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COUPLED FIXED POINT THEOREMS FOR RATIONAL INEQUALITY IN GENERALIZED METRIC SPACES

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Abstract. In modern times, coupled fixed point theorems have been rigorously studied by many researchers in the milieu of partially ordered G-metric spaces using different contractive conditions. In this note, some coupled fixed point theorems using mixed monotone property in partially ordered G-metric spaces are obtained. Furthermore some theorems by omitting the completeness on the space and continuity conditions on function, are obtained. Our results partially generalize some existing results in the present literature. To exemplify our results and to distinguish them from the existing ones, we equip the article with suitable examples.

1. Introduction

Taking into accounts its applications, fixed point theory has received substantial concentration through the last ninety years in many different ways. One of the newest branches of this theory is dedicated to the study of G-metric spaces. The notion of G- metric space was introduced by Mustafa in collaboration with Sims[12]. This was a generalization of metric spaces in which a non-negative real number was assigned to every triplet of elements. Mustafa et al.[13],[14],[15],[16] studied many fixed point results for a self-mapping in G-metric space under certain conditions. In recent times, fixed point theory has extended rapidly in partially ordered metric spaces, that is, metric spaces endowed with a partial ordering. Existence of fixed points in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings[19], and then by Nieto and Lopez[18]. After wards fixed point problems have also been considered in partially ordered probabilistic metric spaces[8], partially ordered G-metric spaces[3],[20], partially ordered cone metric spaces[10].
Mixed monotone operators were pioneered by Guo and Lakshmikantham in [9]. Their study has not only important theoretical meaning but also wide applications in engineering and many other fields. Particularly, a coupled fixed point result in partially ordered metric spaces was recognized by Bhaskar and

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Lakshmikantham [4]. After the publication of this effort, several coupled fixed point and coincidence point results have materialized in the recent literature. Works noted in [5],[7],[1],[11],[2] are some relevant and noticeable examples.

2. Preliminaries

Definition 1. [12] Let X be a nonempty set, and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:
(G-1) $G(x, y, z) = 0$ if $x = y = z$.
(G-2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$.
(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$
(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables,
(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$
Then the function G is called a generalized or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2. [12] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to $x \in X$ if $\lim\limits_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is for any $\epsilon > 0$, there is $k \in N$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq k$. We call x the limit of the sequence and write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n \to x$

Definition 3. [12] Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is a $k \in N$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq k$; that is $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Lemma 2.1. [12] *Let (X, G) be a G-metric space. Then the following are equivalent:*

1. *$\{x_n\}$ is G-convergent to x;*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$;*
3. *$G(x_n, x, x) \to 0$ as $n \to \infty$;*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$;*

Lemma 2.2. [17] *Let (X, G) be a G-metric space. Then the following are equivalent:*

1. *$\{x_n\}$ is G-cauchy;*
2. *for every $\epsilon > 0$, there is $k \in N$, $G(x_n, x_m, x_m) < \epsilon$ for all $n, m \geq k$;*

Lemma 2.3. [12] *If (X, G) is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.*

Combining Lemmas(2.2) and Lemmas(2.3) we have the following result.

Lemma 2.4. [12] *If (X, G) is a G-metric space then $\{x_n\}$ is G-Cauchy sequence if and only if for every $\epsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \epsilon$, for all $m > n \geq N$.*

Definition 4. [12] Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-Sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

Definition 5. [12] A G-metric space (X, G) is called symmetric G-metric space if

$$G(x, y, y) = G(y, x, x) \ for \ all \ x, y \in X.$$

Definition 6. [12] A G-metric space is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 7. [4] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is monotone non decreasing in x and is monotone non increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$$

Definition 8. [4] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x, y) = x, \qquad F(y, x) = y$$

Definition 9. [6] Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y then $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

3. Main Results

Our main result runs as follows:

Theorem 3.1. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Suppose that there exists a $k \in [0, 1)$ such that*

