimas and Sim (2003). A general discrete optimization problem can be stated as $\min\{\mathbf{c}^T\mathbf{x} : \mathbf{x} \in X\}$, where X is a set of feasible solutions. The uncertainty set proposed in the paper is the following; each parameter has its value in an interval, that is, $c_j \in [\overline{c}_j, \overline{c}_j + d_j]$ but the maximum number of possible deviations from the nominal value $\overline{c}_j$ is restricted to no more than Γ. Under this uncertainty set, the robust discrete optimization problem can be stated as $\min\{\overline{\mathbf{c}}^T\mathbf{x} + \max_{\{S:\, S \subseteq N,\, |S| \le \Gamma\}} \sum_{j \in S} d_j x_j : \mathbf{x} \in X\}$, where $N = \{1, \cdots, n\}$. In this case, the robust counterpart can be shown to be solvable by solving no more than $n+1$ instances of the nominal problem. Hence if the nominal problem can be solved efficiently, the robust counterpart can also be solved efficiently. As an application of the model, Lee *et al*. (2012) applied the result to a telecommunication network design problem with demand uncertainty.

4. RELATIONS WITH CHANCE CONSTRAINTS, DISTRIBUTION-ROBUSTNESS, AND RISK MEASURES

In this section, we consider relations of the robust optimization approach with various other relevant optimization approaches under uncertainty: chance constraints in stochastic optimization models, distribution-robust optimization, and risk measures used in risk-averse optimization models. The results are helpful to get a sound understanding on the nature of the robust optimization approach. Moreover, they can give some insight on how to construct suitable uncertainty sets.

4.1 Chance Constraints: Probability Guarantee and Safe Tractable Approximation

As in the previous section 2, we will focus on a single linear constraint $\mathbf{a}^T\mathbf{x} \le b$ for the sake of simplicity. Suppose $\mathbf{a}$ is a random vector that follows a probability law $\mathbf{P}$. The chance constraint (see Birge and Louveaux, 1997, for background materials) is defined as

$$\text{(ChC)} \qquad \mathbf{P}[\mathbf{a}^T\mathbf{x} > b] \le \epsilon, \tag{11}$$

where ϵ is a small positive number.

The meaning of (ChC) is that we want the solution to be feasible except in the case of some rare events. Though the rationale behind the chance constraint is intuitively clear, it usually renders the problem computationally intractable (usually the feasible solution set is not even convex). Moreover, in many cases, the probability law cannot be known precisely but only partial information on it is available. Hence it is more appropriate to consider the following distribution-ambiguous chance constraint:

$$\max_{\mathbf{P} \in \Im} \mathbf{P}[\mathbf{a}^T\mathbf{x} > b] \le \epsilon, \tag{12}$$

where $\Im$ is a family of probability laws. The meaning of the above constraint is that the solution should be feasible except in the case of some rare events *for all probability laws* in the family.

A relevant issue in robust optimization is the probability guarantee, which is the problem of finding a family $\Im$ of probability measures and a real number ε satisfying

$$\mathbf{a}^T x \le b, \forall \mathbf{a} \in U \Rightarrow \mathbf{P}[\mathbf{a}^T x > b] \le \epsilon, \forall \mathbf{P} \in \Im. \tag{13}$$

If an answer of the above problem can be found, we can use it to construct an approximation (in the form of robust optimization problem) of the (distribution-ambiguous) chance constraint. Moreover, it can serve as a guide to choosing a suitable uncertainty set. There are many results on the probability guarantee of the robust feasible solutions, for example, see Ben-Tal and Nemirovski (2000), Bertsimas and Sim (2004), and Chen *et al*. (2007).

Here we consider the following simple case found in Ben-Tal and Nemirovski (2000). Let $\mathbf{a}_j = (1+\mathbf{u}_j)\bar{\mathbf{a}}_j$, where $\bar{\mathbf{a}}$ is a nominal (deterministic) value and $\mathbf{u}=(\mathbf{u}_j)_{j=1,\cdots,n}$ is a random vector. Assume the random variables $\{\mathbf{u}_i\}_{i=1,\cdots,n}$ are independent with mean 0 and each of which support is $[-1,1]$. Consider the following robust constraint

$$\sum_j \bar{a}_j x_j + r\sqrt{\sum_j \bar{a}_j^2 x_j^2} \le b\ , \tag{14}$$

which is a robust counterpart corresponding to the uncertainty set $U=\{\mathbf{u}:\|\mathbf{u}\|_2 \le r\}$. Then it can be shown that the robust solution violates the chance constraint with probability at most $\exp(-r^2/2)$ for any probability law specified above (that is, independence and finite support). Since the inequality (14) is a second-order cone constraint, the robust counterpart is tractable. Hence we can say that the robust counterpart is a *safe tractable approximation* of the chance constraint (specifically we can set $r=\sqrt{-2\ln\varepsilon}$ to guarantee (12)).

