ON EVALUATIONS OF THE MODULAR $j$–INARIANT BY MODULAR EQUATIONS OF DEGREE 2

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Abstract. We derive modular equations of degree 2 to establish explicit relations for the parameterizations for the theta functions $\phi$ and $\psi$. We then find specific values of the parameterizations to evaluate some new values of the modular $j$–invariant in terms of $J_n$.

1. Introduction

The invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H} = \{\tau : \text{Im} \tau > 0\}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \quad \text{and} \quad j(\tau) = 1728J(\tau),$$

where,

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau),$$

$$g_2(\tau) = 60 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m\tau + n)^4},$$

and

$$g_3(\tau) = 140 \sum_{m,n=-\infty}^{\infty} \frac{1}{(m\tau + n)^6}.$$

Moreover, the function $\gamma_2(\tau)$ is defined by ([5, p. 249])

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)},$$

where the principal branch is chosen. Ramanujan defined a function $J_n$ by

$$J_n = \frac{1 - 16\alpha_n(1 - \alpha_n)}{8(4\alpha_n(1 - \alpha_n))^{1/3}},$$

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where \( \alpha_n = k_n^2 \) and \( n \) is a natural number. Here, as usual, in the theory of elliptic functions, let \( k, 0 < k < 1 \), denote the modulus, then the singular modulus \( k_n \) is defined by \( k_n = k(e^{-\pi \sqrt{n}}) \). To identify \( J_n \) with the class invariant \( G_n \), we first, as usual, set

\[
(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1,
\]

and let

\[
\chi(q) = (-q; q^2)_{\infty}.
\]

Furthermore, for \( |ab| < 1 \), Ramanujan’s general theta function \( f(a, b) \) is defined by

\[
f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}.
\]

Then, the classical theta functions \( \varphi, \psi, \) and \( f \) are defined by, for \( |q| < 1 \),

\[
\varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty}},
\]

\[
\psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^3)_{\infty}},
\]

and

\[
f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty}.
\]

From \( (q; q)_{\infty} = (q; q^2)_{\infty} (q^2; q^2)_{\infty} \), it is easily seen that

\[
\chi(-q) = \frac{f(-q)}{f(-q^2)}.
\]

If \( q_n = e^{-\pi \sqrt{n}} \) and \( n \) is a positive rational number, then the class invariant \( G_n \) is defined by

\[
G_n = 2^{-1/4} q_n^{-1/24} \chi(q_n).
\]

Since, by [2, p. 124],

\[
\chi(q) = 2^{1/6} \left( \frac{q}{\alpha(1 - \alpha)} \right)^{1/24},
\]

it follows from (1.4) that

\[
G_n = (4\alpha_n (1 - \alpha_n))^{-1/24}.
\]

Hence, by (1.2) and (1.5), we find that

\[
J_n = \frac{1}{8} G_n^8 (1 - 4G_n^{-24}).
\]
We now identify $J_n$ with $\gamma_2$. From [5, Theorem 12.17], for $q = e^{2\pi i r}$,

\begin{equation}
\gamma_2(\tau) = 2^8 q^{2/3} f^{16}(-q^2) + \frac{f^8(-q)}{q^{1/3} f^8(-q^2)}.
\end{equation}

Setting $\tau = \frac{3 + \sqrt{-n}}{2}$, we deduce from (1.3), (1.4), and (1.7) that

\begin{equation}
\gamma_2\left(\frac{3 + \sqrt{-n}}{2}\right) = -4G_n^8(1 - 4G_n^{-24}).
\end{equation}

Hence, by (1.1) and (1.6), we have

\begin{equation}
J_n = -\frac{1}{32} \gamma_2\left(\frac{3 + \sqrt{-n}}{2}\right) = -\frac{1}{32} \sqrt{j\left(\frac{3 + \sqrt{-n}}{2}\right)}.
\end{equation}

