SUFFICIENT CONDITION FOR THE EXISTENCE OF THREE DISJOINT THETA GRAPHS

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Abstract. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. We show that every graph of order \( n \geq 12 \) and size at least \( \left\lfloor \frac{11n - 18}{2} \right\rfloor \) contains three disjoint theta graphs. As a corollary, every graph of order \( n \geq 12 \) and size at least \( \left\lfloor \frac{11n - 18}{2} \right\rfloor \) contains three disjoint cycles of even length.

1. Terminology and Introduction

In this paper, we only consider finite undirected graphs, without loops or multiple edges. We use [1] for the notation and terminology not defined here. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. Let \( n \) be a positive integer, let \( K_n \) denote the complete graph of order \( n \) and \( K_n^{-4} \) be the graph obtained by removing exactly one edge from \( K_4 \). For a graph \( G \), we denote its vertex set, edge set, minimum degree by \( V(G) \), \( E(G) \) and \( \delta(G) \), respectively. The order and size of a graph \( G \), are defined by \( |V(G)| \) and \( |E(G)| \), respectively. A set of subgraphs is said to be vertex-disjoint or independent, if no two of them have any common vertex in \( G \), and we use disjoint to stand for vertex-disjoint throughout this paper. If \( u \) is a vertex of \( G \) and \( H \) is either a subgraph of \( G \) or a subset of \( V(G) \), we define \( N_H(u) \) to be the set of neighbors of \( u \) contained in \( H \), and \( d_H(u) = |N_H(u)| \). For a subset \( U \) of \( V(G) \), \( G[U] \) denotes the subgraph of \( G \) induced by \( U \). In particular, we often use \([U]\) to stand for \( G[U] \). If \( S \) is a set of subgraphs of \( G \), we write \( G \supseteq S \) it means that \( S \) is isomorphic to a subgraph of \( G \), in particular, we use \( mS \) to represent a set of \( m \) vertex-disjoint copies of \( S \). When \( S = \{x_1, x_2, \ldots, x_t\} \), we may also use \([x_1, x_2, \ldots, x_t]\) to denote \([\{x_1, x_2, \ldots, x_t\}]. \) Let \( V_1, V_2 \) be two disjoint subsets or subgraphs of \( G \), we use \( E(V_1, V_2) \) to denote the set of edges in \( G \) with one end-vertex in \( V_1 \), while the other in \( V_2 \). For simplicity, let \( E(x, V_2) \) stand for \( E(\{x\}, V_2) \), \( E(V_1, x) \)
for $E(V_1, \{x\})$, respectively. A path of order $n$ is denoted by $P_n$. Throughout this paper, we consider that any cycle has a fixed orientation. Let $C$ be a cycle of $G$. For $x, y \in V(C)$, we denote by $\overrightarrow{C}[x, y]$ the path from $x$ to $y$ on $\overrightarrow{C}$. A vertex $u$ is called a leaf of $G$ if $d_G(u) = 1$.

Corrádi and Hajnal [3] proved the following well-known result on the existence of vertex-disjoint cycles in graphs.

**Theorem 1.1 ([3]).** Let $k$ be a positive integer and $G$ be a graph with order $n \geq 3k$. If $\delta(G) \geq 2k$, then $G$ contains $k$ disjoint cycles.

Later, Wang [10] and independently Enomoto [5] proved a result stronger than Theorem 1.1 as follows.

**Theorem 1.2 ([10]).** Let $k$ be a positive integer and $G$ be a graph with order $n \geq 3k$. Suppose for any pair of nonadjacent $u$ and $v$ in $G$, $d_G(u) + d_G(v) \geq 4k - 1$, then $G$ contains $k$ disjoint cycles.

Given a cycle $C$ of a graph $G$, a chord of $C$ is an edge of $G - E(C)$ which joins two vertices of $C$. A cycle is called a chorded cycle if it has at least one chord. A theta graph is the union of three internally disjoint paths that have the same two distinct end vertices. A chorded cycle is a simple example of a theta graph but, in general a theta graph need not be a chorded cycle. It is obvious that $K^-_4$ is the theta graph with minimum order and every theta graph contains a cycle of even length. Pósa [9] proved that any graph with minimum degree at least three contains a chorded cycle. Motivated by these results, Finkel et al. [6] and Chiba et al. [3] obtained the following results analogous to Theorem 1.2, respectively.

**Theorem 1.3 ([6]).** If $G$ is a graph of order $n \geq 4k$ and $\delta(G) \geq 3k$, then $G$ contains $k$ disjoint chorded cycles.

**Theorem 1.4 ([3]).** Let $r, s$ be two nonnegative integers and let $G$ be a graph with order $n \geq 3r + 4s$. Suppose for any pair of nonadjacent $u$ and $v$ in $G$, $d_G(u) + d_G(v) \geq 4r + 6s - 1$, then $G$ contains $r + s$ disjoint cycles such that $s$ of them are chorded cycles.

Kawarabayashi [8] considered the minimum degree to ensure the existence of disjoint copies of $K^-_4$ in a general graph $G$, which can be seen as a specified version of disjoint chorded cycles.

**Theorem 1.5 ([8]).** Let $k$ be a positive integer and $G$ be a graph with order $n \geq 4k$. If $\delta(G) \geq \lceil \frac{n + k}{2} \rceil$, then $G$ contains $k$ disjoint copies of $K^-_4$.

In this paper, we determine the edge number for a graph to contain three disjoint theta graphs. Our research is motivated by the conjecture put forward by Gao and Ji [7].
Conjecture 1.6 ([7]). Let \( k \geq 2 \) be an integer. Every graph of order \( n \) and size at least \( f(n, k) + 1 \) contains \( k \) disjoint theta graphs, when

\[
f(n, k) = \max \left\{ \left( \binom{4k-1}{2} - 3(n-4k+1) \right), \left( \frac{2(k-1)(2k-1) + (4k-1)(n-2k+1)}{2} \right) \right\}.
\]

If the conjecture is true, then the bound on size is best possible, which can be seen as following examples in [7]: Let \( G_1 \) be \( K_{4k-2} \cup \frac{n-4k+1}{2} K_2 \). The order of \( G_1 \) is \( n \) and size \( \binom{4k-1}{2} + \frac{2}{3}(n-4k+1) \), but \( G_1 \) does not contain \( k \) disjoint theta graphs. Also, let \( n \) be an integer such that \( n - (2k-1) \) is even. Let \( l_1 = \frac{n - (2k-1)}{2} \), \( F = K_{2k-1} \), \( H_1 = l_1 K_2 \) and \( G_2 = F + H_1 \). It is obvious that \( G_2 \) has order \( n \), \(|E(G_1)| = (k-1)(2k-1) + (4k-1)l_1 = (k-1)(2k-1) + \frac{(4k-1)(n-2k+1)}{2} = \left[ \frac{2(k-1)(2k-1) + (4k-1)(n-2k+1)}{2} \right] \). Gao and Ji [7] verified Conjecture 1.6 for the case \( k = 2 \).