4.2 Distribution-Robustness

Distribution-robustness has been used to cope with the problem of incomplete information on the relevant probability measure. In a broad sense, the distribution-robust optimization can be viewed as a generalization of the robust optimization approach. Formally, the distribution-robust optimization problem can be stated as

$$\min_{\mathbf{x}\in X} \max_{\mathbf{P}\in\Im} \mathbf{E}_{\mathbf{P}}[f(\mathbf{u},\mathbf{x})],$$

where $\Im$ is a family of probability measures and $\mathbf{u}$ is a random vector. $\mathbf{E}_{\mathbf{P}}$ is expectation evaluated with respect to the probability law $\mathbf{P}$. Note the relevant probability law is not specified exactly but it is assumed to be a member of a family of probability measures. A usual choice of the family is a set of probability measures whose moments up to some order are given. The research on the distribution-robust optimization is vast and we only consider some results that are directly related to the results given in the previous sections.

Using the results in Bertsimas and Popescu (2004), it can be shown that

$$\sup \mathbf{P}[\mathbf{a}^T\mathbf{x} > b] = \mathbf{x}^T\mathbf{\Sigma}\mathbf{x}/((b-\boldsymbol{\mu}^T\mathbf{x})_+^2 + \mathbf{x}^T\mathbf{\Sigma}\mathbf{x}),$$

where $\mathbf{a}$ is a random vector with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ and $a_+ = \max(a,0)$. Now consider a (distribution-ambiguous) chance-constraint $\sup \mathbf{P}[\mathbf{a}^T\mathbf{x} > b] \le \epsilon$. Then it can be shown to be equivalent to

$$\mu^T x + \kappa(\epsilon)\sqrt{x^T\Sigma x} \le b\ ,$$

where $\kappa(\varepsilon)=\sqrt{\varepsilon/(1-\varepsilon)}$. It can be easily seen that the above constraint is a robust constraint with an ellipsoidal uncertainty set. Hence the robust optimization with an ellipsoidal uncertainty set can be viewed as a distribution-robust optimization problem with partial moment information. For more general results, see El Ghaoui *et al*. (2003), Popescu (2007) and Xu *et al*. (2010).

4.3 Coherent Risk Measures: Constructing Uncertainty Set from the Preference

Apparently, the choice of the uncertainty set should be problem-specific. However the commonly used uncertainty sets considered in Section 2 are basically ad-hoc. In an effort to present a systematic method of choosing an uncertainty set, Bertsimas and Brown (2009) propose an approach based on the decision-maker's preference. The approach is based on the following result on the representation of coherent risk measure (Artzner *et al*., 1999). A given functional μ on the space of random variables is a coherent risk measure if and only if there exists a family of probability measures $\Im$ such that

$$\mu(X) = \sup_{\mathbf{P}\in\Im} \mathbf{E}_{\mathbf{p}}(-X). \tag{15}$$

Now consider a linear constraint $\mathbf{a}^T\mathbf{x} \ge b$, where $\mathbf{a}$ is a random vector. We can define a risk constraint for it as $\mu(\mathbf{a}^T\mathbf{x}-b)\le 0$, where μ is a coherent risk functional. Then by using the above result (15), we have the following

$$\mu(\mathbf{a}^T\mathbf{x}-b)\le 0 \Leftrightarrow \inf_{\mathbf{P}\in\Im} \mathbf{E}_{\mathbf{p}}[\mathbf{a}^T\mathbf{x}] \ge b. \tag{16}$$

The right-hand side of the above relation (16) can be expressed as a robust constraint

$$\mathbf{u}^T\mathbf{x} \ge b,\ \forall \mathbf{u}\in U = \mathrm{conv}(\{\mathbf{E}_{\mathbf{p}}[\mathbf{a}] : \mathbf{P}\in\Im\}). \tag{17}$$

The above result is particularly useful when the support of $\mathbf{a}$ is a finite set (the case when empirical data is used) and the corresponding family of the probability measures is also finite. For example, the well-known coherent risk measure CVaR (conditional value at risk) induces an uncertainty set that is a bounded polyhedron, see Bertsimas and Brown (2009). This type of results provides a linkage between the uncertainty set and the type of user's risk preference.

