

MEAN VALUES OF THE HOMOGENEOUS DEDEKIND SUMS

XIAOYING WANG AND XIAXIA YUE

ABSTRACT. Let a, b, q be integers with $q > 0$. The homogeneous Dedekind sum is defined by

$$S(a, b, q) = \sum_{r=1}^q \left(\left(\frac{ar}{q} \right) \right) \left(\left(\frac{br}{q} \right) \right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

In this paper we study the mean value of $S(a, b, q)$ by using mean value theorems of Dirichlet L -functions, and give some asymptotic formula.

1. Introduction

For integers a and $q > 0$, the classical Dedekind sum is defined by

$$S(a, q) = \sum_{r=1}^q \left(\left(\frac{r}{q} \right) \right) \left(\left(\frac{ar}{q} \right) \right),$$

Received September 1, 2015. Revised October 13, 2015. Accepted October 15, 2015.

2010 Mathematics Subject Classification: 11F20.

Key words and phrases: Dedekind sum, character sum, Dirichlet L -function mean value.

This work was supported by the Science and Technology Program of Shaanxi Province of China under Grant No. 2013JM1017.

© The Kangwon-Kyungki Mathematical Society, 2015.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer,} \\ 0, & \text{if } x \text{ is an integer.} \end{cases}$$

The sum $S(a, q)$ plays an important role in the transformation theory of the Dedekind η function (see [6] and Chapter 3 of [1] for details).

Let a, b, q be integers with $q > 0$. According to [2], the homogeneous Dedekind sum is defined by

$$S(a, b, q) = \sum_{r=1}^q \left(\left(\frac{ar}{q} \right) \right) \left(\left(\frac{br}{q} \right) \right).$$

Z. Zheng [9] proved a generalized Knopp’s identity for $S(a, b, q)$ as following:

$$\sum_{d|n} \sum_{r_1=1}^d \sum_{r_2=1}^d S\left(\frac{n}{d}a + r_1q, \frac{n}{d}b + r_2q, dq\right) = n\sigma(n)S(a, b, q),$$

where n is a positive integer, and $\sigma(n) = \sum_{d|n} d$.

H. Liu [4] studied the mean values of the homogeneous Dedekind sums in short intervals $\left[1, \frac{p}{3}\right]$ and $\left[1, \frac{p}{4}\right]$, and showed some asymptotic formulae.

PROPOSITION 1.1. *Let $p \geq 5$ be a prime. We have*

$$\begin{aligned} \sum_{a \leq \frac{p}{3}} \sum_{b \leq \frac{p}{3}} S(a, b, p) &= \frac{1}{18}p^2 + O(p^{1+\epsilon}), \\ \sum_{a \leq \frac{p}{4}} \sum_{b \leq \frac{p}{4}} S(a, b, p) &= \frac{131}{2304}p^2 + O(p^{1+\epsilon}), \\ \sum_{a \leq \frac{p}{5}} \sum_{b \leq \frac{p}{4}} S(a, b, p) &= \frac{3}{64}p^2 + O(p^{1+\epsilon}). \end{aligned}$$

In this paper we further study the mean values of the homogeneous Dedekind sums, and give some formulae. Our main results are the following:

THEOREM 1.2. *Let $p \geq 5$ be a prime. We have*

$$\sum_{a \leq \frac{p}{3}} \sum_{b \leq \frac{p}{3}} abS(a, b, p) = \left(\frac{1}{1296} + \frac{\sqrt{3}L^2(3, \chi_3)}{1296\pi L(5, \chi_3)} + \frac{135L^2(3, \chi_3)}{832\pi^6} \right) p^4 + (p^{3+\epsilon}),$$

$$\sum_{a \leq \frac{p}{4}} \sum_{b \leq \frac{p}{4}} abS(a, b, p) = \left(\frac{1}{4608} + \frac{L^2(3, \chi_4)}{2304\pi L(5, \chi_4)} + \frac{5L^2(3, \chi_4)}{24\pi^6} \right) p^4 + (p^{3+\epsilon}),$$

$$\sum_{a \leq \frac{p}{3}} \sum_{b \leq \frac{p}{4}} abS(a, b, p) = \left(\frac{1}{13824} + \frac{13\sqrt{3}L^2(3, \chi_3)}{202752\pi L(5, \chi_3)} + \frac{23L^2(3, \chi_4)}{26352\pi L(5, \chi_4)} \right. \\ \left. + \frac{\sqrt{3}L(3, \chi_3)L(3, \chi_4)L(4, \chi_3\chi_4)}{96\pi^4 L(6, \chi_3\chi_4)} \right) p^4 + O(p^{3+\epsilon}),$$

where χ_3 is the non-principal character modulo 3, χ_4 is the non-principal character modulo 4, and $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ is the Dirichlet L -function.

2. Identities involving certain Dirichlet series

THEOREM 2.1. *Let k and l be fixed non-negative integers, and let $d(n)$ be the divisor function. Then we have*

$$\sum_{n=1}^{\infty} \frac{d(2^k n)d(3^l n)}{n^2} = \frac{(3k + 5)(4l + 5)\pi^4}{360}. \tag{2.1}$$

Proof. By the property of the divisor function we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d(2^k n)d(3^l n)}{n^2} &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{n=1 \\ 2^i | n, 2^{i+1} \nmid n \\ 3^j | n, 3^{j+1} \nmid n}} \frac{d(2^k n)d(3^l n)}{n^2} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{n=1 \\ (n,6)=1}} \frac{d(2^{k+i} 3^j n)d(2^i 3^{l+j} n)}{(2^i 3^j n)^2} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{d(2^{k+i})d(3^j)d(2^i)d(3^{l+j})}{2^{2i} 3^{2j}} \sum_{\substack{n=1 \\ (n,6)=1}}^{\infty} \frac{d^2(n)}{n^2} \\ &= \sum_{i=0}^{\infty} \frac{(k+i+1)(i+1)}{2^{2i}} \sum_{j=0}^{\infty} \frac{(j+1)(l+j+1)}{3^{2j}} \sum_{\substack{n=1 \\ (n,6)=1}}^{\infty} \frac{d^2(n)}{n^2}. \end{aligned}$$