Theorem 1.7 ([7]). Every graph of order \( n \geq 8 \) and size at least \( f(n) \) contains two disjoint theta graphs, if

\[
f(n) = \begin{cases} 
23 & \text{if } n = 8 \\
\left\lceil \frac{7n-14}{2} \right\rceil & \text{if } n \geq 9.
\end{cases}
\]

Based on Theorem 1.7, in this paper, we give a sufficient condition for the existence of three disjoint theta graphs.

Theorem 1.8. Every graph of order \( n \geq 12 \) and size at least \( \left\lceil \frac{11n-18}{2} \right\rceil \) contains three disjoint theta graphs.

Note that there is a small gap on the lower bound of size between Theorem 1.8 and Conjecture 1.6 for \( k = 3 \). However, the following corollary follows from Theorem 1.8.

Corollary 1.9. Every graph of order \( n \geq 12 \) and size at least \( \left\lceil \frac{11n-18}{2} \right\rceil \) contains three disjoint cycles of even length.

2. Basic lemma

Lemma 2.1. Let \( G \) be a graph of order 12 and size at least 57. Then \( G \) contains three disjoint copies of \( K_4^- \).

Proof. Suppose that \( G \) does not contain three disjoint copies of \( K_4^- \). If \( \delta(G) \geq 8 \), then by Theorem 1.5, \( G \supseteq 3K_4^- \), a contradiction. Hence, we may assume that \( \delta(G) \leq 7 \). Let \( v_0 \in V(G) \) such that \( d_G(v_0) = \delta(G) \). Suppose that \( d_G(v_0) = 1 \), then 56 = \(|E(G)| < 57 \), a contradiction. Thus, \( d_G(v_0) \geq 2 \) and let \( v_1, v_2 \in N_G(v_0) \). Suppose that \( d_G(v_0) = 2 \), then choose \( w \in V(G - \{v_0, v_1, v_2\}) \), since \(|E(G - \{v_0\})| \geq 55 \), it is obvious that \( \{v_0, v_1, v_2, w\} \supseteq K_4^- \) and \(|E(G - \{v_0, v_1, v_2, w\})| \supseteq 2K_4^- \), a contradiction. Hence, we may assume that \( d_G(v_0) \geq 3 \). Furthermore, since \( G - \{v_0\} \) can be obtained from \( K_{11} \) by removing at most five edges, it follows that \([N_G(v_0)]\) contains a path of order three, denoted by \( P_3 \). That is, \( P_3 + \{v_0\} \) contains a subgraph \( Q \cong K_4^- \). Note
that $|E(G - V(Q) - \{v_0\})| \geq 57 - 7 - (10 + 9 + 8) = 23$, by Theorem 1.7, $G - V(Q) - \{v_0\}$ contains two disjoint copies of $K_4^-$, which disjoints from $Q$, this implies that $G \supseteq 3K_4^-$, a contradiction. This proves Lemma 2.1.

\[ \square \]

3. Proof of Theorem 1.8

If $n = 12$, then Lemma 2.1 gives us the required conclusion. Hence, it is sufficient to prove that every graph of order $n \geq 13$ and size at least $\lceil \frac{11n - 18}{2} \rceil$ contains three disjoint theta graph. We employ induction on $n$.

Assume that for all integers $k$ with $12 \leq k < n$, every graph of order $k$ and size at least $\lceil \frac{11k - 18}{2} \rceil$ contains three disjoint theta graphs. In the following proof, we always let $G$ be any graph of order $n$ and size at least $\lceil \frac{11n - 18}{2} \rceil$. By way of contradiction, we suppose that

\[(1) \quad G \text{ does not contain three disjoint theta graphs.} \]

Claim 3.1. $6 \leq \delta(G) \leq 8$.

Proof. By Theorem 1.3, we have $\delta(G) \leq 8$. Suppose that $\delta(G) \leq 5$ and let $v_0 \in V(G)$ such that $d_G(v_0) = \delta(G)$. The graph $G - v_0$ is of order $n - 1$ and size $\lceil \frac{11n - 18}{2} \rceil - d_G(v_0) \geq \lceil \frac{11n - 18}{2} \rceil - 5 \geq \frac{11n - 19 - 10}{2} = \frac{11(n - 1) - 18}{2} \geq \left\lceil \frac{11(n - 1) - 18}{2} \right\rceil$,

by induction hypothesis, $G - v_0$ contains three disjoint theta graphs, and so does $G$, which contradicts (1). Therefore, $\delta(G) \geq 6$. \[ \square \]

Let $v_0$ be a vertex in $G$ such that $d_G(v_0) = \delta(G)$. In what following, we always assume that $N_G(v_0) = \{v_1, \ldots, v_l\}$ and $H = [v_1, \ldots, v_l]$, where $l = d_G(v_0)$. By Claim 3.1, $6 \leq l \leq 8$. If $l = 6$, then let $\varepsilon_1 = 1$; if $l = 7$, then let $\varepsilon_1 = 2$; if $l = 8$, then let $\varepsilon_1 = 3$. Note that $l = 5 + \varepsilon_1$.

Claim 3.2. For each $1 \leq i \leq l$, $d_H(v_i) \geq l - \varepsilon_i$.

Proof. Suppose that there exists $1 \leq i \leq l$ such that $d_H(v_i) \leq l - \varepsilon_i - 1 = (l - 1) - \varepsilon_i$. Without loss of generality, we may assume that $i = l$, and we may also assume that $v_jv_l \notin E(G)$ for each $1 \leq j \leq \varepsilon_l$ (otherwise, we can relabel the index of $V(H)$). Define the edge set $X = \{v_jv_l : 1 \leq j \leq \varepsilon_l\}$ and construct the graph $G' = (G - v_0) + X$, which is a graph with order $n - 1$ and size $\lceil \frac{11n - 18}{2} \rceil - l + \varepsilon_l \geq \frac{11n - 19}{2} - l + \varepsilon_l = \frac{11(n - 1) - 18}{2} \geq \left\lceil \frac{11(n - 1) - 18}{2} \right\rceil$, because of $l = 5 + \varepsilon_1$. By induction hypothesis, $G'$ contains three disjoint theta graphs, say $T_1$, $T_2$ and $T_3$, respectively. Clearly, at least two of them, say $T_1$ and $T_2$, do not contain vertex $v_l$, since $T_1$, $T_2$ and $T_3$ are disjoint theta graphs, then $E(T_1) \cap X = \emptyset$, $E(T_2) \cap X = \emptyset$ and by (1), $E(T_3) \cap X = \emptyset$.