Therefore, in general, the value of $J_n$ can be obtained in terms of $G_n$ or $j\left(\frac{3 + \sqrt{-n}}{2}\right)$ for any natural number $n$. For 15 values of $n$ such that $n \equiv 3 \pmod{4}$, Ramanujan indicated the corresponding 15 values of $J_n$, although some are not given very explicitly. There are 13 cases when the class number of the order in an imaginary quadratic fields equals 1 ([5, p. 260]). In such an instance, the value of $j$–invariant is known to be an integer. In these cases, Ramanujan gave 7 values of $J_n$ for $n = 3, 11, 19, 27, 43, 67,$ and $163$. See [1, pp. 310–311] for more details. Formula (1.6) can be used to evaluate some of these values such as $J_3 = 0$ and $J_{27} = 5 \cdot 3^{1/3}$, but in most instances the value of $G_n$ is unavailable. For the rest of cases of degree 1, the values of $J_n$ can be evaluated by using the relation (1.8) and the corresponding $j$–invariant given in [5, p. 261]. It is also known that there are 29 cases when the degree of $j\left(\frac{3 + \sqrt{-n}}{2}\right)$ equals 2. Ramanujan dealt with 6 of these: the values of $J_n$ for $n = 35, 51, 75, 91, 99,$ and $115$, even though he did not record the value of $J_{99}$ explicitly. See also [1, pp. 311–312] for more details.

As an instance of using the values of $J_n$, Ramanujan further defined a function $t_n$ by, for $q = e^{-\pi \sqrt{n}}$,

\begin{equation}
t_n = \sqrt{3} q^{1/18} \frac{f(q^{1/3}) f(q^3)}{f^4(q)}
\end{equation}

and asserted that

\begin{equation}
t_n = \left(2 \sqrt{64 J_n^2 - 24 J_n + 9} - (16 J_n - 3)\right)^{1/6}.
\end{equation}

Ramanujan then considered the polynomials $p_n(t)$ satisfied by $t_n$ for $n = 11, 35, 59, 83,$ and $107$. These polynomials are extremely simple, whereas the corresponding polynomials of the same degrees satisfied by $J_n$ are more complicated. Refer [4] to see that if $n$ is square free, $n \equiv 11 \pmod{24}$, and the class number of the Hilbert class field is odd, then $t_n$ and $J_n$ satisfy irreducible polynomials of the same degree.

Since modular equations are crucial in this study on evaluations of the modular $j$–invariant in terms of $J_n$, we now give a definition of a modular equation.
Let \( a, b, \) and \( c \) be arbitrary complex numbers except that \( c \) cannot be a non-positive integer. Then, for \( |z| < 1 \), the Gaussian or ordinary hypergeometric function \( {}_2F_1(a, b; c; z) \) is defined by

\[
{}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n,
\]

where \((a)_0 = 1\) and \((a)_n = a(a+1)(a+2) \cdots (a+n-1)\) for each positive integer \( n \).

The complete elliptic integral of the first kind \( K(k) \) is defined by

\[
K(k) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right) = \frac{\pi}{2} \varphi^2 \left( e^{-\pi \frac{K'}{K}} \right),
\]

where \( 0 < k < 1, K' = K(k'), \) and \( k' = \sqrt{1 - k^2} \). The number \( k \) is called the modulus of \( K \) and \( k' \) is called the complementary modulus. Let \( K, K', L, \) and \( L' \) denote complete elliptic integrals of the first kind associated with the moduli \( k, k', l, \) and \( l' \), respectively, where \( 0 < k < 1 \) and \( 0 < l < 1 \). Suppose that

\[
\frac{L'}{L} = n \frac{K'}{K}
\]

holds for some positive integer \( n \). Then a relation between \( k \) and \( l \) induced by (1.10) is called a modular equation of degree \( n \). If we set

\[
q = e^{-\pi \frac{K'}{K}} \quad \text{and} \quad q' = e^{-\pi \frac{K'}{K}},
\]

then (1.10) is equivalent to the relation \( q^n = q' \). Hence a modular equation can be viewed as an identity involving theta functions at the arguments \( q \) and \( q^n \). Set \( \alpha = k^2 \) and \( \beta = l^2 \), then we say that \( \beta \) has degree \( n \) over \( \alpha \).