It is not hard to show that

$$\sum_{i=0}^{\infty} \frac{(k+i+1)(i+1)}{2^{2i}} = \frac{16(3k+5)}{27} \quad \text{and}$$

$$\sum_{j=0}^{\infty} \frac{(j+1)(l+j+1)}{3^{2j}} = \frac{81(4l+5)}{256}.$$

On the other hand, from Euler product formula we have

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,6)=1}}^{\infty} \frac{d^2(n)}{n^2} &= \prod_{p \nmid 6} \left(1 + \frac{d^2(p)}{p^2} + \frac{d^2(p^2)}{p^4} + \cdots \right) \\ &= \prod_{p \nmid 6} \left(1 + \frac{2^2}{p^2} + \frac{3^2}{p^4} + \cdots \right) \\ &= \prod_{p \nmid 6} \frac{1 + \frac{1}{p^2}}{\left(1 - \frac{1}{p^2}\right)^3} = \prod_p \frac{1 + \frac{1}{p^2}}{\left(1 - \frac{1}{p^2}\right)^3} \prod_{p \mid 6} \frac{\left(1 - \frac{1}{p^2}\right)^3}{1 + \frac{1}{p^2}} \\ &= \frac{\zeta^4(2)}{\zeta(4)} \cdot \frac{16}{75} = \frac{2\pi^4}{135}, \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function satisfying $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$. Therefore

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d(2^k n)d(3^l n)}{n^2} &= \frac{16(3k+5)}{27} \cdot \frac{81(4l+5)}{256} \cdot \frac{2\pi^4}{135} \\ &= \frac{(3k+5)(4l+5)\pi^4}{360}. \end{aligned}$$

□

THEOREM 2.2. *Let $p \geq 5$ be a prime, and let k be fixed non-negative integer. Suppose that χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4. Define $\tau(n) = \sum_{t|n} \frac{\chi_3(t)}{t}$ and*

$\rho(n) = \sum_{t|n} \frac{\chi_4(t)}{t}$. Then we have

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2} = \frac{45L^2(3, \chi_3)}{26}, \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{\rho^2(n)}{n^2} = \frac{5L^2(3, \chi_4)}{3}, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \frac{d(n)\tau(2^k n)}{n^2} = \left(1 + \frac{(-1)^k}{2^{k-1} \cdot 9}\right) \frac{\pi^4 L^2(3, \chi_3)}{44L(5, \chi_3)}, \tag{2.4}$$

$$\sum_{n=1}^{\infty} \frac{d(n)\rho(3^k n)}{n^2} = \left(\frac{1}{4} + \frac{(-1)^k}{3^{k-1} \cdot 49}\right) \frac{49\pi^4 L^2(3, \chi_4)}{549L(5, \chi_4)}, \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)\rho(n)}{n^2} = \frac{\pi^2 L(3, \chi_3)L(3, \chi_4)L(4, \chi_3\chi_4)}{6L(6, \chi_3\chi_4)}. \tag{2.6}$$

Proof. We only prove (2.2) since similarly we can get others. Noting that $\tau(n)$ is a multiplicative function, we can write

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2} = \sum_{j=0}^{\infty} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(3^j n)}{(3^j n)^2} = \sum_{j=0}^{\infty} \frac{1}{9^j} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2} = \frac{9}{8} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2}.$$

For the summation

$$\sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2},$$

by using the Euler product formula we have

$$\begin{aligned} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2} &= \prod_{p \neq 3} \left(1 + \frac{\tau^2(p)}{p^2} + \frac{\tau^2(p^2)}{p^4} + \dots\right) \\ &= \prod_{p \neq 3} \left(1 + \left(\frac{1 + \frac{\chi_3(p)}{p}}{p}\right)^2 + \left(\frac{1 + \frac{\chi_3(p)}{p} + \left(\frac{\chi_3(p)}{p}\right)^2}{p^2}\right)^2 + \dots\right) \end{aligned}$$

$$\begin{aligned}
&= \prod_{p \neq 3} \sum_{n=0}^{\infty} \left(\frac{1 + \frac{\chi_3(p)}{p} + \left(\frac{\chi_3(p)}{p}\right)^2 + \dots + \left(\frac{\chi_3(p)}{p}\right)^n}{p^n} \right)^2 \\
&= \prod_{p \neq 3} \sum_{n=0}^{\infty} \left(\frac{\frac{1 - \left(\frac{\chi_3(p)}{p}\right)^{n+1}}{\frac{1 - \chi_3(p)}{p}}}{p^n} \right)^2 \\
&= \prod_{p \neq 3} \frac{1}{\left(1 - \frac{\chi_3(p)}{p}\right)^2} \sum_{n=0}^{\infty} \left(\frac{1}{p^{2n}} - \frac{2(\chi_3(p))^{n+1}}{p^{3n+1}} + \frac{1}{p^{4n+2}} \right) \\
&= \prod_{p \neq 3} \frac{1}{\left(1 - \frac{\chi_3(p)}{p}\right)^2} \left(\frac{p^2}{p^2 - 1} - \frac{2\chi_3(p)p^2}{p^3 - \chi_3(p)} + \frac{p^2}{p^4 - 1} \right) \\
&= \prod_{p \neq 3} \frac{p^2}{\left(1 - \frac{\chi_3(p)}{p}\right)^2} \times \frac{(p^2 + 1)(p^3 - \chi_3(p)) - 2\chi_3(p)(p^4 - 1) + p^3 - \chi_3(p)}{(p^4 - 1)(p^3 - \chi_3(p))} \\
&= \prod_{p \neq 3} \frac{p^4}{\left(1 - \frac{\chi_3(p)}{p}\right)^2} \times \frac{p^3 - 2p^2\chi_3(p) + 2p - \chi_3(p)}{(p^4 - 1)(p^3 - \chi_3(p))} \\
&= \prod_{p \neq 3} \frac{p^6(p - \chi_3(p))(p + \chi_3(p))(p^2 - \chi_3(p)p + (\chi_3(p))^2)}{(p - \chi_3(p))^2(p + \chi_3(p))(p^4 - 1)(p^3 - \chi_3(p))} \\
&= \prod_{p \neq 3} \frac{p^6(p^3 + \chi_3(p))}{(p^2 - 1)(p^4 - 1)(p^3 - \chi_3(p))} = \prod_{p \neq 3} \frac{1 - \frac{1}{p^6}}{\left(1 - \frac{1}{p^2}\right)\left(1 - \frac{1}{p^4}\right)\left(1 - \frac{\chi_3(p)}{p^3}\right)^2} \\
&= \frac{80\zeta(2)\zeta(4)L^2(3, \chi_3)}{91\zeta(6)} = \frac{20L^2(3, \chi_3)}{13},
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function satisfying $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$. Therefore

$$\sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2} = \frac{9}{8} \sum_{\substack{n=1 \\ (n,3)=1}}^{\infty} \frac{\tau^2(n)}{n^2} = \frac{45L^2(3, \chi_3)}{26}.$$

