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ON REFLEXIVE MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring and U an R-module. The aim of this paper is to study the duality between U-reflexive (pre)envelopes and U-reflexive (pre)covers of R-modules.

1. Introduction and preliminaries

Let R be a commutative ring and U an R-module. For R-modules M and N, $\mathrm{Hom}(M,N)$ means $\mathrm{Hom}_R(M,N)$, and $\mathrm{Ext}^n(M,N)$ means $\mathrm{Ext}^n_R(M,N)$ for an integer $n \geq 1$. For an R-module M, $\mathrm{Hom}(M,U)$ is called the dual module of M with respect to U and denoted by M^*. For a homomorphism f between R-modules, we put $f^* = \mathrm{Hom}(f,U)$. Let $\delta_M : M \to M^{**}$ via $\delta_M(x)(f) = f(x)$ for any $x \in M$ and $f \in M^*$ be the canonical evaluation homomorphism. If δ_M is an isomorphism, then M is called a U-reflexive module. We denote by $\mathcal{R}_U$ the class of U-reflexive modules.

Let R be a Noetherian ring. A finitely generated R-module M is said to have Gorenstein dimension (abbr. G-dimension) zero [2] if $M \cong \mathrm{Hom}(\mathrm{Hom}(M,R),R)$ (i.e., M is R-reflexive); and $\mathrm{Ext}^i(M,R) = 0 = \mathrm{Ext}^i(\mathrm{Hom}(M,R),R)$ for all $i \geq 1$. We denote by $\mathcal{G}(R)$ the class of R-modules having G-dimension zero.

Let $\mathcal{F}$ be a class of R-modules and M an R-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{F}$ is called an $\mathcal{F}$-preenvelope of M [10] if for any homomorphism f: $M \to F^{'}$ with $F^{'} \in \mathcal{F}$, there is a homomorphism $g : F \to F^{'}$ such that $g\phi = f$. Moreover, if the only such g are automorphisms of F when $F^{'} = F$ and $f = \phi$, the $\mathcal{F}$-preenvelope ϕ is called an $\mathcal{F}$-envelope of M. An $\mathcal{F}$-envelope $\phi : M \to F$ is said to have the unique mapping property [9] if for any homomorphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a unique homomorphism $g : F \to F'$ such that $g\phi = f$. Dually we have the definitions of an $\mathcal{F}$-precover and an $\mathcal{F}$-cover (with the unique mapping property). A homomorphism of R-modules $f : M_1 \to M_2$ is said to be an $\mathcal{F}$-covering (or covering) morphism [12] if M_1 and M_2 have $\mathcal{F}$-covers $\varphi_1 : F_1 \to M_1$ and $\varphi_2 : F_2 \to M_2$ and

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some (so every) lifting $g : F_1 \to F_2$ with $\varphi_2 g = f\varphi_1$ is an isomorphism. A homomorphism is said to be enveloping if the dual situation holds. Covering and enveloping morphisms were introduced in [12], and studied in [11, 13, 14]. They are useful in developing a categorical theory analogous to the Galois theory of fields.

U-reflexive modules (such as $U = R$ or $U = E$, an injective cogenerator of the category of R-modules) have been extensively explored by many authors (see, for example, [3, 4, 5, 6, 24, 25]).

In this paper, we continue the study of U-reflexive modules. Section 2 is devoted to investigating the duality between U-reflexive (pre)envelopes and (pre)covers of R-modules. Let M be an R-module and G a U-reflexive module, and let $\gamma : G \to M^*$ be a homomorphism. It is proved that γ is an $\mathcal{R}_U$-(pre)cover if and only if $\gamma^*\delta_M$ is an $\mathcal{R}_U$-(pre)envelope (Theorem 2.1). For a homomorphism $\mu : M \to G$, it is proved that μ is an $\mathcal{R}_U$-(pre)envelope if and only if μ^* is an $\mathcal{R}_U$-(pre)cover (Theorem 2.2). As corollaries, we obtain that each finitely generated R-module has a $\mathcal{G}(R)$-preenvelope when R is Gorenstein (i.e., R is Noetherian and has finite self-injective dimension) (Corollary 2.3), and that if R is a local Cohen-Macaulay ring admitting a dualizing module, then each finitely generated R-module has a maximal Cohen-Macaulay envelope and each finitely generated R-module of finite Gorenstein dimension has a $\mathcal{G}(R)$-envelope (Corollary 2.4). At the end of this section, for a homomorphism of R-modules $f : M_1 \to M_2$, we prove that f is an $\mathcal{R}_U$-enveloping morphism if and only if f^* is an $\mathcal{R}_U$-covering morphism (Proposition 2.5).

