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DIVISION PROBLEM IN GENERALIZED GROWTH SPACES ON THE UNIT BALL IN $\mathbb{C}^n$

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Abstract. Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. For a weight function ω, we define the generalized growth space $A^\omega(\mathbb{B})$ by the space of holomorphic functions f on $\mathbb{B}$ such that

$$|f(z)| \leq C\omega(|\rho(z)|, \quad z \in \mathbb{B}.$$

Our main purpose in this note is to get the corona type decomposition in generalized growth spaces on $\mathbb{B}$.

1. Introduction and statement of results

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $H^\infty(\Omega)$ denote the space of all bounded holomorphic functions on Ω. Suppose that $G_1, G_2, \ldots, G_m \in H^\infty(\Omega)$ have no common zeroes, so that $|G|^2 = \sum |G_j|^2 > 0$. Then we can state the corona problem : Do there exist functions $u_1, u_2, \ldots, u_m \in H^\infty(\Omega)$ such that

$$\sum G_j u_j \equiv 1 \quad \text{on} \quad \Omega\ ?$$

This problem has been solved by L. Carleson [8] when $n = 1$ and Ω is the unit disk. It remains an open problem whether there are versions of the corona theorem for every planar domain or higher dimensional domains.

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z|^2 < 1\}$. For any holomorphic function ϕ on $\mathbb{B}$ one can consider holomorphic functions $u_1, u_2, \ldots, u_m$ on $\mathbb{B}$ such that

$$\sum G_j u_j \equiv \phi \quad \text{on} \quad \mathbb{B}.$$

Formulas for explicit solutions of such division problems were studied by many authors in various situations and norms (see [1], [2], [3], [4], [5], [11], [12], [14], [15], [16], [17], [18]). In particular, the H^p-corona problem asks for the condition on holomorphic n-tuples $G = (G_1, G_2, \ldots, G_m)$ such that the map $\mathcal{M}_G$ given by $\mathcal{M}_G(u) = \sum G_j u_j$ sends $H^p \times H^p \times \cdots \times H^p$ onto H^p. Of course,

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the conditions $G_1, G_2, \dots, G_m \in H^p$ and

$$|G|^2 = \sum |G_j|^2 > 0,$$

are necessary.

Our main purpose in this note is to consider solutions of such division problem in holomorphic growth type spaces on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$.

Let $\rho(z) = |z|^2 - 1$. Let $A^{-\alpha}(\mathbb{B})$ be the growth space of holomorphic functions f satisfying

$$|f(z)| \le C\frac{1}{|\rho(z)|^\alpha}, \quad z \in \mathbb{B}.$$

We define the growth space $A^{\log}(\mathbb{B})$ to be the space of holomorphic functions such that

$$|f(z)| \le C \log\left(\frac{1}{|\rho(z)|}\right), \quad z \in \mathbb{B}.$$

We denote by $\mathcal{B}(\mathbb{B})$ the usual Bloch space on $\mathbb{B}$. Then the following inclusions between the above spaces are known [10]:

$$H^\infty(\mathbb{B}) = A^{-0}(\mathbb{B}) \subsetneq \mathcal{B}(\mathbb{B}) \subsetneq A^{\log}(\mathbb{B}) \subsetneq A^{-\alpha}(\mathbb{B}). \tag{1}$$

In [9], they proved the embedding of Hardy spaces into weighted Bergman spaces on a general bounded domain in $\mathbb{C}^n$ by using the growth spaces.

Now we introduce a notion of the general weight function.

Let $\omega(t)$ be a positive real-valued function. We say that $\omega(t)$ is almost increasing (or decreasing, resp.), if there exists $C > 0$ such that

$$\omega(t) \le C\omega(\tau) \quad (\text{or, } C\omega(t) \ge \omega(\tau), \text{ resp.}) \quad \text{for} \quad t < \tau.$$

Definition 1. ([7]) Let $\omega(t)$ be a positive real-valued function defined on $(0, 1]$. Then ω is called *a weight function of order* α if there exists a constant α such that

$$\begin{aligned}\alpha &= \sup\left\{\gamma : \frac{\omega(t)}{t^\gamma} \text{ is almost increasing on } (0,1]\right\}\\ &= \inf\left\{\delta : \frac{\omega(t)}{t^\delta} \text{ is almost decreasing on } (0,1]\right\}.\end{aligned}$$

In this case we write $\operatorname{ord}(\omega) = \alpha$.