This proves (2.2). □

3. Mean values of the Dirichlet L - functions

In this section, we shall prove some mean values of the Dirichlet L - functions, which will be used to prove Theorem 1.1.

THEOREM 3.1. *Let $p \geq 5$ be a prime, and let k and l be fixed non-negative integers. Then we have*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\bar{\chi}(3^l) |L(1, \chi)|^4 = \frac{(3k + 5)(4l + 5)\pi^4}{2^k 3^l \cdot 720} p + O(p^\epsilon),$$

where $L(s, \chi) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$ is the Dirichlet L -function, and $\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}$ denotes the summation over all odd Dirichlet characters modulo p .

Proof. Let N be an integer with $p \leq N < p^4$, and let $d(n)$ be the divisor function. By Abel's identity we have

$$L^2(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)d(n)}{n} = \sum_{n=1}^N \frac{\chi(n)d(n)}{n} + \int_N^{\infty} \frac{\sum_{N < n \leq y} \chi(n)d(n)}{y^2} dy.$$

Hence,

$$\begin{aligned} & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\bar{\chi}(3^l) |L(1, \chi)|^4 \\ &= \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{n_1=1}^N \frac{\chi(n_1)d(n_1)}{n_1} + \int_N^{\infty} \frac{\sum_{N < n_1 \leq y_1} \chi(n_1)d(n_1)}{y_1^2} dy_1 \right) \\ & \quad \times \bar{\chi}(3^l) \left(\sum_{n_2=1}^N \frac{\bar{\chi}(n_2)d(n_2)}{n_2} + \int_N^{\infty} \frac{\sum_{N < n_2 \leq y_2} \bar{\chi}(n_2)d(n_2)}{y_2^2} dy_2 \right) \\ &= \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{n_1=1}^N \frac{\chi(n_1)d(n_1)}{n_1} \right) \bar{\chi}(3^l) \left(\sum_{n_2=1}^N \frac{\bar{\chi}(n_2)d(n_2)}{n_2} \right) \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{n_1=1}^N \frac{\chi(n_1)d(n_1)}{n_1} \right) \\
& \qquad \qquad \qquad \bar{\chi}(3^l) \left(\int_N^\infty \frac{\sum_{N < n_2 \leq y_2} \bar{\chi}(n_2)d(n_2)}{y_2^2} dy_2 \right) \\
& + \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\int_N^\infty \frac{\sum_{N < n_1 \leq y_1} \chi(n_1)d(n_1)}{y_1^2} dy_1 \right) \\
& \qquad \qquad \qquad \bar{\chi}(3^l) \left(\sum_{n_2=1}^N \frac{\bar{\chi}(n_2)d(n_2)}{n_2} \right) \\
& + \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) \left(\int_N^\infty \frac{\sum_{N < n_1 \leq y_1} \chi(n_1)d(n_1)}{y_1^2} dy_1 \right) \\
& \qquad \qquad \qquad \bar{\chi}(3^l) \left(\int_N^\infty \frac{\sum_{N < n_2 \leq y_2} \bar{\chi}(n_2)d(n_2)}{y_2^2} dy_2 \right) \\
& := M_1 + M_2 + M_3 + M_4. \tag{3.1}
\end{aligned}$$

From the Pólya-Vinogradov inequality we get

$$\begin{aligned}
\sum_{N < n \leq y} \chi(n)d(n) &= \sum_{N < nm \leq y} \chi(nm) \\
&= 2 \sum_{n \leq \sqrt{y}} \chi(n) \sum_{m \leq \frac{y}{n}} \chi(m) - \left(\sum_{n \leq \sqrt{y}} \chi(n) \right)^2
\end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{n \leq \sqrt{N}} \chi(n) \sum_{m \leq \frac{y}{n}} \chi(m) + \left(\sum_{n \leq \sqrt{N}} \chi(n) \right)^2 \\
 & \ll y^{\frac{1}{2}} p^{\frac{1}{2}} \log p + p(\log p)^2 + N^{\frac{1}{2}} p^{\frac{1}{2}} \log p + p(\log p)^2 \\
 & \ll y^{\frac{1}{2}} p^{\frac{1}{2}} (\log p)^2.
 \end{aligned}$$

So we have

$$M_2 \ll \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \sum_{n_1=1}^N \frac{d(n_1)}{n_1} \int_N^\infty \frac{y_2^{\frac{1}{2}} p^{\frac{1}{2}} (\log p)^2}{y_2^2} dy_2 \ll p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}, \quad (3.2)$$

$$M_3 \ll \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \int_N^\infty \frac{y_1^{\frac{1}{2}} p^{\frac{1}{2}} (\log p)^2}{y_1^2} dy_1 \sum_{n_2=1}^N \frac{d(n_2)}{n_2} \ll p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}, \quad (3.3)$$

and

$$M_4 \ll \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \int_N^\infty \frac{y_1^{\frac{1}{2}} p^{\frac{1}{2}} (\log p)^2}{y_1^2} dy_1 \int_N^\infty \frac{y_2^{\frac{1}{2}} p^{\frac{1}{2}} (\log p)^2}{y_2^2} dy_2 \ll p^{2+\epsilon} N^{-1}. \quad (3.4)$$