In Section 3, we mainly study the duality between (pre)envelopes and (pre)covers of U-reflexive modules. Let $\mathcal{F}$ and $\mathcal{C}$ be subclasses of $\mathcal{R}_U$ such that $\mathcal{F}^* \subseteq \mathcal{C}$ and $\mathcal{C}^* \subseteq \mathcal{F}$, where $\mathcal{F}^* = \{X^* \mid X \in \mathcal{F}\}$ and $\mathcal{C}^* = \{X^* \mid X \in \mathcal{C}\}$. Let M be a U-reflexive module and $\phi : F \to M$ a homomorphism with $F \in \mathcal{F}$. It is shown that ϕ is an $\mathcal{F}$-(pre)cover if and only if ϕ^* is a $\mathcal{C}$-(pre)envelope (Proposition 3.1). For an R-module M and a homomorphism $\psi : M \to C$ with $C \in \mathcal{C}$, we prove that ψ is a $\mathcal{C}$-(pre)envelope if and only if ψ^* is an $\mathcal{F}$-(pre)cover (Proposition 3.2). In particular, [25, Theorem 6] is obtained as a corollary of the propositions above. Finally, if R is a commutative Noetherian ring and M a finitely generated R-reflexive module, we prove that M is in $\mathcal{G}(R)$ if and only if M has a $\mathcal{G}(R)$-envelope with the unique mapping property if and only if M has a $\mathcal{G}(R)$-cover with the unique mapping property (Corollary 3.7).

2. *U*-reflexive (pre)envelopes and (pre)covers

We start with the following:

Theorem 2.1. *Let M be an R-module and $G \in \mathcal{R}_U$, and let $\gamma : G \to M^*$ be a homomorphism. Then*

(1) *γ is an $\mathcal{R}_U$-precover if and only if $\gamma^*\delta_M$ is an $\mathcal{R}_U$-preenvelope.*
(2) *γ is an $\mathcal{R}_U$-cover if and only if $\gamma^*\delta_M$ is an $\mathcal{R}_U$-envelope.*

Proof. First we note that there is a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\gamma"] \arrow[d, "\delta_G"'] & M^* \arrow[d, shift left, "\delta_{M^*}"] \\ G^{**} \arrow[r, "\gamma^{**}"] & M^{***} \arrow[u, shift left, "(\delta_M)^*"] \end{tikzcd}$$

such that $\delta_{M^*}\gamma = \gamma^{**}\delta_G$ and $(\delta_M)^*\delta_{M^*} = 1_{M^*}$ by [1, Proposition 20.14]. So $\gamma = (\delta_M)^*\gamma^{**}\delta_G$.

(1) Suppose that γ is an $\mathcal{R}_U$-precover. Let $A \in \mathcal{R}_U$. Since $A \cong A^{**}$, by the "swap" isomorphism ([8, p. 12]), we have the following natural equivalences of functors

$$\mathrm{Hom}(-, A) \simeq \mathrm{Hom}(-, A^{**}) \simeq \mathrm{Hom}(A^*, (-)^*).$$

So we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(G^*, A) \arrow[r, "\mathrm{Hom}(\gamma^*, A)"] \arrow[d, "\cong"] & \mathrm{Hom}(M^{**}, A) \arrow[r, "\mathrm{Hom}(\delta_M, A)"] \arrow[d, "\cong"] & \mathrm{Hom}(M, A) \arrow[d, "\cong"] \\ \mathrm{Hom}(A^*, G^{**}) \arrow[r, "\mathrm{Hom}(A^*, \gamma^{**})"] & \mathrm{Hom}(A^*, M^{***}) \arrow[r, "\mathrm{Hom}(A^*, (\delta_M)^*)"] & \mathrm{Hom}(A^*, M^*) \end{tikzcd}$$