Suppose that $|E(T_3) \cap X| = 1$, we may assume that $E(T_3) \cap X = \{v_lv_l\}$. Then $T_3' = (T_3 - \{v_lv_l\}) + \{v_1v_0, v_lv_l\}$ is a theta graph in $G$, $T_1$, $T_2$ and $T_3'$ are disjoint in $G$, which contradicts (1). Therefore, it remains the case
\[ E(T_3) \cap X = \{v_1v_1, v_2v_1\} \text{ or } E(T_3) \cap X = \{v_1v_1, v_2v_1, v_3v_1\}, \text{ as } \varepsilon_1 \leq 3. \]

Let

\[
T' = \begin{cases} 
(T_3 - \{v_1v_1, v_2v_1\}) + \{v_0v_1, v_0v_2\}, & \text{if } d_{T_3}(v_1) = 2 \\
(T_3 - \{v_1v_1, v_2v_1\}) + \{v_0v_1, v_0v_2, v_0v_3\}, & \text{if } d_{T_3}(v_1) = 3 \text{ and } E(T_3) \cap X = \{v_1v_1, v_2v_1\} \\
(T_3 - \{v_1v_1, v_2v_1\}) + \{v_0v_1, v_0v_2, v_0v_3\}, & \text{otherwise.}
\end{cases}
\]

It is obvious that \(T_1, T_2\) and \(T'_3\) are three disjoint theta graphs in \(G\), which
contradicts (1). \(\Box\)

By Claim 3.2, Theorem 1.5 and the definition of \(\varepsilon_l\), when \(7 \leq l \leq 8\), for each subset \(S\) of \(V(H)\) with \(|S| \geq 7\), we obtain

\[ [\{v_0\} \cup S] \geq 2K_4^- \quad \text{(2)} \]

In particular, if \(l = 6\), then

\[ [\{v_0\} \cup V(H)] \cong K_7. \quad \text{(3)} \]

We take a vertex \(v \in V(G - H - \{v_0\})\) such that \(|E(v, V(H))|\) is maximum. When \(l = 6\), by (3) and the definition of \(v\), denote \(W = V(H) \cup \{v\}\), we claim that

\[ [\{v_0\} \cup W] \geq 2K_4^- \quad \text{(4)} \]

**Proof.** By way of contradiction, suppose that \([\{v_0\} \cup W]\) does not contain two disjoint \(K_4^-\). By (3) and the assumption that \([\{v_0\} \cup W] \not\geq 2K_4^-\), for each \(w \in V(G - \{v_0\} - V(H))\), there is at most one edge between \(w\) and \(V(H)\). If \(n = 13\), then \(62 \leq |E(G)| \leq \frac{2 \times 6}{2} + 6 + \frac{6 \times 6}{2} = 42\), a contradiction. If \(n = 14\), then \(68 \leq |E(G)| \leq \frac{2 \times 6}{2} + 7 + \frac{7 \times 6}{2} = 49\), a contradiction. If \(n = 15\), then \(73 \leq |E(G)| \leq \frac{2 \times 6}{2} + 8 + \frac{8 \times 5}{2} = 57\), a contradiction. If \(n = 16\), then \(84 \leq |E(G)| \leq \frac{2 \times 6}{2} + 9 + \frac{9 \times 7}{2} = 66\), a contradiction. Therefore, we see that \(n \geq 17\).

Since

\[
|E(G - \{v_0\} - V(H))| \geq |E(G)| - \frac{7 \times 6}{2} - (n - 7) \\
\geq \frac{11n - 19}{2} - n - 14 \\
\geq \frac{7n - 13}{2},
\]

by Theorem 1.7, \(G - \{v_0\} - V(H)\) contains two disjoint theta graphs, together with (3), \(G\) contains three disjoint theta graphs, a contradiction. \(\Box\)

Let

\[
G^* = \begin{cases} 
G - \{v_0\} \cup V(H), & \text{if } 7 \leq l \leq 8 \\
G - \{v_0, v\} \cup V(H), & \text{if } l = 6.
\end{cases}
\]

Let \(F^*\) be the set of components of \(G^*\). By (2) and (4), it follows from (1) that every graph in \(F^*\) contains no theta graph. In the following proof, let \(F\) denote arbitrary component in \(F^*\), then, each block of \(F\) is either a \(K_2\) or a cycle.
Claim 3.3. Let \( F \in F^* \) with \( |V(F)| \geq 4 \). Then each end block of \( F \) is isomorphic to \( K_2 \).

Proof. Otherwise, suppose that there exists an end block \( B \) of \( F \), such that \( B \) is a cycle. Let \( C \) denote the set of cut vertices of \( F \). Let \( u_1 \) and \( u_2 \) be two distinct vertices in \( V(B) - C \). Next, we choose two distinct vertices \( u_3 \) and \( u_4 \) (both are distinct with \( u_1 \) and \( u_2 \)) as follows: If \( F = B \), then let \( \{u_3, u_4\} \subseteq V(F - \{u_1, u_2\}) \); otherwise, \( F \) contains another end blocks \( B' \) which is different from \( B \), let \( u_3 \in V(B') \) such that \( u_3 \notin C \) and choose \( u_4 \in V(F) \setminus C \) if possible, unless \( F \) contains exactly two end blocks \( B \) and \( B' \), such that \( B \) is a triangle and \( B' \cong K_2 \). For each \( i \) with \( 1 \leq i \leq 3 \), since \( d_F(u_i) \leq 2 \), if \( 7 \leq l \leq 8 \), then \( |E(u_i, V(H))| \geq \delta(G) - 2 = l - 2 \); if \( l = 6 \), then \( |E(u_i, V(H)) \cup \{v_i\}| \geq l - 2 \). This implies that there exists a vertex \( v' \in V(H) \setminus \{u_i\} \). As \( B \) is a cycle, it is easy to see that \( [B \cup \{v'\}] \) contains a theta graph. When \( F = B \), without loss of generality, we may assume that \( u_1, u_2, u_3 \) and \( u_4 \) occur along the direction of \( B \).