We now turn to evaluations of \( J_n \) by using a parametrization \( r_{k,n} \) for the theta function \( f \). In [6], \( r_{k,n} \) is defined by, for any positive real numbers \( k \) and \( n \),

\[
r_{k,n} = \frac{f(-q)}{k^{1/4} q^{(k-1)/24} f(-q^k)},
\]

where \( q = e^{-2\pi \sqrt{n/k}} \). For convenience, we write \( r_n \) instead of \( r_{2,n} \). Then, \( G_n \) can be written in terms of \( r_n \) ([6, Theorem 2.2.3])

\[
G_n = \frac{r_{2n}}{2^{1/4} r_{n/2}}.
\]

Hence (1.6) can also be written in terms of \( r_n \)

\[
J_n = \frac{1}{32} \left( \frac{r_{2n}}{r_{n/2}} \right)^8 \left( 1 - 2^8 \left( \frac{r_{n/2}}{r_{2n}} \right)^{24} \right).
\]
Using the formula (1.12), the explicit values of \( J_1, J_2, \ldots, J_{10} \) were evaluated in [6]. In particular,

\[
J_2 = \frac{5}{16} \left( -19 + 13\sqrt{2} \right),
\]

\[
J_4 = \frac{3}{32} \left( -724 + 513\sqrt{2} \right),
\]

\[
J_8 = \frac{(1 + \sqrt{2})^3 \left( 4 + \sqrt{2} + 10\sqrt{2} \right)^3 - 256}{32 (1 + \sqrt{2})^2 \left( 4 + \sqrt{2} + 10\sqrt{2} \right)^2}.
\]

Meanwhile the relation (1.6) can also be written in terms of parametrizations \( h'_{k,n} \) and \( l'_{k,n} \) for the theta function \( \varphi \) and \( \psi \), respectively. In [7, 8], \( h'_{k,n} \) and \( l'_{k,n} \) are defined by, for any positive real numbers \( k \) and \( n \),

\[
h'_{k,n} = \frac{\varphi(-q)}{k^{1/4} \varphi(-q^k)},
\]

where \( q = e^{-2\pi\sqrt{n/k}} \) and

\[
l'_{k,n} = \frac{\psi(q)}{k^{1/4} q^{(k-1)/8} \psi(q^k)},
\]

where \( q = e^{-\pi\sqrt{n/k}} \). For convenience, we write \( h'_{n} \) and \( l'_{n} \) instead of \( h'_{2,n} \) and \( l'_{2,n} \), respectively, throughout this paper. Then it follows from [8, Theorem 6.3] that

\[
l'_{2n} = \frac{r_{2n}^2}{r_{n/2}} \quad \text{and} \quad h'_{2n} = \frac{r_{n/2}^2}{r_{2n}}.
\]

Hence (1.12) can be written in the alternative form

\[
(1.13) \quad J_n = \frac{1}{32} \left( \frac{l'_{2n}}{h'_{n/2}} \right)^{8/3} \left( 1 - 2^8 \left( \frac{h'_{n/2}}{l'_{2n}} \right)^{8/3} \right).
\]

In addition, in [7], \( h_{k,n} \) is defined by, for any positive real numbers \( k \) and \( n \),

\[
h_{k,n} = \frac{\varphi(q)}{k^{1/4} \varphi(q^k)},
\]

where \( q = e^{-\pi\sqrt{n/k}} \). For convenience, we also write \( h_{n} \) in stead of \( h_{2,n} \) throughout this paper.