Now from (3.1)-(3.4) and the orthogonality relations for characters we get

$$\begin{aligned}
 & \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \chi(2^k) \bar{\chi}(3^l) |L(1, \chi)|^4 \\
 & = \sum_{n_1=1}^N \frac{d(n_1)}{n_1} \sum_{n_2=1}^N \frac{d(n_2)}{n_2} \sum_{\substack{\chi \pmod{p} \\ \chi(-1)=-1}} \chi(2^k n_1) \bar{\chi}(3^l n_2) \\
 & \quad + O\left(p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}\right) + O\left(p^{2+\epsilon} N^{-1}\right) \\
 & = \frac{1}{2} \sum_{n_1=1}^N \frac{d(n_1)}{n_1} \sum_{n_2=1}^N \frac{d(n_2)}{n_2} \sum_{\chi \pmod{p}} (1 - \chi(-1)) \chi(2^k n_1) \bar{\chi}(3^l n_2) \\
 & \quad + O\left(p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}\right) + O\left(p^{2+\epsilon} N^{-1}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{p-1}{2} \sum_{\substack{n_1=1 \\ (n_1,p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}}^N \sum_{\substack{n_2=1 \\ (n_2,p)=1}}^N \frac{d(n_1)d(n_2)}{n_1 n_2} - \frac{p-1}{2} \sum_{\substack{n_1=1 \\ (n_1,p)=1 \\ 2^k n_1 \equiv -3^l n_2 \pmod{p}}}^N \sum_{\substack{n_2=1 \\ (n_2,p)=1}}^N \frac{d(n_1)d(n_2)}{n_1 n_2} \\
 &\quad + O\left(p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}\right) + O\left(p^{2+\epsilon} N^{-1}\right) \\
 &:= \frac{p-1}{2} M_{11} - \frac{p-1}{2} M_{12} + O\left(p^{\frac{3}{2}+\epsilon} N^{-\frac{1}{2}}\right) + O\left(p^{2+\epsilon} N^{-1}\right). \tag{3.5}
 \end{aligned}$$

First we consider M_{11} . We have

$$\begin{aligned}
 M_{11} &= \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k} - 1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{1 \leq n_2 \leq \frac{p}{3^l} - 1}} \frac{d(n_1)d(n_2)}{n_1 n_2} + \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k} - 1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{\frac{p}{3^l} \leq n_2 \leq N \\ (n_2,p)=1}} \frac{d(n_1)d(n_2)}{n_1 n_2} \\
 &\quad + \sum_{\substack{\frac{p}{2^k} \leq n_1 \leq N \\ (n_1,p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{1 \leq n_2 \leq \frac{p}{3^l} - 1}} \frac{d(n_1)d(n_2)}{n_1 n_2} + \sum_{\substack{\frac{p}{2^k} \leq n_1 \leq N \\ (n_1,p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{\frac{p}{3^l} \leq n_2 \leq N \\ (n_2,p)=1}} \frac{d(n_1)d(n_2)}{n_1 n_2} \\
 &:= \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4. \tag{3.6}
 \end{aligned}$$

It is not hard to show that

$$\begin{aligned}
 \Omega_1 &= \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k} - 1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{1 \leq n_2 \leq \frac{p}{3^l} - 1}} \frac{d(n_1)d(n_2)}{n_1 n_2} = \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k} - 1 \\ 3^l | n_1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{1 \leq n_2 \leq \frac{p}{3^l} - 1 \\ 2^k | n_2}} \frac{d(n_1)d(n_2)}{n_1 n_2} \\
 &= \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k 3^l} - 1 \\ 2^k 3^l n_1 \equiv 2^k 3^l n_2 \pmod{p}}} \sum_{\substack{1 \leq n_2 \leq \frac{p}{2^k 3^l} - 1}} \frac{d(3^l n_1)d(2^k n_2)}{3^l n_1 2^k n_2} = \sum_{1 \leq n \leq \frac{p}{2^k 3^l} - 1} \frac{d(3^l n)d(2^k n)}{3^l 2^k n^2} \\
 &= \frac{1}{2^k 3^l} \sum_{n=1}^{\infty} \frac{d(2^k n)d(3^l n)}{n^2} + O\left(p^{\epsilon-1}\right), \tag{3.7}
 \end{aligned}$$

$$\Omega_2 \ll p^\epsilon \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k} - 1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{\frac{p}{3^l} \leq n_2 \leq N \\ (n_2,p)=1}} \frac{1}{n_1 n_2} = p^\epsilon \sum_{\substack{1 \leq n_1 \leq \frac{p}{2^k} - 1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{\frac{p}{3^l} \leq n_2 \leq N \\ (n_2,p)=1}} \frac{2^k 3^l}{2^k n_1 3^l n_2}$$

$$\begin{aligned}
 &= p^\epsilon \sum_{1 \leq r_1 \leq p-1} \sum_{\substack{1 \leq l_2 \leq \frac{3^l N}{p} \\ r_1 \equiv l_2 p + r_2 \pmod{p}}} \sum_{1 \leq r_2 \leq p-1} \frac{2^k 3^l}{r_1(l_2 p + r_2)} \\
 &\ll p^\epsilon \sum_{1 \leq r \leq p-1} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{r(l_2 p + r)} \\
 &\ll p^\epsilon \sum_{1 \leq r \leq p-1} \frac{1}{r} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{l_2 p} \ll p^{\epsilon-1}, \tag{3.8}
 \end{aligned}$$

$$\begin{aligned}
 \Omega_3 &\ll p^\epsilon \sum_{\substack{\frac{p}{2^k} \leq n_1 \leq N \\ (n_1, p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{1 \leq n_2 \leq \frac{p}{3^l} - 1} \frac{1}{n_1 n_2} = p^\epsilon \sum_{\substack{\frac{p}{2^k} \leq n_1 \leq N \\ (n_1, p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{1 \leq n_2 \leq \frac{p}{3^l} - 1} \frac{2^k 3^l}{2^k n_1 3^l n_2} \\
 &= p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r_1 \leq p-1} \sum_{\substack{1 \leq r_2 \leq p-1 \\ l_1 p + r_1 \equiv r_2 \pmod{p}}} \frac{2^k 3^l}{(l_1 p + r_1) r_2} \\
 &\ll p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \frac{1}{(l_1 p + r) r} \\
 &\ll p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \frac{1}{l_1 p} \sum_{1 \leq r \leq p-1} \frac{1}{r} \ll p^{\epsilon-1}, \tag{3.9}
 \end{aligned}$$