(1)

$$\begin{aligned} G(F(x,y), F(u,v), F(w,z)) \leq k\Big(\min\Big\{ & \frac{G(x,u,w)+G(y,v,z)}{2}, \\ & \max\Big\{ \frac{G(u,u,w)+G(v,v,z)}{2}, G(x,w,w)+G(y,z,z), \\ & \frac{G(F(x,y),F(u,v),F(x,y)) + 4\sqrt{G(x,u,w).G(y,v,z)}}{2} \Big\}\Big\}\Big) \end{aligned}$$

For all $x, y, z, u, v, w \in X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$ where either $u \neq w$ or $v \neq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point in X, that is, there exist $x, y \in X$ such that

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Proof: Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Let $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Then $x_1 \succeq x_0$ and $y_0 \succeq y_1$. Again let $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we write

$$F^2(x_0, y_0) = F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2$$

and

$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2.$$

Since, F has a mixed monotone property, then we have

$$x_2 = F^2(x_0, y_0) = F(x_1, y_1) \succeq F(x_0, y_0) = x_1 \succeq x_0$$

$$y_2 = F^2(y_0, x_0) = F(y_1, x_1) \preceq F(y_0, x_0) = y_1 \preceq y_0.$$

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n) = F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0))$$

and

$$y_{n+1} = F(y_n, x_n) = F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)).$$

Then for all $n \geq 0$,

$$x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ... \tag{2}$$

$$y_0 \succeq y_1 \succeq y_2 \succeq ... \succeq y_n \succeq y_{n+1} \succeq ... \tag{3}$$

If for some n, we have $(x_{n+1}, y_{n+1}) = (x_n, y_n)$, then $F(x_n, y_n) = x_n$ and $F(y_n, x_n) = y_n$, that is, F has a coupled fixed point. So we assume, $(x_{n+1}, y_{n+1}) \neq (x_n, y_n)$ for all $n \geq 0$, that is, we assume that either $x_{n+1} = F(x_n, y_n) \neq x_n$ or $y_{n+1} = F(y_n, x_n) \neq y_n$.

Now

$$\begin{aligned} G(x_{n+1}, x_{n+1}, x_n) &= G(F(x_n, y_n), F(x_n, y_n), x_n) = G(F^{n+1}(x_0, y_0), F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \\ &\leq k^n \Big(\min \Big\{ \frac{G(x_1, x_1, x_0) + G(y_1, y_1, y_0)}{2}, \\ &\quad \max \Big\{ \frac{G(x_1, x_1, x_0) + G(y_1, y_1, y_0)}{2}, G(x_1, x_0, x_0) + G(y_1, y_0, y_0), \\ &\quad \frac{(G(x_2, x_2, x_2)) + 4\sqrt{G(x_1, x_1, x_0).G(y_1, y_1, y_0)}}{2} \Big\} \Big\} \Big) \end{aligned}$$

Now, using the property $G(x, x, y) \leq 2G(x, y, y)$, from the above inequality, we get

$$G(x_{n+1}, x_{n+1}, x_n) \leq \frac{k^n}{2} [G(x_1, x_1, x_0) + G(y_1, y_1, y_0)] \tag{4}$$

and

(5)

$$G(y_{n+1},y_{n+1},y_n)=G(F(y_n,x_n),F(y_n,x_n),y_n)=G(F^{n+1}(y_0,x_0),F^{n+1}(y_0,x_0),F^n(y_0,x_0))$$
$$\le \frac{k^n}{2}[G(y_1,y_1,y_0)+G(x_1,x_1,x_0)]$$