Definition 2. ([7]) For a weight function ω, we define the generalized growth space $A^\omega(\mathbb{B})$ by the space of holomorphic functions f on $\mathbb{B}$ such that

$$|f(z)| \le C\omega(|\rho(z)|, \quad z \in \mathbb{B}$$

and

$$\|f\|_{A^\omega} = \sup_{z\in\mathbb{B}} \frac{|f(z)|}{\omega(|\rho(z)|)}.$$

The above $\|\cdot\|_{A^\omega}$ is semi norm. Hence, the norm $\|f\|$ is given by $|f(0)| + \|f\|_{A^\omega}$ for all order α.

Example 1.1. (i) For any positive number α, the functions $t^{-\alpha}$ and $\log(\frac{1}{t})$ are the most typical examples of the weight functions of negative order and zero order, respectively. In these cases, the class A^{ω} is the growth space $A^{-\alpha}$, and log-growth space $A^{\log}$, respectively.
(ii) Non-typical examples of weight functions are $\omega_1(t) = t^{-\alpha}\left(\log(\frac{C_D}{t})\right)^{\beta}$ and $\omega_2(t) = t^{-\alpha}\left(2+\cos(\frac{1}{t})\right)$, where $\alpha > 0, \beta \in \mathbb{R}$. Both of $\omega_1(t)$ and $\omega_2(t)$ have ord $= -\alpha$.

Remark 1. Let ω_α and ω_β be weight functions of $\operatorname{ord}(\omega_\alpha) = \alpha$ and $\operatorname{ord}(\omega_\beta) = \beta$.
(i) For $\epsilon > 0$, it follows that

$$\left(\frac{\omega_\beta(t)}{\omega_\alpha(t)}\right) \Big/ t^{\beta-\alpha-\epsilon} = \left(\frac{\omega_\beta(t)}{t^{\beta-\epsilon/2}}\right) \Big/ \left(\frac{\omega_\alpha(t)}{t^{\alpha+\epsilon/2}}\right)$$

is almost increasing and that

$$\left(\frac{\omega_\beta(t)}{\omega_\alpha(t)}\right) \Big/ t^{\beta-\alpha+\epsilon} = \left(\frac{\omega_\beta(t)}{t^{\beta+\epsilon/2}}\right) \Big/ \left(\frac{\omega_\alpha(t)}{t^{\alpha-\epsilon/2}}\right)$$

is almost decreasing. Thus $\operatorname{ord}(\omega_\beta/\omega_\alpha) = \beta - \alpha$.
(ii) Let $\alpha < \beta$. For $\epsilon > 0$, since

$$\left(\frac{\omega_\beta(t)}{\omega_\alpha(t)}\right) \Big/ t^{\beta-\alpha-\epsilon}$$

is almost increasing, it follows that $\omega_\beta(t) \lesssim \omega_\alpha(t)$ on $(0, 1]$. Thus $A^{\omega_\beta} \subset A^{\omega_\alpha}$.
However, if $\alpha = \beta$, then there is no inclusion relation between A^{ω_α} and A^{ω_β}.

Remark 2. Let ω be a weight function of $\operatorname{ord}(\omega) = \alpha$. If $\alpha < 0$, then

$$\omega(t) = \frac{\omega(t)}{t^{\alpha+\epsilon}} \cdot t^{\alpha+\epsilon}$$

is almost decreasing, if $\epsilon > 0$ is chosen such that $\alpha + \epsilon < 0$. However, if $\alpha = 0$, then there is no such information.

Theorem 1.2. *Let $G_1, G_2, \ldots, G_m \in H^\infty(\mathbb{B})$. Let ω be a weight function such that $-1 < \operatorname{ord}(\omega) \le 0$. Let $\phi \in A^{\omega}(\mathbb{B})$. Suppose that*

$$\sum |G_j|^2 \ge \delta^2$$

for some $\delta > 0$. Then there exist $u_1, u_2, \ldots, u_m \in A^{\tilde{\omega}}(\mathbb{B})$ such that

$$G_1 u_1 + G_2 u_2 + \cdots + G_m u_m = \phi \quad \text{on} \quad \mathbb{B},$$

where

$$\tilde{\omega} = \begin{cases} \omega(t), & \text{if} \quad -1 < \operatorname{ord}(\omega) < 0 \\ \omega(t)\log\left(\frac{1}{t}\right), & \text{if} \quad \operatorname{ord}(\omega) = 0 \text{ and } \omega \text{ is almost decreasing.} \end{cases}$$