On the other hand, since

$$\begin{aligned} \mathrm{Hom}(A^*, (\delta_M)^*)\mathrm{Hom}(A^*, \gamma^{**})\mathrm{Hom}(A^*, \delta_G) &= \mathrm{Hom}(A^*, (\delta_M)^*\gamma^{**}\delta_G) \\ &= \mathrm{Hom}(A^*, \gamma) \end{aligned}$$

and $\mathrm{Hom}(A^*, \gamma)$ is epic by hypothesis (for $A^* \in \mathcal{R}_U$), $\mathrm{Hom}(A^*, (\delta_M)^*)\mathrm{Hom}(A^*, \gamma^{**})$ is epic. Thus $\mathrm{Hom}(\delta_M, A)\mathrm{Hom}(\gamma^*, A)$ is epic by the commutative diagram above, and so $\gamma^*\delta_M$ is an $\mathcal{R}_U$-preenvelope.

Conversely, if $\gamma^*\delta_M$ is an $\mathcal{R}_U$-preenvelope, and let $f : A \to M^*$ be a homomorphism with $A \in \mathcal{R}_U$. Then there exists $g : G^* \to A^*$ such that $g\gamma^*\delta_M = f^*\delta_M$. Thus

$$(\delta_M)^* f^{**} = (\delta_M)^*\gamma^{**}g^* = (\delta_M)^*\gamma^{**}\delta_G(\delta_G)^{-1}g^* = \gamma(\delta_G)^{-1}g^*.$$

Let $h = (\delta_G)^{-1}g^*\delta_A : A \to G$. Then $\gamma h = \gamma(\delta_G)^{-1}g^*\delta_A = (\delta_M)^* f^{**}\delta_A = f$. So γ is an $\mathcal{R}_U$-precover.

(2) Suppose that γ is an $\mathcal{R}_U$-cover. Then $\gamma^*\delta_M$ is an $\mathcal{R}_U$-preenvelope by (1). It is enough to show that each endomorphism $f : G^* \to G^*$ with $f\gamma^*\delta_M = \gamma^*\delta_M$ is an automorphism. Since $f\gamma^*\delta_M = \gamma^*\delta_M$, $(\delta_M)^*\gamma^{**}f^* = (\delta_M)^*\gamma^{**}$. On the other hand, let $h = (\delta_G)^{-1}f^*\delta_G$. Then

$$\begin{aligned} \gamma h &= \gamma(\delta_G)^{-1}f^*\delta_G \\ &= ((\delta_M)^*\gamma^{**}\delta_G)(\delta_G)^{-1}f^*\delta_G \text{ (for } (\delta_M)^*\gamma^{**}\delta_G = \gamma) \\ &= (\delta_M)^*\gamma^{**}f^*\delta_G = (\delta_M)^*\gamma^{**}\delta_G \\ &= \gamma. \end{aligned}$$

Thus h is an automorphism (for γ is an $\mathcal{R}_U$-cover), and so is f^*. It follows that $f = (\delta_{G^*})^{-1} f^{**} \delta_{G^*}$ is an automorphism too.

Conversely, if $\gamma^* \delta_M$ is an $\mathcal{R}_U$-envelope, then γ is an $\mathcal{R}_U$-precover by (1). It is enough to show that each endomorphism $g : G \to G$ with $\gamma g = \gamma$ is an automorphism. In this case we have $g^* \gamma^* \delta_M = \gamma^* \delta_M$, and so g^* is an automorphism. Thus $g = (\delta_G)^{-1} g^{**} \delta_G$ is an automorphism, as desired. $\square$

Theorem 2.2. *Let M be an R-module and $G \in \mathcal{R}_U$, and let $\mu : M \to G$ be a homomorphism. Then*

(1) *μ is an $\mathcal{R}_U$-preenvelope if and only if μ^* is an $\mathcal{R}_U$-precover.*
(2) *μ is an $\mathcal{R}_U$-envelope if and only if μ^* is an $\mathcal{R}_U$-cover.*

Proof. (1) Let $A \in \mathcal{R}_U$ and μ be an $\mathcal{R}_U$-preenvelope. By the "swap" isomorphism ([8, p. 12]), we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(A, G^*) \arrow[r, "\mathrm{Hom}(A,\mu^*)"] \arrow[d, "\cong"] & \mathrm{Hom}(A, M^*) \arrow[d, "\cong"] \\ \mathrm{Hom}(G, A^*) \arrow[r, "\mathrm{Hom}(\mu,A^*)"] & \mathrm{Hom}(M, A^*) \end{tikzcd}$$

Since $A^* \in \mathcal{R}_U$ and μ is an $\mathcal{R}_U$-preenvelope, $\mathrm{Hom}(\mu, A^*)$ is epic. Thus $\mathrm{Hom}(A, \mu^*)$ is epic by the commutative diagram above, and hence μ^* is an $\mathcal{R}_U$-precover.