Note that specific values of \( h_{n} \) will play crucial roles in evaluating the corresponding values of \( h'_{n} \) and \( l'_{n} \) later on. This study is motivated by the values of \( J_2, J_4, \) and \( J_8 \) obtained from (1.12). Some values of the modular \( j \)-invariant, as mentioned before, were used to evaluate the values of \( J_n \) for \( n = 11, 19, 43, 67, \) and 163. Hence, instead of evaluating some values of the modular \( j \)-invariant to find the corresponding values of \( J_n \), we first employ (1.13) to evaluate the
values of $J_n$, and then find the corresponding values of $j \left( \frac{3+\sqrt{-n}}{2} \right)$ by the relation (1.8), in the case of when $n$ is of the form $n = 2^{2m-1}$ or $n = 2^{2m}$ for every positive integer $m$. Note that our results contain the values of $J_2$, $J_4$, and $J_8$ as special cases. In order to do so, we first derive modular equations of degree 2 for the theta functions $\varphi$ and $\psi$. We then find explicit relations for the corresponding parameterizations, evaluate some numerical values of $h_n$, $h'_n$, and $l'_n$, and evaluate some new values of $J_n$ so that we have the corresponding values of the modular $j$–invariant.

2. Preliminary results

In this section, we introduce fundamental theta function identities that will play key roles in deriving modular equations of degree 2. Let $k$ be the modulus as in (1.9). Set $x = k^2$ and also set

\begin{equation}
(2.1) \quad x = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}.
\end{equation}

Then

\begin{equation}
(2.2) \quad \varphi^2(q) = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right) = z,
\end{equation}

where

\begin{equation}
(2.3) \quad q = e^{-y} = \exp \left( -\pi \frac{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; 1-x \right)}{2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; x \right)} \right) = \exp \left( -\pi \frac{K(k')}{K(k)} \right).
\end{equation}

**Lemma 2.1** ([3, Theorem 5.4.1]). If $x$, $q$, and $z$ are related by (2.1), (2.2), and (2.3), then

(i) $\varphi(q) = \sqrt{z}$,

(ii) $\varphi(-q) = \sqrt{\frac{1}{2}} (1-x)^{1/4}$,

(iii) $\varphi(q^2) = \sqrt{z} \sqrt{1 + \sqrt{1-x^2}}$,

(iv) $\varphi(-q^2) = \sqrt{z} (1-x)^{1/8}$,

(v) $\varphi(q^4) = \frac{1}{2} \sqrt{z} \left( 1 + (1-x) \right)^{1/4}$.

**Lemma 2.2** ([3, Theorem 5.4.2]). If $x$, $q$, and $z$ are related by (2.1), (2.2), and (2.3), then

(i) $\psi(q) = \sqrt{\frac{1}{2}} \left( \frac{2}{z} \right)^{1/8}$,

(ii) $\psi(q^2) = \frac{1}{2} \sqrt{z} \left( \frac{2}{z} \right)^{1/4}$.

3. Modular equations of degree 2

In this section, we derive modular equations of degree 2 and establish some explicit relations for $h_n$, $h'_n$, and $l'_n$ for some positive real number $n$ by using these modular equations.
Theorem 3.1. If \( P = \frac{\varphi(q)}{\varphi(q^2)} \) and \( Q = \frac{\varphi(q^2)}{\varphi(q^4)} \), then
\[
PQ + \frac{2}{PQ} = \frac{Q}{P} + 2.
\]

Proof. By Lemma 2.1(i), (iii), and (v),
\[
P = \sqrt{\frac{2}{1 + \sqrt{1 - \alpha}}} \quad \text{and} \quad Q = \frac{\sqrt{2}\sqrt{1 + \sqrt{1 - \alpha}}}{1 + (1 - \alpha)^{1/4}}.
\]
Combine and rearrange the last two equalities in terms of \( P \) and \( Q \) to complete the proof. \( \Box \)

For a different proof of Theorem 3.1, see [7, Theorem 4.2]. Now, by the definition of \( h_n \), we have:

Corollary 3.2. For every positive real number \( n \), we have
\[
\sqrt{2}\left(h_n h_{4n} + \frac{1}{h_n h_{4n}}\right) = \frac{h_{4n}}{h_n} + 2.
\]
Note that (3.2) is the same equation as in [7, Theorem 4.6].