If \( l = 8 \), by applying (2) and Theorem 1.5, \( \{v_0 \cup V(H) - \{v'\}\} \) contains two theta graphs, that is, \( G \) contains three disjoint theta graphs, which contradicts (1). If \( l = 7 \), we may assume that \( \{v_2, v_3, v_4, v_5, v_6\} \subseteq N_G(u_3) \) and \( v' \neq v_4, v_5 \) and \( v_6 \), then \( |\{v_4, v_5, v_6, u_3\}| \geq K_4 \) by Claim 3.2. If \( u_3 \notin V(B) \), that is, \( u_3 \), belongs to another end block by our choice, notice that \( |V(H - \{v_4, v_5, v_6, u_3\}) \cup \{v_0\}| \geq K_4 \) and \( |B \cup \{v'\}| \) contains a theta graph, we obtain a contradiction to (1). Therefore, we see that \( u_3 \in V(B) \) and \( F = B \) by our choice. We may assume that \( \{v_1, v_2, v_3\} \subseteq N_G(u_1) \cap N_G(u_2) \) because we don’t use the assumption of \( \{v_2, v_3, v_4, v_5, v_6\} \subseteq N_G(u_3) \). Suppose for the moment, there exists at most one \( v_1 \in \{v_1, v_2, v_3\} \), such that \( v_1 \cup \{v_1, v_2, v_3\} \subseteq E(G) \). Then there exist \( v_p, v_q \in V(H - \{v_1, v_2, v_3\}) \) with \( p \neq q \), such that \( \{v_p, v_q\} \subseteq N_G(v_2) \cap N_G(u_4) \). However, by Claim 3.2, \( \{v_0 \cup V(H - \{v_1, v_2, v_p, v_q\}) \supseteq K_4 \), notice that \( \{\{v_0 \cup V(H - \{v_1, v_2\}) \cup \{v_p, v_q\} \cup V(H - \{u_3, u_4\}) \) contain two disjoint theta graphs, this implies that \( G \) contains three disjoint theta graphs, a contradiction. Thus, without loss of generality, say \( \{v_1, v_2\} \subseteq N_H(u_3) \cap N_H(u_4) \). As \( |E(u_3, V(H))| \geq 5 \), without loss of generality, we may assume that \( v_1, v_2, v_3, v_4, v_5 \in E(G) \). This implies that \( \{v_1, v_2, u_3\} \supseteq K_4 \), notice that \( \{v_0, v_5, v_6, v_7\} \supseteq K_4 \) and \( \{v_2, v_3\} \cup V(H - \{u_1, u_2\}) \) contains a theta graph, then \( G \) contains three disjoint theta graphs, a contradiction. Now, it remains the case \( l = 6 \). As \( d_F(u_i) \leq 2 \) for \( i \in \{1, 2, 3\} \), so \( |E(u_i, V(H) \cup \{v\}| \geq l - 2 = 4 \). Furthermore, by our choice of \( u_4 \), \( d_F(u_4) \leq 3 \) and \( |E(u_4, V(H) \cup \{v\})| \geq 3 \).

Suppose for the moment that \( u_3, u_2, v_3, v \in E(G) \), then \( \{B \cup \{v\}\} \) contains a theta graph. If \( u_3 \notin V(B) \), by the choice of \( u_3 \) and (3), \( H + \{v_0, u_4\} \supseteq 2K_4 \), this implies that \( G \) contains three disjoint theta graphs, which contradicts (1). Thus, \( u_3 \in V(B) \) and so \( F = B \). However, \( \{v\} \cup V(H - \{u_3\}) \) contains a theta graph and \( \{u_4, v_0\} \cup V(H) \supseteq 2K_4 \), a contradiction. Thus, there exists
i ∈ \{1, 2, 3\}, such that \( u, v \notin E(G) \). By the definition of \( v, |E(v, V(H))| \geq 4 \).

Without loss of generality, we assume that \( \{v_1, v_2, v_3, v_4\} \subseteq N_H(v) \).

If \( u_1v, u_2v \in E(G) \), then \( F = B \). Without loss of generality, we may assume that \( v_4u_1 \in E(G) \) and so \( \{v, v_4\} \cup V(\bar{B}[u_1, u_2]) \) contains a theta graph. Notice that \( u_3v \notin E(G) \) and \( u_4v \notin E(G) \), then \( \{v_3, u_4, v_0\} \cup V(H - \{v_4\}) \supseteq 2K_4^{+} \), a contradiction. Therefore, we assume that \( u_1v \notin E(G) \) by symmetry.

Suppose that \( u_3 \notin V(B) \). Then \( |E(u_1,\{v_1, v_2, v_3, v_4\})| \geq 2 \); otherwise, \( u_3v \notin E(G) \) and \( u_4v \notin E(G) \). By pigeonhole principle, there exists \( \{v_p, v_q\} \subseteq V(H) \) such that \( \{v_p, v_q\} \subseteq N_H(u_1) \cap N_H(u_2) \). If \( u_3v_p \in E(G) \), then \( V(\bar{B}[u_1, u_3]) \cup \{v_p\} \) contains a theta graph, notice that \( |V(\bar{B}[u_1, u_3]) \cup \{v, v_0, u_4\}| \geq 2K_4^{+} \), \( \bar{G} \) contains three disjoint theta graphs, a contradiction. Thus, \( u_3v_p \notin E(G) \) and \( u_4v_p \notin E(G) \). This implies that there exist \( v_i, v_j \in V(H) - \{v_p, v_q\} \), such that \( \{v_i, v_j\} \subseteq N_H(u_3) \cap N_H(u_4) \). By (1), we see that \( |\{v_i, v_j\} \cap \{1, 2, 3, 4\}| \leq 1 \) and \( |\{i, j\} \cap \{1, 2, 3, 4\}| \leq 1 \). Therefore, \( \{v_i, v_j\} \cup V(\bar{B}[u_3, u_4]) \) contains two disjoint theta graphs, which contradicts (1). This completes the proof that \( B \) is not an end block, and in particular, we see that every end block of \( F \) is isomorphic to \( K_2 \).