and

$$\begin{aligned}
 \Omega_4 &\ll p^\epsilon \sum_{\substack{\frac{p}{2^k} \leq n_1 \leq N \\ (n_1, p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{\frac{p}{3^l} \leq n_2 \leq N \\ (n_2, p)=1}} \frac{1}{n_1 n_2} = p^\epsilon \sum_{\substack{\frac{p}{2^k} \leq n_1 \leq N \\ (n_1, p)=1 \\ 2^k n_1 \equiv 3^l n_2 \pmod{p}}} \sum_{\substack{\frac{p}{3^l} \leq n_2 \leq N \\ (n_2, p)=1}} \frac{2^k 3^l}{2^k n_1 3^l n_2} \\
 &= p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r_1 \leq p-1} \sum_{\substack{1 \leq l_2 \leq \frac{3^l N}{p} \\ l_1 p + r_1 \equiv l_2 p + r_2 \pmod{p}}} \sum_{1 \leq r_2 \leq p-1} \frac{2^k 3^l}{(l_1 p + r_1)(l_2 p + r_2)} \\
 &\ll p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{(l_1 p + r)(l_2 p + r)}
 \end{aligned}$$

$$\begin{aligned} &\ll p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \frac{1}{l_1 p + r} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{l_2 p} \ll p^{\epsilon-1} \sum_{1 \leq n \leq 2^k N} \frac{1}{n} \\ &\ll p^{\epsilon-1}. \end{aligned} \tag{3.10}$$

Then from (3.6)-(3.10) we have

$$M_{11} = \frac{1}{2^k 3^l} \sum_{n=1}^{\infty} \frac{d(2^k n) d(3^l n)}{n^2} + O(p^{\epsilon-1}). \tag{3.11}$$

Similarly we can get

$$\begin{aligned} M_{12} &= \sum_{\substack{n_1=1 \\ (n_1,p)=1 \\ 2^k n_1 \equiv -3^l n_2 \pmod{p}}}^N \sum_{\substack{n_2=1 \\ (n_2,p)=1 \\ 2^k n_1 \equiv -3^l n_2 \pmod{p}}}^N \frac{d(n_1) d(n_2)}{n_1 n_2} \ll p^\epsilon \sum_{\substack{n_1=1 \\ (n_1,p)=1 \\ 2^k n_1 \equiv -3^l n_2 \pmod{p}}}^N \sum_{\substack{n_2=1 \\ (n_2,p)=1 \\ 2^k n_1 \equiv -3^l n_2 \pmod{p}}}^N \frac{1}{n_1 n_2} \\ &\ll p^\epsilon \sum_{0 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r_1 \leq p-1} \sum_{0 \leq l_2 \leq \frac{3^l N}{p}} \sum_{1 \leq r_2 \leq p-1} \frac{1}{(l_1 p + r_1)(l_2 p + r_2)} \\ &\quad l_1 p + r_1 \equiv -(l_2 p + r_2) \pmod{p} \\ &\ll p^\epsilon \sum_{0 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \sum_{0 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{(l_1 p + r)(l_2 p + p - r)} \\ &= p^\epsilon \sum_{1 \leq r \leq p-1} \frac{1}{r(p-r)} + p^\epsilon \sum_{1 \leq r \leq p-1} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{r(l_2 p + p - r)} \\ &\quad + p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \frac{1}{(l_1 p + r)(p - r)} \\ &\quad + p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{(l_1 p + r)(l_2 p + p - r)} \\ &\ll p^{\epsilon-1} \sum_{1 \leq r \leq p-1} \left(\frac{1}{r} + \frac{1}{p-r} \right) + p^\epsilon \sum_{1 \leq r \leq p-1} \frac{1}{r} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{l_2 p} \\ &\quad + p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \frac{1}{l_1 p} \sum_{1 \leq r \leq p-1} \frac{1}{p-r} + p^\epsilon \sum_{1 \leq l_1 \leq \frac{2^k N}{p}} \sum_{1 \leq r \leq p-1} \frac{1}{l_1 p + r} \sum_{1 \leq l_2 \leq \frac{3^l N}{p}} \frac{1}{l_2 p} \\ &\ll p^{\epsilon-1}. \end{aligned} \tag{3.12}$$

Now taking $N = p^3$ and combining (3.5), (3.11), (3.12) we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k)\bar{\chi}(3^l) |L(1, \chi)|^4 = \frac{p}{2^{k+1}3^l} \sum_{n=1}^{\infty} \frac{d(2^k n)d(3^l n)}{n^2} + O(p^\epsilon) \tag{3.13}$$

Thus from Theorem 2.1 we immediately get Theorem 3.1. □

COROLLARY 3.1. *Let $p \geq 5$ be a prime, and let k be fixed non-negative integer. Then we have*

$$\begin{aligned} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) |L(1, \chi)|^4 &= \frac{(3k + 5)\pi^4}{144 \cdot 2^k} p + O(p^\epsilon), \\ \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) |L(1, \chi)|^4 &= \frac{(4k + 5)\pi^4}{144 \cdot 3^k} p + O(p^\epsilon). \end{aligned}$$

THEOREM 3.2. *Let $p \geq 5$ be a prime. Suppose that χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4. Then we have*

$$\begin{aligned} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 |L(2, \chi\chi_3)|^2 &= \frac{45L^2(3, \chi_3)}{52} p + O(p^\epsilon), \\ \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 |L(2, \chi\chi_4)|^2 &= \frac{5L^2(3, \chi_4)}{6} p + O(p^\epsilon). \end{aligned}$$