Theorem 3.3. If \( P = \frac{\varphi(q)}{\varphi(q^2)} \) and \( Q = \frac{\varphi(-q)}{\varphi(-q^2)} \), then
\[
P^2(Q^4 + 1) = 2.
\]

Proof. By Lemma 2.1(i)–(iv),
\[
P = \sqrt{\frac{2}{1 + \sqrt{1 - \alpha}}} \quad \text{and} \quad Q = (1 - \alpha)^{1/8}.
\]
Combining and rearranging the last two equalities in terms of \( P \) and \( Q \), we complete the proof. \( \Box \)

By the definitions of \( h_n \) and \( h_n' \), we have:

Corollary 3.4. For every positive real number \( n \), we have
\[
h_n^2(2h_{n/4}^4 + 1) = \sqrt{2}.
\]

Theorem 3.5. If \( P = \frac{\varphi(q)}{\varphi(q^2)} \) and \( Q = \frac{\varphi(q)}{q^{1/8}\varphi(q^4)} \), then
\[
4P^4 = (P^2 - 1)Q^8.
\]

Proof. By Lemma 2.1(i) and (iii) and Lemma 2.2,
\[
P = \sqrt{\frac{2}{1 + \sqrt{1 - \alpha}}} \quad \text{and} \quad Q = \frac{\sqrt{2}}{(\alpha^{1/8})}.
\]
Combining and rearranging the last two equalities in terms of \( P \) and \( Q \), we have the required result. \( \Box \)

By the definitions of \( h_n \) and \( l_n' \), we have:
Corollary 3.6. For every positive real number $n$, we have

\begin{equation}
2h_n^4 = (\sqrt{2}h_n^2 - 1) l_n^8.
\end{equation}

4. Specific values of $h_n$, $h_n'$, and $l_n'$

We are now in position to evaluate specific values of $h_n$, $h_n'$, and $l_n'$ for some positive real number $n$ by using the explicit relations established in Section 3.

To begin with, we show how to evaluate the values of $h_2^2$, $h_2^{2m}$, and $l_2^2$ for every positive integer $m$. We only state the instances when $m = 1, 2, 3, 4$.

Theorem 4.1. We have

(i) $h_4 = 1 + \sqrt{2} - \sqrt{1 + \sqrt{2}}$,

(ii) $h_8 = \frac{2 + \sqrt{2}}{1 + 2\sqrt{2}}$,

(iii) $h_{16} = \frac{1 + \sqrt{1 + \sqrt{2}}}{\sqrt{1 + \sqrt{2} + (2(1 + \sqrt{2}))^{1/4}}}$,

(iv) $h_{32} = \frac{2\sqrt{2} + \sqrt{1 + \sqrt{2}}}{2^{5/8} + \sqrt{1 + \sqrt{2}}}$,

(v) $h_{64} = \frac{2^{3/4}(1 + \sqrt{2})^{1/4} + \sqrt{2(1 + \sqrt{2})}}{1 + 2^{7/8}(1 + \sqrt{2})^{3/8} + \sqrt{1 + \sqrt{2}}}$,

(vi) $h_{128} = \frac{2^{7/8} + \sqrt{2 + \sqrt{2}}}{1 + 2^{7/8}(2 + \sqrt{2})^{1/4}}$,

(vii) $h_{256} = \frac{\sqrt{2}}{a + \sqrt{2 - a^2}}$,

(viii) $h_{512} = \frac{\sqrt{2}}{b + \sqrt{2 - b^2}}$,

where

\begin{align*}
a &= \frac{2^{3/4}(1 + \sqrt{2})^{1/4} + \sqrt{2(1 + \sqrt{2})}}{1 + 2^{7/8}(1 + \sqrt{2})^{3/8} + \sqrt{1 + \sqrt{2}}}, \\
b &= \frac{2^{7/8} + \sqrt{2 + \sqrt{2}}}{1 + 2^{1/4} + 2^{11/16}(2 + \sqrt{2})^{1/4}}.
\end{align*}

Proof. For (i), letting $n = 1$ in (3.2) and putting the value of $h_1 = 1$ from [7, Theorem 2.2], we find that

$$\sqrt{2} \left( h_4 + \frac{1}{h_4} \right) = h_4 + 2.$$ 

Solving the last equation for $h_4$ and using the fact that $h_4 < 1$, we complete the proof.