**Claim 3.4.** Let \( F \in F^* \) with \( |V(F)| \geq 4 \). Then each block of \( F \) is isomorphic to \( K_2 \).

**Proof.** Since \( |V(F)| \geq 4 \), \( F \) contains at least two end block, say \( F_1 \) and \( F_2 \). Note \( F_i \cong K_2 \) for each \( 1 \leq i \leq 2 \). Let \( u_1 \in V(F_1) \) such that \( d_{F_1}(u_1) = 1 \) and let \( u_3 \in V(F_2) \) such that \( d_{F_2}(u_3) = 1 \). Suppose that the conclusion of Claim 3.4 is false, we may assume that \( B \) is the nearest block to \( u_1 \) in \( F \), such that \( B \) is a cycle. By Claim 3.3, \( B \) is not an end block of \( F \). We choose two distinct vertices \( u_2 \) and \( u_4 \) such that both of them are distinct with \( u_1 \) and \( u_3 \) as follows: Let \( u_2 \in V(B) \) and \( u_2 \) is not a cut vertex of \( F \), and choose \( u_4 \) such that \( u_4 \) is not a cut vertex of \( F \), unless \( F \) contains exactly three blocks \( F_1, F_2 \) and \( B \cong K_3 \), then choose \( u_4 \in V(F_2) - \{u_3\} \). Notice that if there exists \( u' \) such that \( u_1u' \in E(G) \), then using these blocks of \( F \) from \( F_1 \) to \( B \), we see that \( |V(F - \{u_3\}) \cup \{u_1\}| \geq 2 \), a contradiction. Therefore, we assume that \( u_1u' \notin E(G) \) by symmetry. Then \( |E(u_1, \{v_1, v_2, v_3, v_4\})| \geq 2 \); otherwise, \( u_3v \notin E(G) \) and \( u_4v \notin E(G) \). By pigeonhole principle, there exists \( \{v_p, v_q\} \subseteq V(H) \) such that \( \{v_p, v_q\} \subseteq N_H(u_1) \cap N_H(u_2) \). If \( u_3v_p \in E(G) \), then \( V(\bar{B}[u_1, u_3]) \cup \{v_p\} \) contains a theta graph, notice that \( |V(\bar{B}[u_1, u_3]) \cup \{v, v_0, u_4\}| \geq 2K_4^{+} \), \( \bar{G} \) contains three disjoint theta graphs, a contradiction. Thus, \( u_3v_p \notin E(G) \) and \( u_4v_p \notin E(G) \). This implies that there exist \( v_i, v_j \in V(H) - \{v_p, v_q\} \), such that \( \{v_i, v_j\} \subseteq N_H(u_3) \cap N_H(u_4) \). By (1), we see that \( |\{v_i, v_j\} \cap \{1, 2, 3, 4\}| \leq 1 \) and \( |\{i, j\} \cap \{1, 2, 3, 4\}| \leq 1 \). Therefore, \( \{v_i, v_j\} \cup V(\bar{B}[u_3, u_4]) \) contains two disjoint theta graphs, which contradicts (1). This completes the proof that \( B \) is not an end block, and in particular, we see that every end block of \( F \) is isomorphic to \( K_2 \).
in different blocks, with the same role of $u_1, u_2, u_3$ and $u_4$, we continue part of the process in the proof of Claim 3.3, we can complete the proof. This proves Claim 3.4.

\[ \square \]

**Claim 3.5.** $|V(F)| \leq 3$ for each $F \in F^*$. 

**Proof.** Otherwise, suppose that there exists $F \in F^*$ such that $|V(F)| \geq 4$. By Claim 3.4, $F$ must be a tree.

Suppose for the moment that there exists three distinct leaves in $V(F)$, say $u_1, u_2$ and $u_3$. Then for each $1 \leq i \leq 3$, $|E(u_i, V(H))| \geq l - 1$ if $7 \leq l \leq 8$, and $|E(u_i, V(H) \cup \{v\})| \geq l - 1$ if $l = 6$. As $|V(F)| \geq 4$, by Claim 3.4, we choose $u_4 \in V(F - \{u_1, u_2, u_3\})$ as follows: if $F$ contains at least four leaves, then let $u_4$ denote the leaf different from $u_1, u_2$ and $u_3$; otherwise, let $u_4$ and $u_1$ belongs to the same block of $F$. It is obvious that $|E(u_4, V(H))| \geq l - 3$ if $7 \leq l \leq 8$, and $|E(u_4, V(H) \cup \{v\})| \geq l - 3$ if $l = 6$.

Suppose that $l = 8$. Notice that there exist $v', v'' \in V(H)$ with $v' \neq v''$ and $v'v'' \in E(G)$ such that $\{v', v''\} \subseteq N_H(u_1) \cap N_H(u_2)$. It is obvious that $\{v', v'', u_1, u_2\} \supseteq K_4$. By Claim 3.1, $H - \{v', v''\} + \{v_0, u_3\}$ induces a graph with minimum degree at least five, and therefore contains two disjoint copies of $K_4$ by Theorem 1.5, a contradiction. Next, suppose that $l = 7$, by pigeonhole principle, we can find two distinct vertices $v_i, v_j \in V(H)$ such that $\{v_i, v_j\} \subseteq N_H(u_3) \cap N_H(u_4)$. Since there is a path $P$ in $F$ which connecting $u_3$ and $u_4$, thus, $[V(P) \cup \{v_i, v_j\}]$ contains a theta graph. Notice that there exist $v', v'' \in V(H - \{v_1, v_2\})$ with $v' \neq v''$ and $v'v'' \in E(G)$, such that $\{v', v''\} \subseteq N_H(u_1) \cap N_H(u_2)$. It is obvious that $\{v', v'', u_1, u_2\} \supseteq K_4$. As $|[v_0] \cup V(H - \{v', v'', v_i, v_j\})| \supseteq K_4$, which contradicts (1). Thus, $l = 6$.

We show $N_H(u_1) \cap N_H(u_4) \neq \emptyset$. Suppose not, without loss of generality, we may assume that $N_G(u_1) \cap (V(H) \cup \{v\}) = \{v_1, v_2, v_3, v_4\}$ and $N_G(u_4) \cap (V(H) \cup \{v\}) = \{v_5, v_6\}$. If $u_3v \in E(G)$, then $[V(F - \{u_2\}) \cup \{v\}]$ contains a theta graph, as $[V(H) \cup \{v_0, u_2\}] \supseteq 2K_4$, which contradicts (1). Hence, $u_3v \notin E(G)$ and $u_2v \notin E(G)$ by symmetry. Furthermore, by the choice of $v$, we have $E(v, V(H)) \geq 4$ and so $N_H(v) \cap N_H(u_1) \neq \emptyset$, without loss of generality, say $v_1 \in E(G)$. Then $[v, v_1, u_1, u_4] \supseteq K_4$, since $|N_H(u_2) \cap N_H(u_3)| \geq 3$, it follows that $[V(H - \{v_1\}) \cup \{u_2, u_3, v_0\}] \supseteq 2K_4$, which contradicts (1).