Now we claim that (4) and (5) are true for all $n\ge 0$. For $n=1$, we have

$$\begin{aligned}G(x_2,x_2,x_1)&=G(F(x_1,y_1),F(x_1,y_1),x_n)=G(F^2(x_0,y_0),F^2(x_0,y_0),F(x_0,y_0))\\ &\le k\Big(\min\Big\{\frac{G(x_1,x_1,x_0)+G(y_1,y_1,y_0)}{2},\\ &\quad \max\Big\{\frac{G(x_1,x_1,x_0)+G(y_1,y_1,y_0)}{2},G(x_1,x_0,x_0)+G(y_1,y_0,y_0),\\ &\quad \frac{(G(x_2,x_2,x_2))+4\sqrt{G(x_1,x_1,x_0).G(y_1,y_1,y_0)}}{2}\Big\}\Big\}\Big)\\ &\le \frac{k}{2}[G(x_1,x_1,x_0)+G(y_1,y_1,y_0)]\end{aligned}$$

Similarly we can prove that

$$G(y_2,y_2,y_1)=\frac{k}{2}[G(y_1,y_1,y_0)+G(x_1,x_1,x_0)]$$

Thus (4) and (5) are true for $n=1$.
Now we assume that (4) and (5) are true for $n=m$, that is

$$G(x_{m+1},x_{m+1},x_m)\le\frac{k^m}{2}[G(x_1,x_1,x_0)+G(y_1,y_1,y_0)]$$

and

$$G(y_{m+1},y_{m+1},y_m)\le\frac{k^m}{2}[G(y_1,y_1,y_0)+G(x_1,x_1,x_0)]$$

Now for $n=m+1$, we have

$$\begin{aligned}G(x_{m+2},x_{m+2},x_{m+1})&=G(F(x_{m+1},y_{m+1}),F(x_{m+1},y_{m+1}),F(x_m,y_m)\\ &\le k\Big(\min\Big\{\frac{G(x_{m+1},x_{m+1},x_m)+G(y_{m+1},y_{m+1},y_m)}{2},\\ &\quad\max\Big\{\frac{G(x_{m+1},x_{m+1},x_m)+G(y_{m+1},y_{m+1},y_m)}{2},\\ &\quad G(x_{m+1},x_m,x_m)+G(y_{m+1},y_m,y_m),\\ &\quad\frac{G(x_{m+2},x_{m+2},x_{m+2})+4\sqrt{G(x_{m+1},x_{m+1},x_m).G(y_{m+1},y_{m+1},y_m)}}{2}\Big\}\Big\}\Big)\\ &\le\frac{k}{2}[G(x_{m+1},x_{m+1},x_m)+G(y_{m+1},y_{m+1},y_m)]\\ &\le\frac{k}{2}\Big[\frac{k^m}{2}(G(x_1,x_1,x_0)+G(y_1,y_1,y_0))+\frac{k^m}{2}(G(y_1,y_1,y_0)+G(x_1,x_1,x_0))\Big]\\ &\le\frac{k^{m+1}}{2}[G(x_1,x_1,x_0)+G(y_1,y_1,y_0)]\end{aligned}$$

Similarly, we can prove that

$$G(y_{m+2}, y_{m+2}, y_{m+1}) \leq \frac{k^{m+1}}{2}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)]$$

Then, by induction, (4) and (5) are established for all $n \geq 0$.
Using the rectangle inequality, for all positive integers n, m,where $n < m$, we get

$$\begin{aligned}
G(x_m, x_m, x_n) &\leq G(x_m, x_m, x_{m-1}) + G(x_{m-1}, x_{m-1}, x_n) \\
&\leq G(x_m, x_m, x_{m-1}) + G(x_{m-1}, x_{m-1}, x_{m-2}) + G(x_{m-2}, x_{m-2}, x_n) \\
&\quad ---- \qquad\qquad ---- \qquad\qquad ---- \\
&\leq G(x_m, x_m, x_{m-1}) + G(x_{m-1}, x_{m-1}, x_{m-2}) + ... + G(x_{n+1}, x_{n+1}, x_n) \\
&\leq \frac{k^{m-1}}{2}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)] + \frac{k^{m-2}}{2}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)]+ \\
&\quad + ... + \frac{k^n}{2}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)] \\
&\leq \frac{k^{m-1} + k^{m-2} + ... + k^n}{2}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)] \\
&= \frac{k^n(1 + k + k^2 + ... + k^{m-n-1})}{2}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)] \\
&< \frac{k^n}{2(1-k)}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)]
\end{aligned}$$