For (ii), letting $n = 2$ in (3.2), putting the value of $h_2 = \sqrt{2\sqrt{2} - 2}$ from [7, Theorem 4.7(i)], we find that

\begin{equation}
-(4 + 3\sqrt{2}) h_2^2 - 4\sqrt{1 + \sqrt{2}} h_2 + 2 = 0.
\end{equation}

Solving (4.1) for $h_8$, and using the fact that $h_8 < 1$, we complete the proof.
For (iii), letting \( n = 4 \) in (3.2), putting the value of \( h_4 \) from the result of (i), solving for \( h_{16} \), and using the fact that \( h_{16} < 1 \), we complete the proof.

For (iv)–(viii), repeat the same argument as in the proof of (iii) to complete the proof. \( \square \)

See [7, Theorem 4.7] for different proofs of Theorem 4.1(i) and (ii). Hence \( h_{2m}^2 \) and \( h_{2m+1}^2 \) for \( m = 5, 6, 7, \ldots \) can be evaluated as in the proof of Theorem 4.1. We next evaluate the values of \( h'_{2m-3} \) and \( h'_{2m-1} \) for every positive integer \( m \). We only state the instances when \( m = 1, 2, 3, \) and 4.

**Theorem 4.2.** Let \( a \) and \( b \) be as in Theorem 4.1. Then we have

\[
\begin{align*}
(i) & \quad h_1' = \left( \frac{-1 + \sqrt{2}}{2} \right)^{1/8}, \\
(ii) & \quad h_2' = \left( \frac{1 + \sqrt{2}}{2} \right)^{1/4}, \\
(iii) & \quad h_4' = \left( \frac{2(1 + \sqrt{2})}{\sqrt{2} + \sqrt{1 + \sqrt{2}}} \right)^{1/16}, \\
(iv) & \quad h_8' = \left( \frac{2^{5/32}(1 + \sqrt{2})^{1/8}}{1 + 2^{1/4}} \right)^{1/8}, \\
(v) & \quad h_{16}' = \left( \frac{1 + \sqrt{2} - 1}{2} \right)^{1/4}, \\
(vi) & \quad h_{32}' = \left( \frac{1 + \sqrt{2} - 1}{2} \right)^{1/4}, \\
(vii) & \quad h_{64}' = \left( \frac{a^2(\sqrt{2} - a^2)}{2} \right)^{1/8}, \\
(viii) & \quad h_{128}' = \left( \frac{b^2(\sqrt{2} - b^2)}{2} \right)^{1/8}.
\end{align*}
\]

**Proof.** For (i), letting \( n = 4 \) in (3.4) and putting the value of \( h_4 \) from Theorem 4.1(i), we find that

\[
(4.2) \quad \left( 1 + \sqrt{2} - \sqrt{1 + \sqrt{2}} \right)^2 (2h_4^4 + 1) = \sqrt{2}.
\]

Solving (4.2) for \( h_1' \) and using the fact that \( h_1' > 0 \), we complete the proof.

For (ii)–(viii), use (3.4) and Theorem 4.1(ii)–(viii), respectively, to repeat the same argument as in the proof of (i). \( \square \)

Next we show how to evaluate the values of \( l'_{2m} \) and \( l'_{2m+1} \) for every positive integer \( m \). We only state the instances when \( m = 1, 2, 3, \) and 4.