Conversely, suppose that μ^* is an $\mathcal{R}_U$-precover, and let $A \in \mathcal{R}_U$. By the "swap" isomorphism ([8, p. 12]) again, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(A^*, G^*) \arrow[r, "\mathrm{Hom}(A^*,\mu^*)"] \arrow[d, "\cong"] & \mathrm{Hom}(A^*, M^*) \arrow[d, "\cong"] \\ \mathrm{Hom}(G, A) \arrow[r, "\mathrm{Hom}(\mu,A)"] & \mathrm{Hom}(M, A) \end{tikzcd}$$

Note that $A^* \in \mathcal{R}_U$. Thus $\mathrm{Hom}(A^*, \mu^*)$ is epic (for μ^* is an $\mathcal{R}_U$-precover), and so $\mathrm{Hom}(\mu, A)$ is epic by the commutative diagram above. Therefore μ is an $\mathcal{R}_U$-preenvelope.

(2) Suppose that μ is an $\mathcal{R}_U$-envelope. Then μ^* is an $\mathcal{R}_U$-precover by (1), and it is enough to show that each endomorphism $g : G^* \to G^*$ with $\mu^* g = \mu^*$ is an automorphism. Applying the functor $(-)^*$ to $\mu^* g = \mu^*$ gives $g^* \mu^{**} = \mu^{**}$. Thus $\delta_G \mu = \mu^{**} \delta_M = g^* \mu^{**} \delta_M = g^* \delta_G \mu$, and hence $(\delta_G)^{-1} g^* \delta_G \mu = \mu$. Because μ is an $\mathcal{R}_U$-envelope, $(\delta_G)^{-1} g^* \delta_G$ is an automorphism. It follows that g is an automorphism.

Conversely, suppose that μ^* is an $\mathcal{R}_U$-cover. Then μ is an $\mathcal{R}_U$-preenvelope by (1), and it is enough to show that each endomorphism $g : G \to G$ with $g\mu = \mu$ is an automorphism. In this case we have $\mu^* g^* = \mu^*$. Thus g^* is an automorphism, and so g is also an automorphism. $\square$

Auslander and Bridger [2] introduced the G-dimension for finitely generated modules. Enochs and Jenda [16] defined Gorenstein projective modules (not necessarily finitely generated) over general rings. Also, they defined the Gorenstein projective dimension for arbitrary (non-finitely generated) modules. It is well known that for finitely generated modules over a commutative Noetherian ring, the Gorenstein projective dimension agrees with the G-dimension.

Corollary 2.3. (1) *If R is Noetherian, then each finitely generated R-module M such that* $\mathrm{Hom}(M, R)$ *has finite Gorenstein projective dimension has a $\mathcal{G}(R)$-preenvelope.*

(2) *If R is Gorenstein, then each finitely generated R-module has a $\mathcal{G}(R)$-preenvelope.*

Proof. (1) Let M be a finitely generated R-module with G-dimension n $(< \infty)$. By [21, Theorem 5.5], there exists an exact sequence $0 \to K \to G_0 \to M \to 0$ of R-modules such that the projective dimension of K is $n-1$ and G_0 is Gorenstein projective. It is easy to see that it is a $\mathcal{G}(R)$-precover of M. Now the assertion follows from Theorem 2.1.

(2) is immediate since each finitely generated R-module has finite Gorenstein projective dimension over Gorenstein rings. □

Let R be a local Notherian ring and M a finitely generated R-module. We denote by $\mathcal{M}_{cm}$ the class of maximal Cohen-Macaulay R-modules. We know that if R is Cohen-Macaulay and $M \in \mathcal{G}(R)$, then $M \in \mathcal{M}_{cm}$, and that if R is Gorenstein, then $M \in \mathcal{G}(R)$ if and only if $M \in \mathcal{M}_{cm}$. Furthermore, by [7, Theorem 3.3.10(d)], when R is a local Cohen-Macaulay ring admitting a dualizing module ω, we have that if $M \in \mathcal{M}_{cm}$, then $M^* \in \mathcal{M}_{cm}$, where $M^* = \mathrm{Hom}(M, \omega)$, and that $\mathcal{M}_{cm}$ is a class of ω-reflexive modules.