Proof. Define $\tau(n) = \sum_{t|n} \frac{\chi_3(t)}{t}$ and $\rho(n) = \sum_{t|n} \frac{\chi_4(t)}{t}$. By using Theorem 2.2, Theorem 2.3 and the methods of Theorem 3.1 we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 |L(2, \chi\chi_3)|^2 = \frac{p}{2} \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^2} + O(p^\epsilon) = \frac{45L^2(3, \chi_3)}{52} p + O(p^\epsilon),$$

and

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 |L(2, \chi\chi_4)|^2 = \frac{p}{2} \sum_{n=1}^{\infty} \frac{\rho^2(n)}{n^2} + O(p^\epsilon) = \frac{5L^2(3, \chi_4)}{6} p + O(p^\epsilon).$$

□

THEOREM 3.3. *Let $p \geq 5$ be a prime, and let k be a non-negative integer. Suppose that χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4. Then we have*

$$\begin{aligned} & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) |L(1, \chi)|^2 L(1, \chi)L(2, \bar{\chi}\chi_3) \\ &= \left(1 + \frac{(-1)^k}{2^{k-1} \cdot 9}\right) \frac{\pi^4 L^2(3, \chi_3)}{2^{k+3} \cdot 11L(5, \chi_3)} p + O(p^\epsilon), \\ & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) |L(1, \chi)|^2 L(1, \chi)L(2, \bar{\chi}\chi_3) = \frac{\pi^4 L^2(3, \chi_3)}{3^{k+2} \cdot 8L(5, \chi_3)} p + O(p^\epsilon), \\ & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) |L(1, \chi)|^2 L(1, \chi)L(2, \bar{\chi}\chi_4) = \frac{\pi^4 L^2(3, \chi_4)}{2^{k+3} \cdot 9L(5, \chi_4)} p + O(p^\epsilon), \\ & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(3^k) |L(1, \chi)|^2 L(1, \chi)L(2, \bar{\chi}\chi_4) \\ &= \left(\frac{49}{4} + \frac{(-1)^k}{3^{k-1}}\right) \frac{\pi^4 L^2(3, \chi_4)}{3^k \cdot 1098L(5, \chi_4)} p + O(p^\epsilon), \\ & \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 L(2, \chi\chi_4)L(2, \bar{\chi}\chi_3) \\ &= \frac{\pi^2 L(3, \chi_3)L(3, \chi_4)L(4, \chi_3\chi_4)}{12L(6, \chi_3\chi_4)} p + O(p^\epsilon). \end{aligned}$$

Proof. Define $\tau(n) = \sum_{t|n} \frac{\chi_3(t)}{t}$ and $\rho(n) = \sum_{t|n} \frac{\chi_4(t)}{t}$. It is easy to show that $\tau(3^k n) = \tau(n)$ and $\rho(2^k n) = \rho(n)$. By using Theorem 2.2, Theorem 2.3 and the methods of Theorem 3.1 we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^k) |L(1, \chi)|^2 L(1, \chi)L(2, \bar{\chi}\chi_3) = \frac{p}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n)\tau(2^k n)}{n^2} + O(p^\epsilon)$$

$$\begin{aligned}
 &= \left(1 + \frac{(-1)^k}{2^{k-1} \cdot 9}\right) \frac{\pi^4 L^2(3, \chi_3)}{2^{k+3} \cdot 11L(5, \chi_3)} p + O(p^\epsilon), \\
 \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} \chi(3^k) |L(1, \chi)|^2 L(1, \chi) L(2, \bar{\chi}\chi_3) &= \frac{p}{2 \cdot 3^k} \sum_{n=1}^{\infty} \frac{d(n)\tau(3^k n)}{n^2} + O(p^\epsilon) \\
 &= \frac{p}{2 \cdot 3^k} \sum_{n=1}^{\infty} \frac{d(n)\tau(n)}{n^2} + O(p^\epsilon) = \frac{\pi^4 L^2(3, \chi_3)}{3^{k+2} \cdot 8L(5, \chi_3)} p + O(p^\epsilon), \\
 \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} \chi(2^k) |L(1, \chi)|^2 L(1, \chi) L(2, \bar{\chi}\chi_4) &= \frac{p}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n)\rho(2^k n)}{n^2} + O(p^\epsilon) \\
 &= \frac{p}{2^{k+1}} \sum_{n=1}^{\infty} \frac{d(n)\rho(n)}{n^2} + O(p^\epsilon) = \frac{\pi^4 L^2(3, \chi_4)}{2^{k+3} \cdot 9L(5, \chi_4)} p + O(p^\epsilon), \\
 \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} \chi(3^k) |L(1, \chi)|^2 L(1, \chi) L(2, \bar{\chi}\chi_4) &= \frac{p}{2 \cdot 3^k} \sum_{n=1}^{\infty} \frac{d(n)\rho(3^k n)}{n^2} + O(p^\epsilon) \\
 &= \left(\frac{49}{4} + \frac{(-1)^k}{3^{k-1}}\right) \frac{\pi^4 L^2(3, \chi_4)}{3^k \cdot 1098L(5, \chi_4)} p + O(p^\epsilon),
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 L(2, \chi\chi_4) L(2, \bar{\chi}\chi_3) &= \frac{p}{2} \sum_{n=1}^{\infty} \frac{\tau(n)\rho(n)}{n^2} + O(p^\epsilon) \\
 &= \frac{\pi^2 L(3, \chi_3) L(3, \chi_4) L(4, \chi_3\chi_4)}{12L(6, \chi_3\chi_4)} p + O(p^\epsilon).
 \end{aligned}$$

□

4. Mean values of the homogeneous Dedekind sums

Let $q > 3$ be an integer, and let χ be a Dirichlet character modulo q . The generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$\sum_{a=1}^q \chi(a) \frac{te^{at}}{e^{qt} - 1} = \sum_{n=0}^{\infty} \frac{B_{n,\chi}}{n!} t^n.$$

Let r be a positive integer prime to q , and let $k \geq 0$ be an integer. By (6) of [7] and (2.12) of [3] we have

$$\frac{1}{q^k} \sum_{1 \leq a \leq q/r} a^k \chi(a) = -\frac{1}{q^k(k+1)} B_{k+1, \chi} + \frac{\bar{\chi}(r)}{(rq)^k \phi(r)} \sum_{\psi \pmod r} \bar{\psi}(-q) \sum_{n=1}^{k+1} \frac{1}{k+1} \binom{k+1}{n} q^{k+1-n} B_{n, \chi \psi},$$

where \sum_{ψ} is over all characters ψ modulo r .