**Theorem 4.3.** Let \( a \) and \( b \) be as in Theorem 4.1. Then we have

\[
\begin{align*}
(i) & \quad l_4' = \left( 2(1 + \sqrt{2}) \right)^{1/4}, \\
(ii) & \quad l_6' = \sqrt{2} + \sqrt{3}, \\
(iii) & \quad l_{16}' = 1 + \sqrt{1 + \sqrt{2}}, \\
(iv) & \quad l_{32}' = 4 + 3\sqrt{2} + 2\sqrt{8} + 6\sqrt{2}, \\
(v) & \quad l_{64}' = \left( \frac{a^4}{\sqrt{2a^2 - 1}} \right)^{1/8}.
\end{align*}
\]
4.1(i), we find that

Proof. For (i), letting \( n = 4 \) in (3.6) and putting the value of \( h_4 \) from Theorem 4.1(i), we find that

\[
(4.3) \quad \left( \sqrt{2} \left( 1 + \sqrt{2} \right) - 1 \right)^4 = 2 \left( 1 + \sqrt{2} - \sqrt{1 + \sqrt{2}} \right)^4.
\]

Solving (4.3) for \( t_4' \) and using the fact that \( t_4' > 0 \), we complete the proof.

For (ii)–(viii), use (3.6) and Theorem 4.1(ii)–(viii), respectively, to repeat the same argument as in the proof of (i).

\[\square\]

See [8, Theorem 4.10] for different proofs of Theorem 4.3(i), (ii), and (iii).

5. Evaluations of \( J_n \)

We now turn to evaluations of the modular \( j \)-invariant in terms of \( J_n \) in the case of when \( n \) is of the form \( n = 2^{2m-1} \) or \( n = 2^{2m} \) for every positive integer \( m \). We only show the instances when \( m = 1, 2, 3, \) and 4. Note that we have some new values of \( J_n \) such as \( J_{16}, J_{32}, J_{64}, J_{128}, \) and \( J_{256} \).

Theorem 5.1. Let \( a \) and \( b \) be as in Theorem 4.1. Then we have

(i) \( J_2 = \frac{1}{16} \left( -19 + 13\sqrt{2} \right) \),

(ii) \( J_4 = \frac{1}{16} \left( -724 + 513\sqrt{2} \right) \),

(iii) \( J_8 = \frac{\sqrt{2}3 + 58 + 2\sqrt{2}(1423 + 1073\sqrt{2})}{16 \left( 8(5+4\sqrt{2}) + \sqrt{2}(1817 + 1285\sqrt{2}) \right)} \),

(iv) \( J_{16} = \frac{3(1+\sqrt{2})(731 + 540\sqrt{2} - 120\sqrt{2} + 43\sqrt{2})}{2(2+2\sqrt{2})^4} \),

(v) \( J_{32} = \frac{256}{512} \left( \sqrt{2}a^2 - 1 \right)^2 \left( 2^{1/4}(2+2\sqrt{2})^4 \right) \),

(vi) \( J_{64} = \frac{512}{1024} \left( \sqrt{2}a^2 - 1 \right)^2 \left( 2^{1/4}(2+2\sqrt{2})^4 \right) \),

(vii) \( J_{128} = \frac{256}{512} \left( \sqrt{2}a^2 - 1 \right)^2 \left( 2^{1/4}(2+2\sqrt{2})^4 \right) \),

(viii) \( J_{256} = \frac{512}{1024} \left( \sqrt{2}a^2 - 1 \right)^2 \left( 2^{1/4}(2+2\sqrt{2})^4 \right) \).

Proof. For (i), letting \( n = 2 \) in (1.13) and putting the values of \( h_4' \) from Theorem 4.2(ii) and \( l_4' \) from Theorem 4.3(i), we complete the proof.

For (ii)–(viii), repeat the same argument as in the proof of (i) by using Theorems 4.2 and 4.3.

\[\square\]

See [6, Theorem 7.2.2] for alternative proofs of Theorem 5.1(i), (ii), and (iii). Observe that \( J_{2^{2m}} \) and \( J_{2^{2m}+1} \) for \( m = 5, 6, 7, \ldots \) can be evaluated by repeating the same argument as in the proof of Theorem 5.1.
Corollary 5.2. The value of $j \left( \frac{3+\sqrt{-n}}{2} \right)$ can be evaluated in the case of when $n$ is of the form $n = 2^{2m-1}$ or $n = 2^{2m}$ for every positive integer $m$.

References


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