Auslander's last theorem says that every finitely generated module over a local Gorenstein ring has a minimal Cohen-Macaulay approximation. Yoshino [26] extended Auslander's result to local Cohen-Macaulay rings admitting a dualizing module (or see [19, Corollary 2.4]). Enochs proved that every finitely generated module of finite Gorenstein projective dimension has a finitely generated Gorenstein projective cover over a local Cohen-Macaulay ring admitting a dualizing module ([18, Theorem 5.5]). By Theorem 2.1 and these results, we immediately get the following:

Corollary 2.4. *Let R be a local Cohen-Macaulay ring admitting a dualizing module. Then*

1. *Every finitely generated R-module has an $\mathcal{M}_{cm}$-envelope.*
2. *Every finitely generated R-module of finite G-dimension has a $\mathcal{G}(R)$-envelope.*

We end this section with the following result.

Proposition 2.5. *Let $f : M_1 \to M_2$ be a homomorphism of R-modules. Then f is an $\mathcal{R}_U$-enveloping morphism if and only if f^* is an $\mathcal{R}_U$-covering morphism.*

Proof. "$\Rightarrow$" By hypothesis, there exist $\mathcal{R}_U$-envelopes $\varphi_1 : M_1 \to G_1$ and $\varphi_2 : M_2 \to G_2$ of M_1 and M_2 respectively and an isomorphism $g : G_1 \to G_2$ with $g\varphi_1 = \varphi_2 f$. Thus $\varphi_1^* : G_1^* \to M_1^*$ and $\varphi_2^* : G_2^* \to M_2^*$ are $\mathcal{R}_U$-covers of M_1^* and M_2^* respectively by Theorem 2.2, and $\varphi_1^* g^* = f^* \varphi_2^*$. Since the lifting g^* is an isomorphism, f^* is an $\mathcal{R}_U$-covering morphism.

"$\Leftarrow$" By hypothesis, there exist $\mathcal{R}_U$-covers $\psi_1 : G_1 \to M_1^*$ and $\psi_2 : G_2 \to M_2^*$ of M_1^* and M_2^* respectively and an isomorphism $h : G_2 \to G_1$ with $\psi_1 h = f^* \psi_2$. Thus $\psi_1^* \delta_{M_1} : M_1 \to G_1^*$ and $\psi_2^* \delta_{M_2} : M_2 \to G_2^*$ are $\mathcal{R}_U$-envelopes of M_1 and M_2 respectively by Theorem 2.1, and $h^* \psi_1^* = \psi_2^* f^{**}$. It follows that $\psi_2^* \delta_{M_2} f = h^* \psi_1^* \delta_{M_1}$. Since the extending h^* is an isomorphism, f is an $\mathcal{R}_U$-enveloping morphism. $\square$

3. (Pre)envelopes and (pre)covers of U-reflexive modules

In this section, we mainly study the duality between (pre)envelopes and (pre)covers of U-reflexive modules. In what follows, $\mathcal{F}$ and $\mathcal{C}$ denote subclasses of $\mathcal{R}_U$ such that $\mathcal{F}^* \subseteq \mathcal{C}$ and $\mathcal{C}^* \subseteq \mathcal{F}$, where $\mathcal{F}^* = \{X^* \mid X \in \mathcal{F}\}$ and $\mathcal{C}^* = \{X^* \mid X \in \mathcal{C}\}$.

The following result is a dual version of [15, Theorem 3.1] and [22, Lemma 3.1].