Now, by symmetry, say $v_6 \in N_H(u_1) \cap N_H(u_4)$. If $u_2v_6 \notin E(G)$, then $[\{v_6\} \cup V(F - \{u_2\})]$ contains a theta graph, as $[V(H - \{v_6\}) \cup \{v, u_3\}] \supseteq 2K_4$, which contradicts (1). Thus, $v_6u_2 \notin E(G)$ and $v_6u_3 \notin E(G)$ by symmetry. As $|E(u_3, V(H))| \geq 4$ and $|E(u_3, V(H))| \geq 4$, we may assume that $\{v_1, v_2, v_3, v_4\} \subseteq N_H(u_2)$ and $\{v_1, v_2, v_3\} \subseteq N_H(u_2) \cap N_H(u_3)$. If $u_3v_1 \in E(G)$, then $[\{v_6, v_5, v_3, u_4\}] \supseteq K_4$. Notice that $[V(H - \{v_5, v_6\}) \cup \{v_0, v_2, u_3, u_4\}] \supseteq 2K_4$, by the definition of $v$ and (3), which contradicts (1). Thus, $v_6u_1 \notin E(G)$. If $u_1v_4 \in E(G)$, then $u_2v_6 \in E(G)$. Otherwise, say $u_2v \notin E(G)$. Then $u_2v \in E(G)$ and $E(v, V(H)) \geq 5$ by the choice of $v$. By symmetry, we may assume that $\{v_1, v_2\} \subseteq N_H(v) \cap N_H(u_3)$. Then $[v, v_1, v_2, u_3] \supseteq K_4$, $[u_1, u_4, v_4, v_6] \supseteq K_4$, and $[u_2, v_3, v_5, v_6] \supseteq K_4$, which contradicts (1). Hence,
by (1), \( v_1 \notin E(G) \) for each \( i \in \{1, 2, 3\} \), that is, \( |E(v, V(H))| \leq 3 \), which contradicts the choice of \( v \). Therefore, \( u_1v_4 \notin E(G) \) and so \( \{v_1, v_2, v_3\} \subseteq N_H(u_1) \) and \( u_1v \in E(G) \). By (1) and (3), \( u_2v, u_3v \in E(G) \) and \( |E(v, V(H))| \leq 3 \), which contradicts the choice of \( v \). Consequently, \( F \) contains exactly two leaves and \( F \) must be a path with order at least four.

Let \( F = u_1v_2 \cdots u_{p-1}u_p \) and \( p \geq 4 \). Suppose that \( 7 \leq l \leq 8 \), then continue the process as above, we can find three disjoint theta graphs, a contradiction.

Hence, \( l = 6 \). Then \( |E(u_1, V(H) \cup \{v\})| \geq 5 \), \( |E(u_p, V(H) \cup \{v\})| \geq 5 \), \( |E(u_2, V(H) \cup \{v\})| \geq 4 \) and \( |E(u_{p-1}, V(H) \cup \{v\})| \geq 4 \).

Suppose \( u_1v, u_pv \in E(G) \). Then \( u_2v \notin E(G) \) or \( u_{p-1}v \notin E(G) \), otherwise, \( \{v, u_1, u_2, u_{p-1}\} \supseteq K_4^* \), as \( |V(H) \cup \{v_0, u_p\}| \geq 2K_4 \) by Claim 3.2, which contradicts (1). By symmetry, say \( u_2v \notin E(G) \) and so \( |E(u_2, V(H))| \geq 4 \).

Without loss of generality, by pigeonhole principle, we may assume that \( v_1 \in N_H(u_2) \cap N_H(u_{p-1}) \) and \( \{v_1, v_2, v_3, v_4\} \subseteq N_H(u_2) \). Suppose for a moment that \( |N_H(u_2) \cap N_H(u_{p-1})| \geq 2 \). Without loss of generality, say \( u_pv_{p-1} \in E(G) \). Then \( [u_2, u_{p-1}, v_1, v_2] \supseteq K_4^* \). We prove that \( u_1v \notin E(G) \) and \( u_2v \notin E(G) \). Otherwise, by symmetry, say \( v_1v \in E(G) \). If \( u_1v_1 \in E(G) \), then \( \{v, v_1, u_1, u_2\} \supseteq K_4^* \), since \( |\{u_p, v_0\} \cup (V(H - \{v_1\})| \geq 2K_4 \), which contradicts (1). Hence, \( u_1v_1 \notin E(G) \). Next, we show that \( u_1v_2 \notin E(G) \). Suppose that \( u_1v_2 \in E(G) \). Then \( |V(F - \{u_p\}) \cup \{v_2\}| \) contains a theta graph, as \( |\{v, u_p, v_0\} \cup (V(H - \{v_2\})| \geq 2K_4 \), a contradiction once again. Until now, we see that \( N_H(u_1) = \{v_3, v_4, v_5, v_6\} \). According to this, we have \( u_pv_1 \notin E(G) \) and \( u_pv_2 \notin E(G) \). This implies that \( N_H(u_1) = N_H(u_p) \). If \( u_2v \in E(G) \), then \( \{v, v_1, u_2, u_{p-1}\} \supseteq K_4^* \), notice that \( |V(H - \{v_1, v_2\}) \cup \{u_1, u_2, u_p, v_0\}| \geq 2K_4 \), which contradicts (1). Thus, \( u_2v \notin E(G) \) and it follows that there exists \( i \in \{3, 4\} \) such that \( v_i v \in E(G) \).

Without loss of generality, say \( i = 3 \), then \( \{v_3, v_4, v_5, v_6\} \subseteq N_H(v) \) and so \( |V(H - \{v_1, v_2\}) \cup \{v, u_p, u_1, v_0\}| \geq 2K_4 \), a contradiction. This proves that \( N_H(u_2) \cap N_H(u_{p-1}) = \{v_1\} \) and so \( u_pv_{p-1} \in E(G) \). Suppose that \( v_1u_1 \in E(G) \), then let \( P' = P - \{u_p\} \), then \( |V(P') \cup \{v_1\}| \) contains a theta graph, by (3), \( |V(H - \{v_1\}) \cup \{v, u_p\}| \geq 2K_4 \), which contradicts (1). Thus, \( v_1u_1 \notin E(G) \) and so \( |N_H - v_1(u_1) \cap N_H - v_1(u_p)| \geq 2 \).

If \( u_1v \in E(G) \), then \( |V(P - \{u_1, u_p\}) \cup \{v, v_1\}| \) contains a theta graph, as \( |V(H - \{v_1\}) \cup \{u_1, u_p\}| \geq 2K_4 \), which contradicts (1). Thus, \( u_1v \notin E(G) \).

