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THE EXISTENCE OF SOME METRICS ON RIEMANNIAN WARPED PRODUCT MANIFOLDS WITH FIBER MANIFOLD OF CLASS (B)

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Abstract. In this paper, we prove the existence of warping functions on Riemannian warped product manifolds with some prescribed scalar curvatures according to the fiber manifolds of class (B).

1. Introduction

One of the basic problems in the differential geometry is studying the set of curvature functions which a given manifold possesses.

The well-known problem in differential geometry is that of whether there exists a warping function of warped metric with some prescribed scalar curvature function. One of the main methods of studying differential geometry is by the existence and the nonexistence of Riemannian warped metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. In order to study these kinds of problems, we need some analytic methods in differential geometry.

For Riemannian manifolds, warped products have been useful in producing examples of spectral behavior, examples of manifolds of negative

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curvature (cf. [2, 3, 5, 6, 8, 14, 15]), and also in studying L_2-cohomology (cf. [16]).

In a study [12,13], M.C. Leung have studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and the nonexistence of Riemannian warped metric with some prescribed scalar curvature function.

In this paper, we also study the existence and the nonexistence of Riemannian warped product metric with prescribed scalar curvature functions on some Riemannian warped product manifolds. So, using upper solution and lower solution methods, we consider the solution of some partial differential equations on a warped product manifold. That is, we express the scalar curvature of a warped product manifold $M = B \times_f N$ in terms of its warping function f and the scalar curvatures of B and N.

By the results of Kazdan and Warner (cf. [9–11]), if N is a compact Riemannian $n-$manifold without boundary, $n \geq 3$, then N belongs to one of the following three categories:
(A) A smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is negative somewhere.
(B) A Smooth function on N is the scalar curvature of some Riemannian metric on N if and only if the function is either identically zero or strictly negative somewhere.
(C) Any smooth function on N is the scalar curvature of some Riemannian metric on N.

This completely answers the question of which smooth functions are scalar curvatures of Riemannian metrics on a compact manifold N.

In [9–11], Kazdan and Warner also showed that there exists some obstruction of a Riemannian metric with positive scalar curvature (or zero scalar curvature) on a compact manifold.

For noncompact Riemannian manifolds, many important works have been done on the question how to determine which smooth functions are scalar curvatures of complete Riemannian metrics on open manifold. Results of Gromov and Lawson (cf. [6]) show that some open manifolds cannot carry complete Riemannian metrics of positive scalar curvature, for example, weakly enlargeable manifolds.

Furthermore, they show that some open manifolds cannot even admit complete Riemannian metrics with scalar curvatures uniformly positive outside a compact set and with Ricci curvatures bounded (cf. [6, 14]).

On the other hand, it is well known that each open manifold of dimension bigger than 2 admits a complete Riemannian metric of constant negative scalar curvature (cf. [2]). It follows from the results of Aviles and McOwen (cf. [1]) that any bounded negative function on an open manifold of dimension bigger than 2 is the scalar curvature of a complete Riemannian metric.

In this paper, when N is a compact Riemannian manifold, we discuss the method of using warped products to construct Riemannian metrics on $M = [a, \infty) \times_f N$ with specific scalar curvatures, where a is a positive constant. It is shown that if the fiber manifold N belongs to class (B), then M admits a Riemannian metric with some prescribed scalar curvature outside a compact set. These results are extensions of the results in [7].

Although we will assume throughout this paper that all data (M, metric g, and curvature, etc.) are smooth, this is merely for convenience. Our arguments go through with little or no change if one makes minimal smoothness hypotheses, such as assuming that the given data is Hölder continuous.

