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Optimistic vs Pessimistic Use of Incomplete Weights in Multiple Criteria Decision Making

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ABSTRACT

This note is concerned with the use of incomplete weights in multiple criteria decision making. In an earlier study, an optimistic use of incomplete weights is developed to prioritize decision alternatives, which applies the most favorable set of weights to the alternative to be evaluated. In this note, we develop a method for a pessimistic use, thereby applying the least favorable weight set to the evaluated alternative. This development makes possible a more detailed prioritization of competing alternatives, and hence enhances decision-making powers.

Keywords: Multiple Criteria Decision Making, Incomplete Weights, Optimistic and Pessimistic Performance Prioritization

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1. INTRODUCTION

In multiple criteria decision making (MCDM), there have been a sizable number of methods to deal with incomplete weights, such as ordinal and bounded forms of relative importance on evaluation criteria, which frequently appear in practical decision-making (Kmietowiz and Pearman, 1984; Weber, 1987; Park, 2004). In this note, we revisit the recent developments by Park and Shin (2011) and Park and Cho (2011). Common to these two papers is to develop a most optimistic approach to the evaluation of decision alternatives involving incomplete weights, so it applies the most favorable set of weights among various feasible weights to the alternative to be evaluated. This development was based on extensions to the micro-economic evaluation concept of Debreu (1951) and Farrell (1957).

In this note, we develop a most pessimistic approach, which applies the least favorable set of weights to the evaluated alternative. It is worth mentioning that the previous optimistic approach tends to render two or more alternatives efficient or non-dominated (Park, 2004; Park and Shin, 2011). Thus it is difficult to choose a single best alternative or rank-order those competing alternatives. The current pessimistic approach helps to overcome this difficulty, because it makes possible a more detailed prioritization of such competing alternatives.

In the next section, the optimistic approach is briefly described. We then develop the pessimistic approach, followed by an illustration along with graphical interpretations.

2. OPTIMISTIC APPROACH

Let there be n alternatives $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$, each of which is described by a vector $\mathbf{x} = (x_1, \cdots, x_p)$ of consequences on each criterion $j = 1, \cdots, p$. Let $\mathbf{w} = (w_1, \cdots, w_p)^T$ be the vector of weights w_j. The point is that a set of exact weights cannot be identified in many situations (Park, 2004). We may frequently have incomplete weights, such as ordinal information ($w_1 \geq w_2 \geq \cdots \geq w_p$) and/or ratio bounds ($c^- \leq w_k/w_j \leq c^+$, where c^- and c^+ respectively are the lower and upper bounds) on the marginal rate of substitution for the two criteria k and j. We can therefore assume that the set of admissible weights W is specified by a system of homogeneous linear inequalities, $W = \{\mathbf{w}|\mathbf{A}\mathbf{w} \leq \mathbf{0}, \mathbf{w} \geq \mathbf{0}\}$, where $\mathbf{A}$ is a $q \times p$ matrix composed of q row vectors $\mathbf{a}_1, \cdots, \mathbf{a}_q$.

To measure the optimistic score of alternative $\mathbf{x}_o$, the following linear program has previously been developed:

$$\alpha_o = \max\ \alpha \tag{1}$$

subject to

$$\sum_{i=1}^{n}\mu_i\mathbf{x}_i + \sum_{k=1}^{q}t_k\mathbf{a}_k^{\mathrm{T}} \geq \alpha\mathbf{x}_o \tag{2}$$

$$\sum_{i=1}^{n}\mu_i = 1 \tag{3}$$

$$\mu_i \geq 0\ \forall i; \quad t_k \geq 0\ \forall k \tag{4}$$

where $\alpha \geq 1$. A smaller α_o value is better, and hence the $\mathbf{x}_o$ alternatives having $\alpha_o = 1$ become efficient. Note that α_o is obtained from an assessment in the light of the most favorable weight scenario for $\mathbf{x}_o$.

3. PESSIMISTIC APPROACH

To begin with, we analyze model (1). The feasible region of model (1) can compactly be represented by $\alpha\mathbf{x}_o \in \Omega(X, \mathbf{A}) = C(X) + F(\mathbf{A})$,where

$$C(X) = \left\{\mathbf{x} \in \Re^p \mid \mathbf{x} = \sum_{i=1}^{n}\mu_i\mathbf{x}_i, \sum_{i=1}^{n}\mu_i = 1, \forall\mu_i \geq 0\right\}$$

$$F(\mathbf{A}) = \{\mathbf{f} \in \Re^p \mid \mathbf{f} = \sum_{k=1}^{q}t_k\,\mathbf{a}_k^{\mathrm{T}},\ \forall t_k \geq 0\}$$

The $C(X)$ set is the convex hull of X, and the $F(\mathbf{A})$ set is a convex polyhedral cone and the polar cone of W. The points $\mathbf{a}_k^{\mathrm{T}}$ can be thought of as the generaters of $F(\mathbf{A})$, and each of them generates a half-ray, $H_k = t_k\mathbf{a}_k^{\mathrm{T}}$ with $t_k \geq 0$.The key observation is that these half-rays are the determinant of the optimistic score α_o. That is, model (1) selects at least one H_k that represents the efficient facet of $\mathbf{x}_o$, and then compares the efficient facet and $\mathbf{x}_o$, there by yielding α_o. This observation implies that α_o is obtained from the limited sources of efficient frontier, half-rays H_k, which are formed by the favorable weight scenarios for $\mathbf{x}_o$.