Proposition 3.1. *Let M be a U-reflexive module and $\phi : F \to M$ a homomorphism with $F \in \mathcal{F}$. Then ϕ is an $\mathcal{F}$-(pre)cover if and only if ϕ^* is a $\mathcal{C}$-(pre)envelope.*

Proof. "$\Rightarrow$" Suppose ϕ is an $\mathcal{F}$-precover. Let $g : M^* \to C$ be a homomorphism of R-modules with $C \in \mathcal{C}$. For the homomorphism $(\delta_M)^{-1} g^* : C^* \to M$, there exists a homomorphism $h : C^* \to F$ such that $\phi h = (\delta_M)^{-1} g^*$ since ϕ is an $\mathcal{F}$-precover and $C^* \in \mathcal{F}$ by hypothesis. So $\delta_M \phi h = g^*$, and then $h^* \phi^* (\delta_M)^* = g^{**}$. Since $(\delta_M)^* \delta_{M^*} = 1_{M^*}$ by [1, Proposition 20.14], $h^* \phi^* = g^{**} \delta_{M^*}$. Set $\gamma = (\delta_C)^{-1} h^* : F^* \to \mathcal{C}$. By the following commutative diagram

$$\begin{tikzcd} M^* \arrow[r, "\delta_{M^*}"] \arrow[d, "g"] & M^{***} \arrow[d, "g^{**}"] \\ C \arrow[r, "\delta_C"] & C^{**} \end{tikzcd}$$

we have that $\gamma \phi^* = (\delta_C)^{-1} h^* \phi^* = (\delta_C)^{-1} g^{**} \delta_{M^*} = g$. Therefore ϕ^* is a $\mathcal{C}$-preenvelope.

Furthermore, if ϕ is an $\mathcal{F}$-cover, then ϕ^* is a $\mathcal{C}$-preenvelope by the proof above. For any endomorphism $\varphi : F^* \to F^*$ such that $\varphi \phi^* = \phi^*$, we have $\phi^{**} \varphi^* = \phi^{**}$. Since $\delta_M \phi = \phi^{**} \delta_F$, $\phi = (\delta_M)^{-1} \phi^{**} \delta_F$. Set $\eta = (\delta_F)^{-1} \varphi^* \delta_F$. We have that

$$\begin{aligned} \phi\eta &= \phi(\delta_F)^{-1} \varphi^* \delta_F = ((\delta_M)^{-1} \phi^{**} \delta_F)(\delta_F)^{-1} \varphi^* \delta_F \\ &= (\delta_M)^{-1} \phi^{**} \varphi^* \delta_F = (\delta_M)^{-1} \phi^{**} \delta_F \\ &= \phi. \end{aligned}$$

Thus η is an automorphism, and so is φ^*. It follows that φ is an automorphism. So ϕ^* is a $\mathcal{C}$-envelope.

"$\Leftarrow$" Suppose ϕ^* is a $\mathcal{C}$-preenvelope. Let $f : A \to M$ be a homomorphism of R-modules with $A \in \mathcal{F}$. Then there exists a homomorphism $g : F^* \to A^*$ such that $g\phi^* = f^*$ since ϕ^* is a $\mathcal{C}$-preenvelope and $A^* \in \mathcal{C}$ by hypothesis. So $\phi^{**}g^* = f^{**}$. Set $h = (\delta_F)^{-1}g^*\delta_A$. By the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "\delta_A"] \arrow[d, "f"] & A^{**} \arrow[ld, bend left, "f^{**}"] \\
F \arrow[r, "\phi"] \arrow[d, "\delta_F"] & M \arrow[d, "\delta_M"] & \\
F^{**} \arrow[r, "\phi^{**}"] & M^{**} &
\end{tikzcd}$$

we have that

$$\begin{aligned}
\phi h &= \phi(\delta_F)^{-1}g^*\delta_A = (\delta_M)^{-1}(\delta_M\phi(\delta_F)^{-1})g^*\delta_A \\
&= (\delta_M)^{-1}\phi^{**}g^*\delta_A = (\delta_M)^{-1}f^{**}\delta_A \\
&= (\delta_M)^{-1}(\delta_M f) = f.
\end{aligned}$$

Thus ϕ is an $\mathcal{F}$-precover.

Furthermore, if ϕ^* is a $\mathcal{C}$-envelope, then ϕ is an $\mathcal{F}$-precover by the proof above. For any endomorphism $\alpha : F \to F$ such that $\phi\alpha = \phi$, we have $\alpha^*\phi^* = \phi^*$. Hence α^* is an automorphism. Thus α is also an automorphism for $\delta_F\alpha = \alpha^{**}\delta_F$ and $F \in \mathcal{R}_U$. Therefore ϕ is an $\mathcal{F}$-cover. $\square$

As a special case of [15, Theorem 3.1] and [22, Lemma 3.1], we have the following:

Proposition 3.2. *Let M be an R-module and $\psi : M \to C$ a homomorphism with $C \in \mathcal{C}$. Then ψ is a $\mathcal{C}$-(pre)envelope if and only if ψ^* is an $\mathcal{F}$-(pre)cover.*

As applications, we list two corollaries of the propositions above.