Corollary 1.3. *Let $\phi, G_1, G_2, \dots, G_m \in H^\infty(\mathbb{B})$. Suppose that*

$$\sum |G_j|^2 \geq \delta^2$$

for some $\delta > 0$. Then there exist $u_1, u_2, \dots, u_m \in A^{\log}(\mathbb{B})$ such that

$$G_1 u_1 + G_2 u_2 + \cdots + G_m u_m = \phi \quad \textit{on} \quad \mathbb{B}.$$

2. The solution operator for the division problem

For the construction of the solution operator for the division problem we use the integral representation for the solution of the $\bar{\partial}$-equation introduced by Berndtsson and Andersson [6]. Let

$$Q = -\frac{\partial\rho}{\rho} = \partial\left(\log\frac{1}{-\rho}\right).$$

Then for any $r > 0$ we have the integral kernel for the solution of the $\bar{\partial}$-equation such that

$$K^r(\zeta, z) = \sum_{\nu=0}^{n-1} C_{\nu,r} \frac{|\rho(\zeta)|^{r+\nu}}{|1-\bar{\zeta}z|^{r+\nu}} \frac{\partial_\zeta|\zeta - z|^2 \wedge (\partial_\zeta\overline{\partial}_\zeta|\zeta - z|^2)^{n-1-\nu} \wedge (\overline{\partial}_\zeta Q)^\nu}{|\zeta - z|^{2(n-\nu)}}$$

which induces a solution operator

$$S^r\eta(z) = \int_{\zeta\in\mathbb{B}} \eta(\zeta) \wedge K^r(\zeta, z), \quad z \in \mathbb{B}$$

such that $\overline{\partial}(S^r\eta) = \eta$ for a $\overline{\partial}$-closed (0,1) form η (see [1]). We note that

$$\begin{aligned}\bar{\partial}_\zeta Q &= \partial\bar{\partial}\left(\log\frac{1}{-\rho}\right)\\ &= -\frac{1}{\rho}\bar{\partial}\partial\rho + \frac{1}{\rho^2}\bar{\partial}\rho \wedge \partial\rho\end{aligned}$$

and thus

$$(\bar{\partial}_\zeta Q)^\nu = \mathcal{O}\left(\frac{1}{|\rho|^\nu} + \frac{\overline{\partial}\rho}{|\rho|^{\nu+1}}\right).$$

Since $|\zeta - z|^2 \leq 2|1 - \bar{\zeta}z|$, we have

$$K^r(\zeta, z) = \frac{|\rho(\zeta)|^r \omega_1(\zeta, z)}{|1-\bar{\zeta}z|^r |\zeta - z|^{2n-1}} + \frac{\bar{\partial}\rho(\zeta) \wedge |\rho(\zeta)|^{r-1}\omega_2(\zeta, z)}{|1-\bar{\zeta}z|^{r+1}|\zeta - z|^{2n-3}},$$

where the forms ω_1 and ω_2 have bounded coefficients on $\bar{\mathbb{B}} \times \bar{\mathbb{B}}$.

We have

$$\begin{aligned}S^r\eta(z) &= \int_{\zeta\in\mathbb{B}} \eta(\zeta) \wedge |\rho(\zeta)|^r K_1^r(\zeta, z) + \int_{\zeta\in\mathbb{B}} \eta(\zeta) \wedge \overline{\partial}\rho(\zeta) \wedge |\rho(\zeta)|^{r-1} K_2^r(\zeta, z)\\ &= S_1\eta(z) + S_2\eta(z),\end{aligned}$$

where $K_1^r(\zeta, z)$ and $K_2^r(\zeta, z)$ are defined by

$$K_1^r(\zeta, z) = \frac{\omega_1(\zeta, z)}{|1-\bar{\zeta}z|^r|\zeta - z|^{2n-1}}$$

and

$$K_2^r(\zeta, z) = \frac{\omega_2(\zeta, z)}{|1-\bar{\zeta}z|^{r+1}|\zeta - z|^{2n-3}}.$$

First we solve the division problem for only the case $m = 2$. We may apply Koszul complex theory [11] to extend to general m.