A closely-related result is the study of Natarajan *et al*. (2009) where they considered the reverse direction of the relation (16). Specifically, they considered the issue of inferring the form of risk measures from the structure of uncertainty sets.

5. DYNAMIC ROBUST OPTIMIZATION

The problems considered so far are static in their nature. All relevant decisions should be made *before* any uncertain data is realized (also called single-stage problem or here-and-now problem). In some applications, a part of decisions can be made *after* the uncertainty is resolved. In this case, the corresponding decisions can be adapted to the realization, so that they are functions of the realizations (information gathered up to that time). This type of decision making problem is called a dynamic (multi-stage or sequential) optimization problem.

To gain an insight on the multi-stage robust optimization, consider the following two-stage problem:

$$\min\{\mathbf{c}^T\mathbf{x}_1 : \mathbf{A}_1(\mathbf{u})\mathbf{x}_1 + \mathbf{A}_2(\mathbf{u})\mathbf{x}_2(\mathbf{u}) \le \mathbf{b}, \forall \mathbf{u} \in U\}. \qquad (18)$$

The variable $\mathbf{x}_1$ is the first-stage decision whose value should be specified before the uncertainty is resolved. On the other hand, the variable $\mathbf{x}_2(\mathbf{u})$ is the second-stage decision whose value is a function of the realized value **u**, meaning that their value can be determined after observing the data. Hence the value of the second-stage variables can be adapted to the realized values of uncertainty.

Unfortunately, the sequential robust optimization problem (18) is in general NP-hard. Hence research has been focused on the approximation methods of the problem. For instance, Ben-Tal *et al*. (2003) propose an approximation scheme called an *affinely adjustable robust counterpart* (AARC). In this scheme, instead of an arbitrary form of functions, the second-stage decision is restricted to be an affine function of the disturbance (called linear decision rule in control theory) such as

$$\mathbf{x}_2(\mathbf{u}) = \mathbf{Q}\mathbf{u} + \mathbf{q}. \qquad (19)$$

Then with the newly defined decision variables $(\mathbf{Q}, \mathbf{q})$ together with the first-stage variable, the problem (18) reduces to

$$\min\{\mathbf{c}^T\mathbf{x}_1 : \mathbf{A}_1(\mathbf{u})\mathbf{x}_1 + \mathbf{A}_2(\mathbf{u})[\mathbf{Q}\mathbf{u} + \mathbf{q}] \le \mathbf{b},\ \forall \mathbf{u} \in U\}. \qquad (20)$$

However, the general form of the above approximated version still remains to be NP-hard. This is because the structure of the dependency on the random disturbance is quadratic (see the term $\mathbf{A}_2(\mathbf{u})\mathbf{Q}\mathbf{u}$). Hence to avoid such a problem, the second-stage matrix is assumed to be constant, that is, $\mathbf{A}_2(\mathbf{u}) = \bar{\mathbf{A}}_2$ independently of the random disturbance. In stochastic optimization literature, this is the case of *fixed recourse* (see Birge and Louveaux, 1997). Then the problem becomes a basically robust linear optimization problem. Ben-Tal *et al*. (2003) show that the robust counterpart can be cast as a conic optimization problem if the uncertainty set in itself is conic. Erera *et al*. (2009) propose a closely-related scheme for a mixed-integer programming problem with right-hand side uncertainty.

6. FUTURE RESEARCH DIRECTIONS

In the previous sections, we gave a brief overview of known theoretical results on the robust optimization. The followings are the issues that call for more investigation, which can serve as future research directions.

Tractable Approximation of Dynamic Robust Optimization Dynamic optimization models are very important in solving many practical problems. However, as we saw in the previous section, the current results are quite limited. More research is in order to build a successful theoretical foundation comparable to that found in the static robust optimization.

Extending the Meaning of Robustness One natural question on the robust optimization approach is how the robust optimal solution behaves if the problem data falls out of the specified uncertainty set.

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