By using the above formula H. Liu [5] showed the following lemma.

LEMMA 4.1. *Let χ be a primitive character modulo $q > 3$. Then*

$$\begin{aligned} & \frac{1}{q} \sum_{1 \leq a \leq q/3} a \chi(a) \\ = & \begin{cases} \frac{1}{6\pi i} (1 - \bar{\chi}(3)) \tau(\chi) L(1, \bar{\chi}) + \frac{\sqrt{3}i}{(2\pi i)^2} \tau(\chi) L(2, \bar{\chi} \chi_3), & \text{if } \chi(-1) = -1, \\ \frac{\sqrt{3}}{6\pi} \tau(\chi) L(1, \bar{\chi} \chi_3) + \frac{1}{(2\pi i)^2} \left(3 - \frac{\bar{\chi}(3)}{3}\right) \tau(\chi) L(2, \bar{\chi}), & \text{if } \chi(-1) = 1, \end{cases} \\ & \frac{1}{q} \sum_{1 \leq a \leq q/4} a \chi(a) \\ = & \begin{cases} \frac{1}{8\pi i} (\bar{\chi}(2) - \bar{\chi}(4)) \tau(\chi) L(1, \bar{\chi}) + \frac{2i}{(2\pi i)^2} \tau(\chi) L(2, \bar{\chi} \chi_4), & \text{if } \chi(-1) = -1, \\ \frac{1}{4\pi} \tau(\chi) L(1, \bar{\chi} \chi_4) + \frac{1}{(2\pi i)^2} \left(2 + \frac{\bar{\chi}(2)}{2} - \frac{\bar{\chi}(4)}{4}\right) \tau(\chi) L(2, \bar{\chi}), & \text{if } \chi(-1) = 1, \end{cases} \end{aligned}$$

where $\tau(\chi) = \sum_{a=1}^q \chi(a) e\left(\frac{a}{q}\right)$ denotes the Gauss sum, χ_3 is the non-principal character modulo 3, and χ_4 is the non-principal character modulo 4.

On the other hand, by [8] we know that

$$S(a, p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1) = -1}} \chi(a) |L(1, \chi)|^2.$$

Then we have

$$S(a, b, p) = S(a\bar{b}, p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} \chi(a)\bar{\chi}(b) |L(1, \chi)|^2.$$

Noting that $\tau(\chi)\tau(\bar{\chi}) = p\chi(-1)$. From Lemma 4.1 we have

$$\begin{aligned} \sum_{a \leq \frac{p}{3}} \sum_{b \leq \frac{p}{3}} abS(a, b, p) &= \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 \sum_{a \leq \frac{p}{3}} a\chi(a) \sum_{b \leq \frac{p}{3}} b\bar{\chi}(b) \\ &= \frac{p^4}{18\pi^4(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} (1 - \chi(3)) |L(1, \chi)|^4 \\ &\quad + \frac{\sqrt{3}p^4}{12\pi^5(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} (1 - \chi(3)) |L(1, \chi)|^2 L(1, \chi)L(2, \bar{\chi}\chi_3) \\ &\quad + \frac{3p^4}{16\pi^6(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 |L(2, \chi\chi_3)|^2 \\ &= \frac{p^4}{18\pi^4(p-1)} \left(\frac{5\pi^4 p}{144} - \frac{(4+5)\pi^4 p}{144 \times 3} \right) \\ &\quad + \frac{\sqrt{3}p^4}{12\pi^5(p-1)} \left(\frac{\pi^4 L^2(3, \chi_3)p}{9 \times 8L(5, \chi_3)} - \frac{\pi^4 L^2(3, \chi_3)p}{3^3 \cdot 8L(5, \chi_3)} \right) \\ &\quad + \frac{3p^4}{16\pi^6(p-1)} \left(\frac{45L^2(3, \chi_3)p}{52} \right) + (p^{3+\epsilon}) \\ &= \frac{p^4}{1296} + \frac{\sqrt{3}p^4}{12\pi^5} \left(\frac{1}{9 \times 8} - \frac{1}{27 \times 8} \right) \frac{\pi^4 L^2(3, \chi_3)}{L(5, \chi_3)} + \frac{135L^2(3, \chi_3)p^4}{832\pi^6} + (p^{3+\epsilon}) \\ &= \left(\frac{1}{1296} + \frac{\sqrt{3}L^2(3, \chi_3)}{1296\pi L(5, \chi_3)} + \frac{135L^2(3, \chi_3)}{832\pi^6} \right) p^4 + (p^{3+\epsilon}). \end{aligned}$$

Similarly we get

$$\sum_{a \leq \frac{p}{4}} \sum_{b \leq \frac{p}{4}} abS(a, b, p) = \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 \sum_{a \leq \frac{p}{4}} a\chi(a) \sum_{b \leq \frac{p}{4}} b\bar{\chi}(b)$$