Let $G_1, G_2 \in H^\infty(\mathbb{B})$ and $\delta > 0$ be such that

$$|G_1(\zeta)|^2 + |G_2(\zeta)|^2 \geq \delta, \quad \zeta \in \mathbb{B}.$$

Let

$$\mathcal{G} = \frac{\bar{G}_1\overline{\partial G_2} - \bar{G}_2\overline{\partial G_1}}{|G|^4}.$$

We note that

$$\mathcal{G} = \frac{1}{G_1}\bar{\partial}\left(\frac{\bar{G}_2}{|G|^2}\right) \quad \text{or} \quad -\frac{1}{G_2}\bar{\partial}\left(\frac{\bar{G}_1}{|G|^2}\right).$$

Thus $\mathcal{G}$ is a $\bar{\partial}$-closed $(0,1)$ form. For $\phi \in \mathcal{O}(\mathbb{B})$ we have

$$\bar{\partial}S^r(\phi\mathcal{G}) = \phi\mathcal{G}.$$

Put

$$\gamma_1 = \frac{\bar{G}_1}{|G|^2} \quad \text{and} \quad \gamma_2 = \frac{\bar{G}_2}{|G|^2}.$$

Then

$$\mathcal{G} = \frac{1}{G_1}\bar{\partial}\gamma_2 \quad \text{or} \quad -\frac{1}{G_2}\bar{\partial}\gamma_1.$$

Clearly, $G_1\gamma_1 + G_2\gamma_2 \equiv 1$. Let

$$u_1 = \gamma_1\phi + G_2S^r(\phi\mathcal{G}) \quad \text{and} \quad u_2 = \gamma_2\phi - G_1S^r(\phi\mathcal{G}).$$

Then

$$G_1u_1 + G_2u_2 \equiv \phi.$$

We know that $\bar{\partial}u_1 = \bar{\partial}u_2 = 0$. Thus $u_1, u_2 \in \mathcal{O}(\mathbb{B})$. It remains to prove that $u_1, u_2 \in A^{\tilde{\omega}}(\mathbb{B})$.

Since $\gamma_j, G_j \in H^\infty(\mathbb{B})$ and $\phi \in A^\omega(\mathbb{B}) \subset A^{\tilde{\omega}}(\mathbb{B})$, it is enough to prove that

$$|S^r(\phi\mathcal{G})(z)| \lesssim \tilde{\omega}(|\rho(z)|), \quad z \in \mathbb{B}.$$

For any bounded holomorphic function h on $\mathbb{B}$ we have

$$|\partial h| \lesssim \frac{1}{|\rho|}, \quad |\partial\rho \wedge \partial h| \lesssim \frac{1}{|\rho|^{1/2}}.$$

We recall the Cauchy integral formula

$$h(z) = \int_{\zeta \in \partial\mathbb{B}} \frac{h(\zeta)\partial\rho(\zeta) \wedge (\partial\bar{\partial}\rho(\zeta))^{n-1}}{(1-\bar{\zeta}z)^n}.$$

By the Cauchy integral formula and noting that

$$\partial_z(1-\bar{\zeta}z) = -\partial_z\rho(z) + \partial_z|\zeta - z|^2,$$

we have

$$\partial h(z) = \alpha(z)\partial\rho(z) + \beta(z),$$

where $|\alpha| \lesssim 1/|\rho|$ and $|\beta| \lesssim 1/\sqrt{|\rho|}$.

3. Estimates for integral kernels

Lemma 3.1. *If ω is a weight function of order α with $-1 < \alpha \leq 0$, then it holds that*

$$\int_0^1 \frac{\omega(t)}{|\rho(z)|+t}dt \lesssim \begin{cases} \omega(|\rho(z)|), & \text{if } \quad -1 < \alpha < 0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \text{if } \alpha = 0 \text{ and } \omega \text{ is almost decreasing.} \end{cases}$$

Proof. Write $r = |\rho(z)|$ and

$$\int_0^1 \frac{\omega(t)}{r+t}dt = \int_0^r \frac{\omega(t)}{r+t}dt + \int_r^1 \frac{\omega(t)}{r+t}dt =: I_1 + I_2.$$