Corollary 3.3. *The following statements are equivalent*:

(1) *Every R-reflexive module has a $\mathcal{G}(R)$-(pre)envelope.*
(2) *Every R-reflexive module has a $\mathcal{G}(R)$-(pre)cover.*

Corollary 3.4 ([25, Theorem 6])**.** *Let U be the injective cogenerator of the category of R-modules. Then the following statements are equivalent*:

(1) *Every U-reflexive R-module has a U-reflexive injective envelope.*
(2) *Every U-reflexive R-module has a U-reflexive flat cover.*

Proof. This follows from the fact that a U-reflexive R-module M is flat (injective) if and only if M^* is injective (flat) by [25, Corollary 3]. $\square$

The following proposition may be considered as a dual version of Proposition 2.5 for U-reflexive modules.

Proposition 3.5. *Let $f : M_1 \to M_2$ be a homomorphism of R-modules with M_1 and M_2 being U-reflexive modules. Then f is an $\mathcal{R}_U$-covering morphism if and only if f^* is an $\mathcal{R}_U$-enveloping morphism.*

Proof. "$\Rightarrow$" By hypothesis, there exist $\mathcal{R}_U$-covers $\varphi_1 : G_1 \to M_1$ and $\varphi_2 : G_2 \to M_2$ of M_1 and M_2 respectively and an isomorphism $g : G_1 \to G_2$ with $\varphi_2 g = f\varphi_1$. Thus $\varphi_1^* : M_1^* \to G_1^*$ and $\varphi_2^* : M_2^* \to G_2^*$ are $\mathcal{R}_U$-envelopes of M_1^* and M_2^* respectively by Proposition 3.1, and $\varphi_1^* f^* = g^*\varphi_2^*$. Since the extending g^* is an isomorphism, f^* is an $\mathcal{R}_U$-enveloping morphism.

"$\Leftarrow$" By hypothesis, there exist $\mathcal{R}_U$-envelopes $\psi_1 : M_1^* \to G_1$ and $\psi_2 : M_2^* \to G_2$ of M_1^* and M_2^* respectively and an isomorphism $h : G_2 \to G_1$ with $h\psi_2 = \psi_1 f^*$. Thus $\psi_1^* : G_1^* \to M_1^{**}$ and $\psi_2^* : G_2^* \to M_2^{**}$ are $\mathcal{R}_U$-covers of M_1^{**} and M_2^{**} respectively by Proposition 3.2, and $f^{**}\psi_1^* = \psi_2^* h^*$. Note that M_1 and M_2 are U-reflexive modules, and so $(\delta_{M_1})^{-1}\psi_1^*$ and $(\delta_{M_2})^{-1}\psi_2^*$ are R_U-covers of M_1 and M_2 respectively. It is easy to check that $f(\delta_{M_1})^{-1}\psi_1^* = (\delta_{M_2})^{-1}\psi_2^* h^*$. Since the lifting h^* is an isomorphism, f is an $\mathcal{R}_U$-covering morphism. $\square$

Proposition 3.6. *Let M be an R-module.*

(1) *$\psi : M \to C$ is a $\mathcal{C}$-envelope with the unique mapping property if and only if $\psi^* : C^* \to M^*$ is an $\mathcal{F}$-cover with the unique mapping property.*
(2) *If $M \in \mathcal{R}_U$, then $\phi : F \to M$ is an $\mathcal{F}$-cover with the unique mapping property if and only if $\phi^* : M^* \to F^*$ is a $\mathcal{C}$-envelope with the unique mapping property.*

Proof. (1) By Proposition 3.2, $\psi : M \to C$ is a $\mathcal{C}$-envelope if and only if $\psi^* : C^* \to M^*$ is an $\mathcal{F}$-cover. For any $F \in \mathcal{F}$, by the "swap" isomorphism ([8, p. 12]), we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}(F, C^*) \arrow[r, "\mathrm{Hom}(F,\psi^*)"] \arrow[d, "\cong"] & \mathrm{Hom}(F, M^*) \arrow[d, "\cong"] \\ \mathrm{Hom}(C, F^*) \arrow[r, "\mathrm{Hom}(\psi,F^*)"] & \mathrm{Hom}(M, F^*) \end{tikzcd}$$