2. Main results

Let (N, g) be a Riemannian manifold of dimension n and let $f : [a, \infty) \to R^+$ be a smooth function, where a is a positive number. A Riemannian warped product of N and $[a, \infty)$ with warping function f is defined to be the product manifold $([a, \infty) \times_f N, g')$ with

$$g' = dt^2 + f^2(t) g \tag{2.1}$$

Let $R(g)$ be the scalar curvature of (N, g). Then the scalar curvature $R(t, x)$ of g' is given by the equation

$$R(t, x) = \frac{1}{f^2(t)} \{R(g)(x) - 2nf(t)f''(t) - n(n-1)|f'(t)|^2\} \tag{2.2}$$

for $t \in [a, \infty)$ and $x \in N$ (For details, cf. [5] or [6]).

If we denote

$$u(t) = f^{\frac{n+1}{2}}(t), \quad t > a,$$

then equation (2.2) can be changed into

$$\frac{4n}{n+1}u''(t) + R(t,x)u(t) - R(g)(x)u(t)^{1-\frac{4}{n+1}} = 0. \tag{2.3}$$

In this paper, we assume that the fiber manifold N is nonempty, connected and a compact Riemannian $n-$manifold without boundary.

If N is in class (B), then we assume that N admits a Riemannian metric of zero scalar curvature. In this case, equation (2.3) is changed into

$$\frac{4n}{n+1}u''(t) + R(t,x)u(t) = 0. \tag{2.4}$$

If N admits a Riemannian metric of zero scalar curvature, then we let $u(t) = t^{\alpha}$ in equation(2.4), where $\alpha > 1$ is a constant, and we have

$$R(t,x) = \frac{4n}{n+1}\alpha(1-\alpha)\frac{1}{t^2} < 0, \quad t > a.$$

Thus we have the following theorem.

THEOREM 2.1. *For $n \geq 3$, let $M = [a,\infty) \times_f N$ be the Riemannian warped product $(n+1)$-manifold with N compact n-manifold. Suppose that N is in class (B), then on M there is a Riemannian metric of negative scalar curvature outside a compact set.*

THEOREM 2.2. *Suppose that $R(g) = 0$ and $R(t,x) = R(t) \in C^{\infty}([a,\infty))$. Assume that for $t > t_0$, there exist an upper solution $u_+(t)$ and a lower solution $u_-(t)$ such that $0 < u_-(t) \leq u_+(t)$. Then there exists a solution $u(t)$ of equation (2.4) such that for $t > t_0$, $\quad 0 < u_-(t) \leq u(t) \leq u_+(t)$.*

Proof. See Theorem 2.2 in [7]. □

LEMMA 2.3. *On $[a,\infty)$, there does not exist a positive solution $u(t)$ such that*

$$t^2 u''(t) + \frac{c}{4}u(t) \leq 0 \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. See Lemma 2.2 in [4]. □

We note that the term $\alpha(1-\alpha)$ achieves its maximum when $\alpha = \frac{1}{2}$. And when $u = t^{\frac{1}{2}}$ and N admits a Riemannian metric of zero scalar curvature, we have

$$R = \frac{4n}{n+1}\frac{1}{4}\frac{1}{t^2}, \quad t > a.$$

If $R(t,x)$ is the function of only t-variable, then we have the following theorem.

THEOREM 2.4. *If $R(g) = 0$, then there is no positive solution to equation (2.4) with*

$$R(t) \geq \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

where $c > 1$ and $t_0 > a$ are constants.

Proof. Assume that

$$R(t) \geq \frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} \quad \text{for} \quad t \geq t_0,$$

with $c > 1$. Equation (2.4) gives

$$t^2 u''(t) + \frac{c}{4}u(t) \leq 0.$$

By Lemma 2.3, we complete the proof. □

In particular, if $R(g) = 0$, then using Riemannian warped product it is impossible to obtain a Riemannian metric of uniformly negative scalar curvature outside a compact subset. The best we can do is when $u(t) = t^{\frac{1}{2}}$, or $f(t) = t^{\frac{1}{n+1}}$, where the scalar curvature is negative but goes to zero at infinity.