As \( |E(v, V(H))| \geq 4 \), by the symmetry role of \( v_5 \) and \( v_6 \), we may assume that \( v_5v \in E(G) \), then \( |v, v_5, u_{p-1}, u_p| \supseteq K_4^* \), since \( u_1 \) and \( u_2 \) has at least two common neighbors in \( V(H - \{v_1, v_5, v_6\}) \), \( |V(H - \{v_5\}) \cup \{u_1, u_2\}| \geq 2K_4 \), which contradicts (1). Consequently, we may assume that \( u_1v \notin E(G) \) by symmetry. This gives us \( |E(u_1, V(H))| \geq 5 \) and so \( |E(v, V(H))| \geq 5 \) by the maximality of \( v \). Without loss of generality, we may assume that \( \{v_1, v_2, v_4, v_5\} \subseteq N_H(v) \) and \( \{v_1, v_2, v_3, v_4\} \subseteq N_H(u_1) \cap N_H(v) \). Because of \( |E(u_1, V(H))| \geq 5 \), we divide the proof into two cases.
Case 1. $u_1v_5 \in E(G)$.

Without loss of generality, say $v_4u_2, v_5u_2 \in E(G)$, because of $|E(u_2, V(H))| \geq 3$. If $u_{p-1}v_4 \in E(G)$, then $[u_1, u_2, \ldots, u_{p-1}, v_4]$ contains a theta graph, since $|V(H - \{v_4\}) \cup \{v, u_p, v_0\}| \geq 2K_4^-$, which contradicts (1) and proves that $u_{p-1}v_4 \notin E(G)$. Similarly, $u_{p-1}v_5 \notin E(G)$. If there exists $v \in \{v_1, v_2, v_3\}$, say $i = 1$, such that $v_1u_2 \in E(G)$, then $u_{p-1}v_1 \notin E(G)$, $N_H(u_{p-1}) = \{v_2, v_3, v_6\}$ and $u_{p-1}v \in E(G)$. Suppose that there exist $v_i, v_j \in \{v_1, v_4, v_5\}$ such that $u_p v_i, u_p v_j \in E(G)$, then $[v_1, v_2, u_p, v_6] \supseteq K_4^-$. For simplicity, say $i = 4$ and $j = 5$. Since $[v_4, v_5, v_6, u_{p-1}] \supseteq K_4^-$ and $[v_0, u_1, v_1, v_3] \supseteq K_4^-$, this contradicts (1) and proves that $u_p$ has at most one neighbor in $\{v_1, v_4, v_5\}$. This implies that $u_p v_6, u_p v_4 \in E(G)$. Hence, $[v, u_{p-1}, v_6] \supseteq K_4^-$, notice that $|V(H - \{v_6\}) \cup \{v_0, u_1, u_2\}| \geq 2K_4^-$, a contradiction. This proves that $u_p$ has no neighbor in $\{v_1, v_2, v_3\}$ and so $u_2v_6, u_2v \notin E(G)$. As $|E(u_{p-1}, V(H))| \geq 3$, we may assume that $v_2u_{p-1}, v_2u_{p-1} \in E(G)$. Since $[v, u_4, u_1, u_2] \supseteq K_4^-$ and $[v_0, v_2, v_3, u_{p-1}] \supseteq K_4^-$, $|E(u_p, \{v_1, v_5, v_6\})| \leq 1$ by (1) and (3). Therefore, $\{v_2, v_3, v_4\} \in N_H(u_p)$ and $u_p v \in E(G)$. However, $[v, v_5, v_6, u_{p-1}] \supseteq K_4^-$, $[u_{p-1}, v_2, u_3, u_p] \supseteq K_4^-$ and $[v_0, v_1, v_4, u_1] \supseteq K_4^-$, a contradiction. This proves Case 1.

Case 2. $u_1v_6 \in E(G)$.

Suppose that $u_2v_6 \in E(G)$. Then for each $u_i$ with $1 \leq i \leq 4$, $v_iu_2 \notin E(G)$, otherwise, $[v, u_i, u_2, u_1] \supseteq K_4^-$, it is obvious that $|V(H - \{u_1\}) \cup \{v_0, u_p, u_{p-1}\}| \geq 2K_4^-$, which contradicts (1). However, this gives us $|E(u_2, V(H) \cup \{v\})| \leq 3$, a contradiction. Thus, $u_2 \notin E(G)$ and $u_{p-1}v_4 \notin E(G)$. If there exists $v_i \in \{v_1, v_2\}$, say $i = 1$, such that $v_1u_2 \in E(G)$, then $u_{p-1}v_1 \notin E(G)$, $N_H(u_{p-1}) = \{v_2, v_3, v_6\}$ and $u_{p-1}v \in E(G)$. This together with (1) tell us $u_p$ has at most one neighbor in $\{v_1, v_3, v_4\}$ and thus $\{v_2, v_3, v_6\} \subseteq N_H(u_p)$ and $u_p \notin E(G)$. We see that $[v, u_p, u_{p-1}, v_6] \supseteq K_4^-$, $[v_1, u_2, v_3, v_4] \supseteq K_4^-$ and $[v_0, v_1, v_2, v_5] \supseteq K_4^-$, a contradiction. This proves that $u_2$ has no neighbor in $\{v_1, v_2\}$ and so $u_2v_5, u_2v_6 \in E(G)$. As $|E(u_{p-1}, V(H))| \geq 3$, by the symmetry role of $v_1$ and $v_2$, we may assume that $v_1u_{p-1} \in E(G)$. Suppose that $u_{p-1}v_6 \in E(G)$. If $v_0u_p \notin E(G)$, then $[u_{p-1}, u_p, v_1, v_6] \supseteq K_4^-$, $[v, v_2, v_3, u_1] \supseteq K_4^-$ and $[v_0, u_2, v_4, v_5] \supseteq K_4^-$, a contradiction. Therefore, $v_0u_p \in E(G)$ and then there exist $v_i, v_j \in \{v_2, v_3, v_4, v_5\}$, such that $u_1v_1, u_2v_2 \in E(G)$. If $2 \in \{i, j\}$, then $[v, v_1, v_2, u_p] \supseteq K_4^-$, $[v_1, u_1, u_{p-1}, v_6] \supseteq K_4^-$ and $[V(H - \{v_1, v_1, v_2, v_3\}) \cup \{v_0, u_2\}] \supseteq K_4^-$, a contradiction. Hence, $2 \notin \{i, j\}$. Then $[v_2, v_1, v_2, u_p] \supseteq K_4^-$, $[v_1, u_1, v_{p-1}, v_6] \supseteq K_4^-$ and $[V(H - \{v_1, v_1, v_2, v_3\}) \cup \{v_0, v, v_1\}] \supseteq K_4^-$, a contradiction. This proves that $u_{p-1}v_6 \notin E(G)$ and it follows that $v_2u_{p-1}, v_3u_{p-1} \in E(G)$. By (1), $u_{p}v_5 \notin E(G)$. Since $|V(F - \{u_1, u_p\}) \cup \{v_5, v_6\}|$ contains a theta graph and $u_p$ has at least two neighbors in $\{v_1, v_2, v_3, v_4\}$, we see that $|V(H - \{v_5, v_6\}) \cup \{v, u_1, u_p, v_0\}| \geq 2K_4^-$, a contradiction. This completes the proof of Case 2 and the proof of Claim 3.5. □
Since $n \geq 13$ and $6 \leq |V(H)| \leq 8$, it follows from Claim 3.5 that $|F^*| \geq 2$.