We choose $\epsilon > 0$ so small that $\alpha - \epsilon > -1$. Then we have

$$I_1 = \int_0^r \frac{\omega(t)}{t^{\alpha-\epsilon}} \cdot \frac{t^{\alpha-\epsilon}}{r+t}dt \lesssim \frac{\omega(r)}{r^{\alpha-\epsilon}} \int_0^r \frac{t^{\alpha-\epsilon}}{r+t}dt \leq \frac{\omega(r)}{r^{\alpha-\epsilon}} \int_0^r \frac{t^{\alpha-\epsilon}}{r}dt \lesssim \omega(r),$$

since $\omega(t)/t^{\alpha-\epsilon}$ is almost increasing. For the case of I_2, if $-1 < \alpha < 0$, then we choose $\epsilon > 0$ so small that $\alpha + \epsilon < 0$. Then we have

$$I_2 \leq \int_r^\infty \frac{\omega(t)}{t^{\alpha+\epsilon}} \cdot \frac{t^{\alpha+\epsilon}}{r+t}dt \lesssim \frac{\omega(r)}{r^{\alpha+\epsilon}} \int_r^\infty \frac{t^{\alpha+\epsilon}}{r+t}dt \leq \frac{\omega(r)}{r^{\alpha+\epsilon}} \int_r^\infty \frac{dt}{t^{1-\alpha-\epsilon}} \lesssim \omega(r),$$

since $\omega(t)/t^{\alpha+\epsilon}$ is almost decreasing. If $\alpha = 0$, since $\omega(t)$ is almost decreasing, we have

$$I_2 \lesssim \omega(r)\int_r^1 \frac{dt}{r+t} = \omega(r)\log\left(\frac{r+1}{2r}\right) \leq \omega(r)\log\frac{1}{r}.$$

□

Lemma 3.2. *Let r be sufficiently large. If ω is a weight function of order α with $-1 < \alpha \leq 0$, then it holds that*

$$\int_{\zeta\in\mathbb{B}} \frac{|\rho(\zeta)|^{r-1}\omega(|\rho(\zeta)|)}{|1-\bar{\zeta}z|^r|\zeta - z|^{2n-1}}dV(\zeta) \lesssim \begin{cases} \omega(|\rho(z)|), & \text{if } \quad -1 < \alpha < 0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \text{if } \alpha = 0 \text{ and } \omega \text{ is almost decreasing.} \end{cases}$$

Proof. We have

$$2\mathrm{Re}(1-\bar{\zeta}z) = |\rho(\zeta)| + |\rho(z)| + |\zeta - z|^2, \quad \zeta, z \in \mathbb{B}.$$

Thus

$$|1-\bar{\zeta}z| \gtrsim |\rho(\zeta)| + |\mathrm{Im}(1-\bar{\zeta}z)| + |\zeta - z|^2 + |\rho(z)|, \quad \zeta, z \in \mathbb{B}.$$

We choose local coordinate $t_z(\zeta) = (t_1, t_2, \dots, t_{2n})$ such that

$$t_1 = -\rho(\zeta),\ t_2 = \mathrm{Im}(1-\bar{\zeta}z),\ t_3(z) = \cdots = t_{2n}(z) = 0,\ |t_z(\zeta)| \sim |\zeta - z|.$$

Then we have

$$\begin{aligned}
I(z) &= \int_{\zeta\in\mathbb{B}\cap B(z,\epsilon)} \frac{|\rho(\zeta)|^{r-1}\omega(|\rho(\zeta)|)}{|1-\bar{\zeta}z|^r|\zeta-z|^{2n-1}} dV(\zeta) \\
&\lesssim \int_{|t|<1} \frac{|t_1|^{r-1}\omega(t_1)}{|t|^{2n-1}(|t_1|+|t_2|+|\rho(z)|)^r} dt \\
&\lesssim \int_{|(t_1,t_2)|<1} \frac{\omega(t_1)dt_1dt_2}{(|t_1|+|t_2|)(|t_1|+|t_2|+|\rho(z)|)} \\
&\lesssim \int_0^1 \frac{\omega(t)}{t+|\rho(z)|} dt \\
&\lesssim \begin{cases} \omega(|\rho(z)|), & \text{if } \quad -1<\alpha<0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \text{if } \alpha = 0 \text{ and } \omega \text{ is almost decreasing,} \end{cases}
\end{aligned}$$

by Lemma3.1. □

Lemma 3.3. *Let r be sufficiently large. If ω is a weight function of order α with $-1<\alpha\le 0$, then it holds that*

$$\int_{\zeta\in\mathbb{B}} \frac{|\rho(\zeta)|^{r-3/2}\omega(|\rho(\zeta)|)}{|1-\bar{\zeta}z|^{r+1}|\zeta-z|^{2n-3}} dV(\zeta) \lesssim \begin{cases} \omega(|\rho(z)|), & \textit{if} \quad -1<\alpha<0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \textit{if } \alpha=0 \textit{ and } \omega \textit{ is almost decreasing.} \end{cases}$$