To measure a pessimistic score, we therefore allow the unfavorable weight scenarios for $\mathbf{x}_o$. This leads us to consider rays, $R_k = t_k\mathbf{a}_k^{\mathrm{T}}$ with t_k to be sign free, which expands the limited efficient frontier used for measuring α_o. We then need to select one of the rays R_k that evaluates alternative $\mathbf{x}_o$ most pessimistically. This idea can be generalized in the following form of linear programs: For each $r = 1, \cdots, q$,

$$\beta_r = \max\ \beta \tag{5}$$

subject to

$$\sum_{i=1}^{n}\mu_i\mathbf{x}_i + \sum_{k=1}^{q}t_k\mathbf{a}_k^{\mathrm{T}} \geq \beta\mathbf{x}_o \tag{6}$$

$$\sum_{i=1}^{n}\mu_i = 1 \tag{7}$$

$$\mu_i \geq 0\ \forall i \tag{8}$$

$$t_k \geq 0\ \forall\ k \neq r;\ t_k \text{ free if } k = r \tag{9}$$

Solving these q linear programs, we can obtain the pessimistic score of $\mathbf{x}_o$, $\beta_o = \max\ \{\beta_1, \cdots, \beta_q\}$.The only difference between models (1) and (2) is that $t_r$is sign free in (2).

4. ILLUSTRATIVE EXAMPLE

Suppose that we have five alternatives under evaluation as shown in Table 1. In this evaluation, we assume incomplete weight information, $1 \leq w_2/w_1 \leq 3$. We then have

$$\mathbf{A} = \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{pmatrix} = \begin{pmatrix}1 & -1\\ -3 & 1\end{pmatrix}$$

First, using model (1), we evaluate $\mathbf{x}_1 = (2, 5)$ optimistically as follows:

$$\begin{aligned}&\alpha_o = \max\ \alpha \text{ subject to}\\ &2\mu_1 + 5\mu_2 + 3\mu_3 + 1\mu_4 + 4\mu_5 + 1t_1 - 3t_2 \geq 2\alpha\\ &5\mu_1 + 2\mu_2 + 3\mu_3 + 5\mu_4 + 1\mu_5 - 1t_1 + 1t_2 \geq 5\alpha\\ &\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 = 1\end{aligned}$$

with all variables to be nonnegative. Solving this we have $\alpha_o = 1$. Carrying out this task for the other alternatives, we obtain all the optimistic scores in Table 1. As can be seen, alternatives $\mathbf{x}_1$ and $\mathbf{x}_2$ appear to be efficient.

Table 1. An Example for Evaluating Five Alternatives under Two Criteria

Alternatives		$\mathbf{x}_1$	$\mathbf{x}_2$	$\mathbf{x}_3$	$\mathbf{x}_4$	$\mathbf{x}_5$
Data	x_1	2	5	3	1	4
	x_2	5	2	3	5	1
Optimistic	α_o	1	1	1.17	1.06	1.40
Pessimistic	β_o	1	1.55	1.42	1.17	2.43

Now using model (2), we evaluate $\mathbf{x}_1 = (2, 5)$ pessimistically as follows:

$$\begin{aligned}&\beta_1 = \max\ \beta \text{ subject to}\\ &2\mu_1 + 5\mu_2 + 3\mu_3 + 1\mu_4 + 4\mu_5 + 1t_1 - 3t_2 \geq 2\beta\\ &5\mu_1 + 2\mu_2 + 3\mu_3 + 5\mu_4 + 1\mu_5 - 1t_1 + 1t_2 \geq 5\beta\\ &\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 = 1\\ &t_1 \text{ free}\end{aligned}$$

with all variables to be nonnegative except for β and t_1. Solving this we have $\beta_1 = 1$. Now replacing β_1 and t_1 free respectively with β_2 and t_2 free in the immediately above problem, we then have $\beta_2 = 1$. Thus, the pessimistic score of $\mathbf{x}_1$ becomes $\beta_o = 1 = \max\{1, 1\}$. Carrying out these operations on the other alternatives, we achieve the last row of Table 1. Only $\mathbf{x}_1$ becomes efficient, and $\mathbf{x}_2$, which was efficient under the optimistic evaluation, now turns out to be inefficient. Therefore, the choice-making of a single best alternative succeeds in this example.