**Claim 3.6.** $|V(F)| \leq 2$ for each $F \in F^*$.

**Proof.** By way of contradiction. Suppose that there exists $F \in F^*$ such that $|V(F)| \geq 3$. According to Claim 3.5, $|V(F)| = 3$. If $F$ is a triangle, then the proof of Claim 3.3 works, because of $|F^*| \geq 2$. Thus, $F$ is a path of order three and write $F = u_1u_4u_3$. Let $F' = F^* - F$ and $u_4 \in V(F')$ such that $u_2$ is an end vertex of $F'$. It is obvious that $d_{F'}(u_2) = 1$. Suppose that $7 \leq l \leq 8$. It is obvious that there exists $v_i \in V(H)$, such that $u_1v_i, u_4v_i, u_3v_i \in E(G)$, that is, $\{u_1, u_4, u_3\} \supseteq K_4$, since $|V(H - \{v_i\}) \cup \{v_0, u_2\}| \geq 2K_4$, a contradiction. Thus, $l = 6$, then continue the same proof in Claim 3.5 (when $|F| \geq 4$ and contains at least three leaves).

**Claim 3.7.** For each graph $F \in \mathcal{F}$ such that $|V(F)| = 2$, there exists $S \subset V(H)$ with $|S| = 2$ and $|V(F) \cup S| \geq K_4$.

**Proof.** Let $F \in \mathcal{F}$ such that $|V(F)| = 2$, label $V(F) = \{u_1, u_2\}$. Since $|E(u_1, V(H))| \geq l - 1$ if $7 \leq l \leq 8$ and $|E(u_1, V(H) \cup \{v\})| \geq l - 1$ for each $i$ with $1 \leq i \leq 2$, it follows from the pigeonhole principle that there exists a subset $S \subset V(H)$ with $|S| = 2$ and $S \subseteq N_H(u_1) \cap N_H(u_2)$. By (3), we know $|V(F) \cup S| \geq K_4$.

**Claim 3.8.** For any $u \in V(G^*)$, $|E(u, \{v_0\} \cup V(H))| = |E(u, V(H))| \leq l - 1$ if $7 \leq l \leq 8$; $|E(u, V(H) \cup \{v\})| \leq l$ if $l = 6$.

**Proof.** Suppose that there exists $u \in V(G^*)$ such that $|E(u, V(H))| \geq l$ if $7 \leq l \leq 8$, and $|E(u, V(H) \cup \{v\})| \geq l + 1$ if $l = 6$. By Claim 3.6, we may assume that $F^*$ contains two components $F_1$ and $F_2$ with $|V(F_i)| \leq 2$ for each $1 \leq i \leq 2$, such that $u \in V(F_1)$. Suppose that $|V(F_2)| = 2$ and label $F_2 = u_2u_3$. Note that $|E(u_1, V(H))| \geq l - 1$ for each $i \in \{2, 3\}$. By Claim 3.7, there exist $v_i, v_j \in V(H)$ such that $\{u_2, u_3, v_i, v_j\} \supseteq K_4$. If $7 \leq l \leq 8$, combining with (2) and (3), $|V(H - \{v_i, v_j\}) \cup \{u, v_0\}| \geq 2K_4$, which contradicts (1). Therefore, $l = 6$. By the choice of $v_i, |E(v, V(H))| = 6$. Notice that $v, v_q, v, u \supseteq K_4$. Since $F^* \setminus \{F_1 \cup F_2\} \neq \emptyset$, choose $u_4 \in V(F^* \setminus \{F_1 \cup F_2\})$. By Claim 3.6, $|E(u_4, V(H))| \geq 4$, choose $\{v_p, v_q\} \subseteq N_H(u_4) \cap N_H(v) - \{v_i, v_j\}$ such that $p \neq q$. Now, $|E(u_4, v_0, v_q)| \geq K_4$ and $|E(v, v_q, v_0)| \geq K_4$, which contradicts (1). This shows the order of each components of $F^* \setminus F_1$ is one. Now, note that $|F^* \setminus F_1| \geq 3$, we can choose three different vertices $u_1, u_2, u_3$, such that $|E(u_i, V(H))| \geq 5$ for each $1 \leq i \leq 3$. As above, it is obvious that $|V(H) \cup \{v, u, v_0, u_1, u_2, u_3\}| \geq 3K_4$, a contradiction.

Now we are in the position to complete the proof of Theorem 1.8. By Claim 3.6 and Claim 3.8, $|V(F)| = 2$ for all $F \in F^*$, we have

$$
\sum_{F \in F^*} |E(F)| = \begin{cases} \frac{n-1-l}{2}, & \text{if } 7 \leq l \leq 8 \\ \frac{n-3}{2}, & \text{if } l = 6. \end{cases}
$$
Suppose that \(7 \leq l \leq 8\). We may assume that \(u_1u_2\) and \(u_3u_4\) are two component of \(G^*\), since \(|E(u_i, V(H))| \geq l - 1\), by Claim 3.2, it is obvious that \([V(H) \cup \{v_0, u_1, u_2, u_3, u_4\}] \supseteq 3K_4\), a contradiction. Thus, \(l = 6\), and according to Claim 3.8, we obtain

\[
|E(G)| = |E([\{v_0, v\} \cup V(H)])| + |E(V(G^*), \{v_0, v\} \cup V(H))| + \sum_{F \in F^*} |E(F)| \\
\leq 27 + 5|V(G^*)| + \sum_{F \in F^*} |E(F)| \\
= 27 + 5(n - 8) + \frac{n - 8}{2} \\
= \frac{11n - 34}{2},
\]

this is an obvious contradiction and completes the proof of Theorem 1.8.

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