$$\begin{aligned}
 &= \frac{p^4}{32\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (1 - \chi(2)) |L(1, \chi)|^4 \\
 &\quad + \frac{p^4}{8\pi^5(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (\chi(2) - \chi(4)) |L(1, \chi)|^2 L(1, \chi) L(2, \bar{\chi}\chi_4) \\
 &\quad + \frac{p^4}{4\pi^6(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 |L(2, \chi\chi_4)|^2 \\
 &= \frac{p^4}{32\pi^4(p-1)} \left(\frac{5\pi^4 p}{144} - \frac{(3+5)\pi^4 p}{2 \times 144} \right) \\
 &\quad + \frac{p^4}{8\pi^5(p-1)} \left(\frac{\pi^4 L^2(3, \chi_4) p}{2^4 \cdot 9L(5, \chi_4)} - \frac{\pi^4 L^2(3, \chi_4) p}{2^5 \cdot 9L(5, \chi_4)} \right) \\
 &\quad + \frac{p^4}{4\pi^6(p-1)} \times \frac{5L^2(3, \chi_4) p}{6} + (p^{3+\epsilon}) \\
 &= \frac{p^4}{32} \left(\frac{10-8}{2 \times 144} \right) + \frac{p^4 L^2(3, \chi_4)}{8\pi L(5, \chi_4)} \left(\frac{2-1}{2^5 \times 9} \right) + \frac{5L^2(3, \chi_4) p^4}{24\pi^6} + (p^{3+\epsilon}) \\
 &= \left(\frac{1}{4608} + \frac{L^2(3, \chi_4)}{2304\pi L(5, \chi_4)} + \frac{5L^2(3, \chi_4)}{24\pi^6} \right) p^4 + (p^{3+\epsilon}),
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{a \leq \frac{p}{3}} \sum_{b \leq \frac{p}{4}} abS(a, b, p) &= \frac{p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 \sum_{a \leq \frac{p}{3}} a\chi(a) \sum_{b \leq \frac{p}{4}} b\bar{\chi}(b) \\
 &= \frac{p^4}{48\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (\chi(2) - \chi(4) - \chi(2)\bar{\chi}(3) + \chi(4)\bar{\chi}(3)) |L(1, \chi)|^4 \\
 &\quad + \frac{\sqrt{3}p^4}{32\pi^5(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (\chi(2) - \chi(4)) |L(1, \chi)|^2 L(1, \chi) L(2, \bar{\chi}\chi_3) \\
 &\quad + \frac{p^4}{12\pi^5(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (1 - \chi(3)) |L(1, \chi)|^2 L(1, \chi) L(2, \bar{\chi}\chi_4)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{\sqrt{3}p^4}{8\pi^6(p-1)} \sum_{\substack{\chi \pmod p \\ \chi(-1)=-1}} |L(1, \chi)|^2 L(2, \chi\chi_4)L(2, \bar{\chi}\chi_3) \\
 = & \frac{p^4}{48\pi^4(p-1)} \left(\frac{8\pi^4 p}{144 \times 2} - \frac{11\pi^4 p}{144 \times 4} - \frac{8 \times 9\pi^4 p}{2 \times 3 \times 720} + \frac{11 \times 9\pi^4 p}{4 \times 3 \times 720} \right) \\
 & + \frac{\sqrt{3}p^4}{32\pi^5(p-1)} \left(\left(1 - \frac{1}{9}\right) \frac{\pi^4 L^2(3, \chi_3)p}{2^4 \cdot 11L(5, \chi_3)} - \left(1 + \frac{1}{2 \times 9}\right) \frac{\pi^4 L^2(3, \chi_3)p}{2^5 \cdot 11L(5, \chi_3)} \right) \\
 & + \frac{p^4}{12\pi^5(p-1)} \left(\left(\frac{49}{4} + 3\right) \frac{\pi^4 L^2(3, \chi_4)p}{1098L(5, \chi_4)} - \left(\frac{49}{4} - 1\right) \frac{\pi^4 L^2(3, \chi_4)p}{3 \times 1098L(5, \chi_4)} \right) \\
 & + \frac{\sqrt{3}p^4}{8\pi^6(p-1)} \times \frac{\pi^2 L(3, \chi_3)L(3, \chi_4)L(4, \chi_3\chi_4)p}{12L(6, \chi_3\chi_4)} + O(p^{3+\epsilon}) \\
 = & \frac{p^4}{48 \times 144} \left(4 - \frac{11}{4} - \frac{12}{5} + \frac{33}{20} \right) + \frac{\sqrt{3}p^4 L^2(3, \chi_3)}{32 \times 11 \times 16 \times 9\pi L(5, \chi_3)} \left(8 - \frac{19}{4} \right) \\
 & + \frac{p^4 L^2(3, \chi_4)}{12 \times 4 \times 1098\pi L(5, \chi_4)} (61 - 15) + \frac{\sqrt{3}p^4 L(3, \chi_3)L(3, \chi_4)L(4, \chi_3\chi_4)}{96\pi^4 L(6, \chi_3\chi_4)} \\
 & + O(p^{3+\epsilon}) \\
 = & \left(\frac{1}{13824} + \frac{13\sqrt{3}L^2(3, \chi_3)}{202752\pi L(5, \chi_3)} + \frac{23L^2(3, \chi_4)}{26352\pi L(5, \chi_4)} \right. \\
 & \left. + \frac{\sqrt{3}L(3, \chi_3)L(3, \chi_4)L(4, \chi_3\chi_4)}{96\pi^4 L(6, \chi_3\chi_4)} \right) p^4 + O(p^{3+\epsilon}).
 \end{aligned}$$

This proves Theorem 1.1.

References

- [1] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, Springer-Verlag, New York, 1976.
- [2] R. R. Hall and M. N. Huxley, *Dedekind sums and continued fractions*, Acta Arithmetica, **63** (1993), 79–90.
- [3] S. Kanemitsu, H. Li and N. Wang, *Weighted short-interval character sums*, Proceedings of the American Mathematical Society, **139** (2011), 1521–1532.
- [4] H. Liu, *On the mean values of the homogeneous Dedekind sums and Cochrane sums in short intervals*, Journal of the Korean Mathematical Society, **44** (2007), 1243–1254.

- [5] H. Liu, *On the mean values of Dirichlet L -functions*, Journal of Number Theory, **147** (2015), 172–183.
- [6] H. Rademacher, *Dedekind Sums*, Carus Mathematical Monographs, Mathematical Association of America, 1972.
- [7] J. Szmidt, J. Urbanowicz and D. Zagier, *Congruences among generalized Bernoulli numbers*, Acta Arithmetica, **71** (1995), 273–278.
- [8] W. Zhang, *On the mean values of Dedekind sums*, Journal de Theorie des Nombres, **8** (1996), 429–442.
- [9] Z. Zheng, *The Petersson-Knopp identity for the homogeneous Dedekind sums*, Journal of Number Theory, **57** (1996), 223–230.

Xiaoying Wang
School of Mathematics
Northwest University
Xi'an 710127, Shaanxi, P. R. China
E-mail: xywang@nwu.edu.cn

Xiaxia Yue
School of Mathematics
Northwest University
Xi'an 710127, Shaanxi, P. R. China
E-mail: 464642269@qq.com