If $\psi : M \to C$ is a $\mathcal{C}$-envelope with the unique mapping property, then $\mathrm{Hom}(\psi, F^*)$ is an isomorphism. So $\mathrm{Hom}(F, \psi^*)$ is an isomorphism by the commutative diagram above. Thus $\psi^* : C^* \to M^*$ is an $\mathcal{F}$-cover with the unique mapping property. Conversely, the proof is similar since for any $C' \in \mathcal{C}$ we have that $C' \cong F^*$ for some $F \in \mathcal{F}$ by assumption.

(2) By Proposition 3.1, when $M \in \mathcal{R}_U$, the homomorphism $\phi : F \to M$ is an $\mathcal{F}$-cover if and only if $\phi^* : M^* \to F^*$ is a $\mathcal{C}$-envelope. For any $C \in \mathcal{C}$, by assumption and the "swap" isomorphism, we have the following commutative

diagram:

$$\begin{tikzcd}
\mathrm{Hom}(F^*, C) \arrow[r, "\mathrm{Hom}(\phi^*, C)"] \arrow[d, "\cong"] & \mathrm{Hom}(M^*, C) \arrow[d, "\cong"] \\
\mathrm{Hom}(F^*, C^{**}) \arrow[r, "\mathrm{Hom}(\phi^*, C^{**})"] \arrow[d, "\cong"] & \mathrm{Hom}(M^*, C^{**}) \arrow[d, "\cong"] \\
\mathrm{Hom}(C^*, F^{**}) \arrow[r, "\mathrm{Hom}(C^*, \phi^{**})"] & \mathrm{Hom}(C^*, M^{**}) \\
\mathrm{Hom}(C^*, F) \arrow[r, "\mathrm{Hom}(C^*, \phi)"] \arrow[u, "\cong"] & \mathrm{Hom}(C^*, M) \arrow[u, "\cong"]
\end{tikzcd}$$

Using a proof similar to that of (1), and noting that for any $F' \in \mathcal{F}$ we have $F' \cong C^*$ for some $C \in \mathcal{C}$, we get that $\phi : F \to M$ is an $\mathcal{F}$-cover with the unique mapping property if and only if $\phi^* : M^* \to F^*$ is a $\mathcal{C}$-envelope with the unique mapping property. □

By [23], there exist finitely generated R-reflexive modules which are not in $\mathcal{G}(R)$ over a commutative Noetherian local ring R. Now we get the following corollary.

Corollary 3.7. *Let R be a Noetherian ring and M a finitely generated R-reflexive module. Then the following statements are equivalent*:

(1) *M has a $\mathcal{G}(R)$-envelope with the unique mapping property.*
(2) *M has a $\mathcal{G}(R)$-cover with the unique mapping property.*
(3) *$M \in \mathcal{G}(R)$.*

Proof. (3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (2) are clear.

(1) $\Rightarrow$ (3) Let $\phi : M \to G$ be a $\mathcal{G}(R)$-envelope with the unique mapping property. Then $\phi^* : G^* \to M^*$ is an isomorphism for $R \in \mathcal{G}(R)$, and so $M^{**} \cong G^{**}$. Thus $M \in \mathcal{G}(R)$ since M is R-reflexive.

(2) $\Rightarrow$ (3) Suppose $\psi : G \to M$ is a $\mathcal{G}(R)$-cover with the unique mapping property. Then $\psi^* : M^* \to G^*$ is a $G(R)$-envelope with the unique mapping property by Proposition 3.6(2). So $M^* \in \mathcal{G}(R)$ by the equivalence of (1) and (3). Thus $M \in \mathcal{G}(R)$ by the R-reflexivity of M. □

By an argument similar to that of Corollary 3.7 we have the following result.

Corollary 3.8. *Let R be a local Cohen-Macaulay ring admitting a dualizing module ω and M a finitely generated ω-reflexive module. Then the following statements are equivalent*:

(1) *M has an $\mathcal{M}_{cm}$-envelope with the unique mapping property.*
(2) *M has an $\mathcal{M}_{cm}$-cover with the unique mapping property.*
(3) *$M \in \mathcal{M}_{cm}$.*

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