Moreover, having obtained both optimistic and pessimistic scores, we can provide a more informative evaluation result. We can classify alternatives into three categories: When $\alpha_o = \beta_o = 1$, this alternative is the best performer because it is always efficient, meaning that it is efficient under both optimistic and pessimistic evaluations ($\mathbf{x}_1$ is the case in the above example). When $\alpha_o = 1$ but $\beta_o > 1$, this alternative is the second best performer because it is sometimes efficient ($\mathbf{x}_2$ is the case). When $\alpha_o > 1$ and $\beta_o > 1$, this alternative is the least performer because it is always inefficient.

Finally, we provide graphical interpretations of models (1) and (2). For model (1), Figure 1 shows how the set $\Omega(X, \mathbf{A}) = C(X) + F(\mathbf{A})$ is constructed on the left, and the resulting value efficient frontier on the right. As an example, the optimistic score of $\mathbf{x}_3$ is the ratio $d(\mathbf{0}, \mathbf{x}_3')/d(\mathbf{0}, \mathbf{x}_3)$ which was 1.17 in Table 1. Here, d is the Euclidean distance function. It is obvious that the optimistic scores of $\mathbf{x}_1$ and $\mathbf{x}_2$ are all unity.

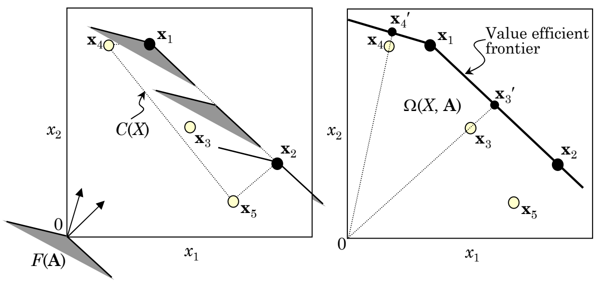

Figure 1. Graphical Interpretation of the Optimistic Evaluation

Figure 2 shows a graphical interpretation of model (2). We have two rays R_1 and R_2. The R_1 that is associated with $\mathbf{a}_1 = (1, -1)$ serves a value efficient frontier, when we use the weights satisfying $w_2/w_1 = 1$ that is feasible for the given weights $1 \le w_2/w_1 \le 3$. The R_2 directs to $\mathbf{a}_2 = (-3, 1)$, corresponding to the feasible weights satisfying $w_2/w_1 = 3$. Now consider alternative $\mathbf{x}_3$. One performance score is the ratio $d(\mathbf{0}, \mathbf{x}_3^1)/d(\mathbf{0}, \mathbf{x}_3)$ when using R_1. Another is $d(\mathbf{0}, \mathbf{x}_3^2)/d(\mathbf{0}, \mathbf{x}_3)$ when using R_2. The maximum of the two values becomes the pessimistic score of $\mathbf{x}_3$, which is $d(\mathbf{0}, \mathbf{x}_3^2)/d(\mathbf{0}, \mathbf{x}_3)$. It is apparent that only $\mathbf{x}_1$ is efficient.

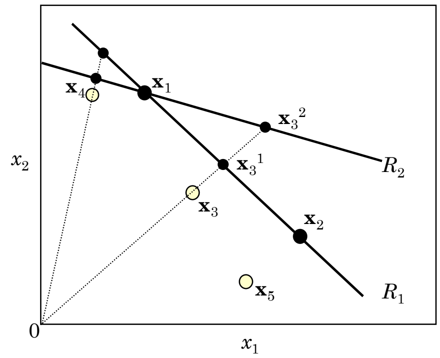

Figure 2. Graphical Interpretation of the Pessimistic Evaluation

REFERENCES

Debreu, G., "The coefficient of resource utilization," *Econometrica* 19 (1951), 273-292.

Farrell, M. J., "The measurement of productive efficiency," *Journal of the Royal Statistical Society, Series A* 120 (1957), 253-281.

Kmietowitz, Z. W. and A. D. Pearman, "Decision theory, linear partial information and statistical dominance," *Omega* 12 (1984), 391-399.

Park, K. S., "Mathematical programming models for characterizing dominance and potential optimality when multicriteria alternative values and weights are simultaneously incomplete," *IEEE Trans. Syst., Man, Cybern., Part A* 34 (2004), 601-614.

Park, K. S. and J. W. Cho, "Pro-efficiency: data speak more than technical efficiency," *European Journal of Operational Research* 215 (2011), 301-308.

Park, K. S. and D. E. Shin, "How to deal with imprecise weights in multiple criteria decision making," *Global Business and Management Research: An International Journal* 3 (2011), 200-214.

Weber, M., "Decision making with incomplete information," *European Journal of Operational Research* 28 (1987), 44-57.