Proof. As the proof of Lemma 3.2, we have

$$\begin{aligned}
J(z) &= \int_{\zeta\in\mathbb{B}\cap B(z,\epsilon)} \frac{|\rho(\zeta)|^{r-3/2}\omega(|\rho(\zeta)|)}{|1-\bar{\zeta}z|^{r+1}|\zeta-z|^{2n-3}} dV(\zeta) \\
&\lesssim \int_{|t|<1} \frac{|t_1|^{r-3/2}\omega(t_1)}{|t|^{2n-3}(|t_1|+|t_2|+|t|^2+|\rho(z)|)^{r+1}} dt \\
&\lesssim \int_{\xi\in\mathbb{C},|\xi|<1} \int_0^1\int_0^1 \frac{t_1^{r-3/2}\omega(t_1)dt_1dt_2d\xi}{|\xi|(t_1+t_2+|\xi|^2+|\rho(z)|)^{r+1}} \\
&\lesssim \int_0^1\int_0^1 \frac{t_1^{r-3/2}\omega(t_1)dt_1dt_2}{(t_1+t_2+|\rho(z)|)^{r+1/2}} \\
&\lesssim \int_0^1 \frac{\omega(t_1)}{t_1+|\rho(z)|} dt_1 \\
&\lesssim \begin{cases} \omega(|\rho(z)|), & \text{if } \quad -1<\alpha<0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \text{if } \alpha = 0 \text{ and } \omega \text{ is almost decreasing,} \end{cases}
\end{aligned}$$

by Lemma3.1. □

4. Proof of Theorem 1.2

We assume that $G_1, G_2 \in H^\infty(\mathbb{B})$ have no common zeroes, so that $|G|^2 = \sum |G_j|^2 > 0$. We will prove that

$$|S_1^r(\phi\mathcal{G})(z)|, |S_2^r(\phi\mathcal{G})(z)| \lesssim \tilde{\omega}(|\rho(z)|), \quad z \in \mathbb{B}.$$

Since $G_1, G_2 \in H^\infty$, we have

$$|\mathcal{G}| \lesssim \frac{1}{|G|^4}(|G_1||\partial G_2| + |G_2||\partial G_1|) \lesssim \frac{1}{|\rho|}$$

and

$$|\mathcal{G} \wedge \bar{\partial}\rho| \lesssim \frac{1}{|G|^4}(|G_1||\partial G_2 \wedge \partial\rho| + |G_2||\partial G_1 \wedge \partial\rho|) \lesssim \frac{1}{|\rho|^{1/2}}.$$

Thus we have

$$\begin{aligned}
|S_1^r(\phi\mathcal{G})(z)| &\lesssim \int_{\zeta\in\mathbb{B}} |\phi(\zeta)||\mathcal{G}(\zeta)||\rho(\zeta)|^r |K_1^r(\zeta,z)| dV(\zeta) \\
&\lesssim \int_{\zeta\in\mathbb{B}} \frac{|\rho(\zeta)|^{r-1}\omega(|\rho(\zeta)|)}{|1-\bar{\zeta}z|^r|\zeta - z|^{2n-1}} dV(\zeta) \\
&\lesssim \begin{cases} \omega(|\rho(z)|), & \text{if} \quad -1 < \alpha < 0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \text{if } \alpha = 0 \text{ and } \omega \text{ is almost decreasing,} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
|S_2^r(\phi\mathcal{G})(z)| &\lesssim \int_{\zeta\in\mathbb{B}} |\phi(\zeta)||\mathcal{G}(\zeta)\wedge\bar{\partial}\rho(\zeta)||\rho(\zeta)|^{r-1} |K_2^r(\zeta,z)| dV(\zeta) \\
&\lesssim \int_{\zeta\in\mathbb{B}} \frac{|\rho(\zeta)|^{r-3/2}\omega(|\rho(\zeta)|)}{|1-\bar{\zeta}z|^{r+1}|\zeta - z|^{2n-3}} dV(\zeta) \\
&\lesssim \begin{cases} \omega(|\rho(z)|), & \text{if} \quad -1 < \alpha < 0, \\ \omega(|\rho(z)|)\log\frac{1}{|\rho(z)|}, & \text{if } \alpha = 0 \text{ and } \omega \text{ is almost decreasing,} \end{cases}
\end{aligned}$$

by Lemma 3.2 and Lemma 3.3, respectively.

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