THEOREM 2.5. *Suppose that $R(g) = 0$. Assume that $R(t,x) = R(t) \in C^\infty([a,\infty))$ is a function such that*

$$\frac{4n}{n+1}\frac{c}{4}\frac{1}{t^2} > R(t) \geq -\frac{4n}{n+1}e^{\alpha t} \quad \text{for} \quad \mathrm{t} > \mathrm{t}_0,$$

where $t_0 > a$, $\alpha > 0$ and $0 < c < 1$ are constants. Then equation (2.4) has a positive solution on $[a,\infty)$.

Proof. Since $R(g) = 0$, put $u_+(t) = t^{\frac{1}{2}}$. Then $u''_+(t) = \frac{-1}{4} t^{\frac{1}{2}-2}$. Hence

$$\begin{aligned}
& \frac{4n}{n+1} u''_+(t) + R(t) u_+(t) \\
\leq & \frac{4n}{n+1} u''_+(t) + \frac{4n}{n+1} \frac{c}{4} \frac{1}{t^2} u_+(t) \\
= & \frac{4n}{n+1} \frac{-1}{4} t^{\frac{1}{2}-2} + \frac{4n}{n+1} \frac{c}{4} \frac{1}{t^2} t^{\frac{1}{2}} \\
= & \frac{4n}{n+1} \frac{1}{4} t^{\frac{1}{2}-2} [-1 + c] \\
< & \; 0.
\end{aligned}$$

Therefore $u_+(t)$ is our (weak) upper solution.

And put $u_-(t) = e^{-e^{\beta t}}$, where β is a positive large constant. Then $u''_-(t) = -\beta^2 e^{\beta t} e^{-e^{\beta t}} + \beta^2 e^{2\beta t} e^{-e^{\beta t}}$. Hence

$$\begin{aligned}
& \frac{4n}{n+1} u''_-(t) + R(t) u_-(t) \\
\geq & \frac{4n}{n+1} u''_-(t) - \frac{4n}{n+1} e^{\alpha t} u_-(t) \\
= & \frac{4n}{n+1} e^{-e^{\beta t}} [-\beta^2 e^{\beta t} + \beta^2 e^{2\beta t} - e^{\alpha t}] \\
= & \frac{4n}{n+1} e^{-e^{\beta t}} [\beta^2 e^{\beta t} (-1 + e^{\beta t}) - e^{\alpha t}] \\
> & \; 0
\end{aligned}$$

for large β. Thus, for large β, $u_-(t)$ is a (weak) lower solution and $0 < u_-(t) < u_+(t)$. So, by Theorem 2.2, equation (2.4) has a (weak) positive solution $u(t)$ such that $0 < u_-(t) \leq u(t) \leq u_+(t)$ for large t. □

REMARK 2.6. In case that $R(g) = 0$, the results in Theorem 2.4 and Theorem 2.5 are almost sharp because if $u(t) = t^{\frac{1}{2}}$, then $R(t) = \frac{4n}{n+1} \frac{1}{4} \frac{1}{t^2}$.

In fact, we have only to solve the following equation.

EXAMPLE 2.7. If $R(g) = 0$ and $R(t) = -\frac{4n}{n+1} \frac{2}{t^2}$, then there is a positive solution to equation (2.4).

$$t^2 u''(t) - 2u(t) = 0. \tag{2.5}$$

By Euler-Cauchy equation method, we put $u(t) = t^m$ then

$$m(m-1)t^{m-2}t^2 - 2t^m = 0$$

and

$$(m^2 - m - 2)t^m = 0,$$

so $m = 2, -1$. Thus $u(t) = c_1 t^2 + c_2 t^{-1}$ is solution of equation (2.5), where c_1 and c_2 are constants.

Therefore $u(t) = c_2 t^{-1}$ is our (weak) solution in the sense of Theorem 2.5 such that $0 < u_-(t) \leq u(t) \leq u_+(t)$.

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