That is,

$$G(x_m, x_m, x_n) \leq \frac{k^n}{2(1-k)}[G(x_1, x_1, x_0) + G(y_1, y_1, y_0)]$$

Thus,

$$\lim_{m,n\to\infty} G(x_m, x_m, x_n) = 0$$

Thus, by Lemma(2.4), $\{x_n\}$, that is, $\{F^n(x_0, y_0)\}$ is a Cauchy sequence and hence is convergent in the complete G-metric space X.
Let $x_n \to x$ (say) as $n \to \infty$.
Similarly, $\{y_n\}$, that is, $\{F^n(y_0, x_0)\}$ is a Cauchy sequence and hence is convergent in the complete G-metric space X.
Let $y_n \to y$ (say) as $n \to \infty$.
Now we show that F has a coupled fixed point in X.
By (4), we have

$$G(x_{n+1}, x_{n+1}, x_n) = G(F(x_n, y_n), F(x_n, y_n), x_n) \leq \frac{k^n}{2}[G(x_1, x_1, x_0)+G(y_1, y_1, y_0)]$$

Taking the limit as $n \to \infty$ and using the fact that F is continuous, we have $G(F(x, y), F(x, y), x) \leq 0$, which implies that $F(x, y) = x$.
Similarly, we get $F(y, x) = y$.

This gives an end to the proof.
In the next theorem, we use the following definition.

Definition 10. Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X. We say that $(X, G, \preceq)$ is regular if the following conditions hold:

(1) If the non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all n.
(2) If the non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y_n \succeq y$ for all n.

Theorem 3.2. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preceq)$ is regular. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Suppose that there exists a $k \in [0, 1)$ such that*

(6)
$$G(F(x,y), F(u,v), F(w,z)) \leq k\Big(\min\Big\{ \frac{G(x,u,w) + G(y,v,z)}{2}, \max\Big\{ \frac{G(u,u,w) + G(v,v,z)}{2}, G(x,w,w) + G(y,z,z), \frac{G(F(x,y), F(u,v), F(x,y)) + 4\sqrt{G(x,u,w).G(y,v,z)}}{2} \Big\}\Big\}\Big)$$

For all $x, y, z, u, v, w \in X$ with $w \preceq u \preceq x$ and $y \preceq v \preceq z$ where either $u \neq w$ or $v \neq z$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point in X, that is, there exist $x, y \in X$ such that

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Proof: Proceeding exactly as in Theorem(3.1), we have that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in the complete G-metric space (X, G). Then, there exist $x, y \in X$ such that $x_n \to x$ and $y_n \to y$. Since $\{x_n\}$is non-deceasing and $\{y_n\}$ is non-increasing, using the regularity of $(X, G, \preceq)$, we have $x_n \preceq x$ and $y_n succeq y$ for all $n \geq 0$. If $(x_n, y_n) = (x, y)$ for some $n \geq 0$, then by construction, $x_{n+1} = x$ and $y_{n+1} = y$, which implies that (x, y) is a coupled fixed point. So we assume that either $x_n \neq x$ or $y_n \neq y$. Using the rectangle inequality and (1), we get

$$\begin{aligned}
G(F(x,y),x,x) &\leq G(F(x,y),x_{n+1},x_{n+1}) + G(x_{n+1},x,x)\\
&= G(F(x_n,y_n),F(x_n,y_n),F(x,y)) + G(F(x_n,y_n),x,x)\\
&\leq k\Big(\min\Big\{\frac{G(x_n,x_n,x)+G(y_n,y_n,y)}{2},\\
&\quad \max\Big\{\frac{G(x_n,x_n,x)+G(y_n,y_n,y)}{2}, G(x_n,x,x)+G(y_n,y,y),\\
&\quad \frac{G(x_{n+1},x_{n+1},x_{n+1})+4\sqrt{G(x_n,x_n,x).G(y_n,y_n,y)}}{2}\Big\}\Big\}\Big) + G(x_{n+1},x,x)\\
&\leq \frac{k}{2}[G(x_n,x_n,x)+G(y_n,y_n,y)] + G(x_{n+1},x,x)
\end{aligned}$$

Taking as $n \to \infty$ in the above inequality we obtain $G(F(x,y),x,x) = 0$, that is, $F(x,y) = x$.
Similarly, we can show that $F(y,x) = y$.
This complete the proof of the theorem.

Theorem 3.3. *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X and such that $F(x,y) \preceq F(y,x)$ whenever $x \preceq y$. Suppose that there exists a $k \in [0,1)$ such that for all $x,y,z,u,v,w \in X$, the inequality(1)holds, whenever $w \preceq u \preceq x$ and $y \preceq v \preceq z$ and $x \prec y$ where either $u \neq w$ or $v \neq z$.*
If there exists $x_0, y_0 \in X$ such that $x_0 \preceq y_0$, $x_0 \preceq F(x_0,y_0)$ and $F(y_0,x_0) \preceq y_0$, then F has a coupled fixed point in X, that is, there exist $x, y \in X$ such that

$$F(x,y) = x \text{ and } F(y,x) = y.$$

Proof: By the condition of the theorem there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $F(y_0,x_0) \preceq y_0$.
We define $x_1, y_1 \in X$ as $x_1 = F(x_0,y_0) \succeq x_0$ and $y_1 = F(y_0,x_0) \preceq y_0$.
Since $x_0 \preceq y_0$, we have, by a condition of the theorem, $F(x_0,y_0) \preceq F(y_0,x_0)$.
Hence, $x_0 \preceq x_1 = F(x_0,y_0) \preceq F(y_0,x_0) = y_1 \preceq y_0$.
Continuing the above procedure we have two sequences $\{x_n\}$ and $\{y_n\}$ recursively as follows. For all $n \geq 1$,

$$x_n = F(x_(n-1), y_(n-1)) \text{ and } y_n = F(y_(n-1), x_(n-1)), \tag{7}$$

Such that

$$x_0 \preceq F(x_0,y_0) = x_1 \preceq ... \preceq F(x_(n-1), y_(n-1)) = x_n \preceq ... \preceq y_n = F(y_(n-1), x_(n-1)) \preceq ... \preceq y_1 = F(y_0,x_0) \tag{8}$$

In particular, we have for all $n \geq 0$,

$$x_n \preceq F(x_n,y_n) = x_{n+1} \preceq y_{n+1} = F(y_n,x_n) \preceq y_n.$$

If $x_n = y_n = c$ (say) for some n, then $c \preceq F(c,c) \preceq F(c,c) \preceq c$.
This shows that $c = F(c,c)$.

Thus (c, c) is a coupled fixed point.
Hence we assume that

$$x_n \prec y_n, \ for \ all \ n \ \geq \ 0. \tag{9}$$

Further, for the same reason as stated in Theorem(3.1), we assume that $(x_n, y_n) \neq (x_{n+1}, y_{n+1})$.
Then, in view of (9), for all $n \geq 0$, the inequality(1) will hold with
$x = x_{n+2}$, $u = x_{n+1}$, $w = x_n$, $y = y_n$, $v = y_{n+1}$ and $z = y_{n+2}$
The rest of the proof is completed by repeating the same steps as in Theorem(3.1).

Theorem 3.4. *If in Theorem(3.3), in place of the continuity of F, we assume that the conditions of Definition(10) hold, then F has a coupled fixed point.*

Remark 1. Since every G-metric on X defines a metric space d_G on X by

$$d_G = G(x, y, y) + G(y, x, x) \ \ for \ \ all \ \ x, y \in X.$$

But, due to the condition that $u \neq w$ and $v \neq z$, the inequality(1) does not reduce to any metric inequality with metric d_G. Thus, all the above results do not reduce to fixed point problems in the corresponding metric space (X, d_G).

Following example substantiates the validity of hypothesis of Theorem(3.1).
Example 1: Let $X = R$ with usual ordering. Define $G : X^3 \to X$ by

$$G(x, y, z) = \max \{|x - y|, |y - z|, |x - z|\}, \ \ for \ \ all \ \ x, y, z \in X$$

Then $(X, G, \leq)$ is a complete partially ordered G-metric space. Let $F : X \times X \to X$, defined by

$$F(x, y) = \frac{x - y}{16}$$

Clearly, the mapping F has the mixed monotone property. Let $x, u, v, y, z, w \in X$ be such that $x \geq u \geq w$ and $y \leq v \leq z$. Now, we will check that inequality(1) of Theorem(3.1) is fulfilled for all $x, u, v, y, z, w \in X$.
From the L.H.S. of inequality(1), we have

$$\begin{aligned} G(F(x, y), F(u, v), F(w, z)) &= G\Big(\frac{x - y}{16}, \frac{u - v}{16}, \frac{w - z}{16}\Big) \\ &= \frac{1}{16}(|(x - y) - (w - z)|) \\ &= \frac{1}{16}(|(x - w) + (z - y)|) \\ &\leq \frac{1}{16}(|x - w| + |y - z|) \end{aligned} \tag{10}$$

From the R.H.S. of inequality(1), we have

(11)

$$k\Big(\min\Big\{\frac{G(x,u,w)+G(y,v,z)}{2},\max\Big\{\frac{G(u,u,w)+G(v,v,z)}{2},G(x,w,w)+G(y,z,z)$$
$$\frac{G(F(x,y),F(u,v),F(x,y))+4\sqrt{G(x,u,w).G(y,v,z)}}{2}\Big\}\Big\}\Big)$$
$$=k\Big(\min\Big\{\frac{\max\{|x-u|,|u-w|,|x-w|\}+\max\{|y-v|,|v-z|,|y-z|\}}{2},$$
$$\max\Big\{\frac{\max\{|u-u|,|u-w|,|u-w|\}+\max\{|v-v|,|v-z|,|v-z|\}}{2},$$
$$\max\{|x-w|,|w-w|,|x-w|\}+\max\{|y-z|,|z-z|,|y-z|\}$$
$$\frac{G\Big(\frac{x-y}{16},\frac{u-v}{16},\frac{x-y}{16}\Big)+4\sqrt{\max\{|x-u|,|u-w|,|x-w|\}.\max\{|y-v|,|v-z|,|y-z|\}}}{2}\Big\}\Big\}\Big)$$
$$=k\Big(\min\Big\{\frac{|x-w|+|y-z|}{2},\max\Big\{\frac{|u-w|+|v-z|}{2},|x-w|+|y-z|$$
$$\frac{1}{32}G(x-y,u-v,x-y)+2\sqrt{|x-w|.|y-z|}\Big\}\Big\}\Big)$$
$$=k\Big(\min\Big\{\frac{|x-w|+|y-z|}{2},\max\Big\{\frac{|u-w|+|v-z|}{2},|x-w|+|y-z|$$
$$\frac{1}{32}(|(x-y)-(u-v)|)+2\sqrt{|x-w|.|y-z|}\Big\}\Big\}\Big)$$
$$=k\Big(\min\Big\{\frac{|x-w|+|y-z|}{2},|x-w|+|y-z|\Big\}\Big)$$
$$=k\frac{|x-w|+|y-z|}{2}$$

Hence, from (10) and (11), the contractive condition of Theorem(3.1) is satisfied for $k\in[\frac{1}{8},1)$.
Thus, all the conditions of Theorem(3.1) are fulfilled and F has a coupled fixed point in X. (which is $(0,0)$).

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