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A CONVERGENCE OF OPTIMAL INVESTMENT STRATEGIES FOR THE HARA UTILITY FUNCTIONS

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Abstract. An explicit expression of the optimal investment strategy corresponding to the HARA utility function under the constant elasticity of variance (CEV) model has been given by Jung and Kim [6]. In this paper we give an explicit expression of the optimal solution for the extended logarithmic utility function. And we prove an a.s. convergence of the HARA solutions to the extended logarithmic one.

1. Introduction

A manager of the surplus by pension funds wants to maximize the expected utility of the surplus $V(T)$ of his insurance company at the maturity time T before retirement of the policyholders. In the case of no investment during the period $[0, T]$, we assume that the surplus process $(V(t))_{t\in[0,T]}$ of the company is given by the following form:

$$\begin{cases} dV(t) = \mu_0 dt \\ V(0) = V_0, \end{cases} \tag{1.1}$$

where the constant $V_0 > 0$ is the initial surplus and the constant $\mu_0 > 0$ is the continuous rate of contribution. And we assume that all of the surplus is invested in a financial market which consists of two securities, named B and S, whose prices are given by the following differential equations:

$$dB(t) = rB(t)dt \tag{1.2}$$

and

$$dS(t) = \mu S(t)dt + kS^{1+\gamma}(t)dW(t), \tag{1.3}$$

where r, μ, k and γ are some constants with $0 < r < \mu$ and $\gamma \leq 0$, and $(W(t))_{t\in[0,T]}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Here r is a rate of return of the risk-free

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asset B, μ is an expected instantaneous rate of return of the risky asset S and γ is the elasticity parameter. In this case we call (B, S) a financial market with the constant elasticity variance (CEV) model.

We denote by $\beta(t)$ the proportion invested in the risky security S at time $t \in [0, T]$. We disallow leverage and short-sales. In this case it holds that $0 \leq \beta(t) \leq 1$ for all $t \in [0, T]$. Therefore, at any time $0 \leq t < T$, a nominal amount $V(t)(1 - \beta(t))$ is allocated to the risk-free asset B. We treat the proportion $\beta(t)$ of the surplus at time t as control parameter. Then the surplus process $(V(t))_{t\in[0,T]}$ is given by the following stochastic differential equations:

$$\begin{cases} dV(t) = [V(t)\{\beta(t)(\mu - r) + r\} + \mu_0]dt + V(t)k\beta(t)S^{\gamma}(t)dW(t) \\ V(0) = V_0. \end{cases} \tag{1.4}$$

Given a strategy $\beta(\cdot)$, the solution $(V^{\beta}(t))_{t\in[0,T]}$ of (1.4) is called the surplus process corresponding to $\beta(\cdot)$.

The hyperbolic absolute risk aversion (HARA) utility function with parameters η, p and q is given by

$$U_{hara}(\eta, p, q; v) = \frac{1-p}{qp}\left(\frac{qv}{1-p} + \eta\right)^p, \tag{1.5}$$

where $\eta \geq 0, p < 1, p \neq 0, q > 0$ and $\frac{-(1-p)\eta}{q} < v < \infty$. From this we can get the power utility function

$$U_{hara}(0, p, 1-p; v) = \frac{1}{p}v^p \equiv U_{power}(p; v) \tag{1.6}$$

and the exponential utility function

$$\lim_{p\to-\infty} U_{hara}(1, p, q; v) = -\frac{1}{q}e^{-qv} \equiv U_{exp}(q; v). \tag{1.7}$$

The modified HARA utility function with parameters η, p and q is given by

$$U_{mhara}(\eta, p, q; v) = \frac{1-p}{qp}\left[\left(\frac{qv}{1-p} + \eta\right)^p - 1\right]. \tag{1.8}$$

It holds

$$\lim_{p\to 0} U_{mhara}(\eta, p, q; v) = \frac{1}{q}\ln(qv + \eta). \tag{1.9}$$

The function defined by

$$U_{log}(\eta, q; v) = \frac{1}{q}\ln(qv + \eta) \tag{1.10}$$

is called the extended logarithmic utility function with parameters η and q. In the case that $\eta = 0$ and $q = 1$ in (1.10), we get the logarithmic utility function $U_{log}(0, 1; v) = \ln v$.

Xaio et al. [8] found an explicit expression for the optimal asset allocation which maximizes the expected logarithmic utility of the final annuity fund at retirement. To do this they used the Legendre transform and dual theory.

And Gao [3] solved the same problem as Xaio et al. [8] with the power and exponential utility functions. See Devolder et al. [2] for the case that $\gamma = 0$. Jung and Kim [6] found an explicit expression of the optimal investment strategy for the HARA utility function and proved that a class of the optimal solutions corresponding to the HARA utility functions converges a.s. to the solution corresponding to the exponential utility function as $p \to -\infty$.

In this paper we give an explicit expression of the optimal investment strategy for the extended logarithmic utility function and prove that a sequence of the optimal solutions corresponding to the HARA utility functions converges a.s. to the solution corresponding to the extended logarithmic utility function as $p \to 0$. Grasselli [4] investigated these problems in another financial market model with $\gamma = 0$.

The structure of the paper is as follows. In Section 2 we formulate our optimal problem and drive a formula for optimal investment strategy by using the theory on stochastic optimal control theory (see [1], [7]) and properties for a Legendre transform (see [3], [5], [8]). In Section 3 we give an explicit expression of the optimal investment strategy corresponding to the extended logarithmic utility function. In Section 4, we prove that a sequence of the optimal investment strategies corresponding to the HARA utility functions converges a.s. to the one corresponding to the exponential utility function.

2. Formulation of the problem and theory background

A control function $\beta(\cdot)$ in (1.4) is said to be admissible if $(\beta(t))_{t\geq 0}$ is $\mathcal{F}_t$-adapted process satisfying $0 \leq \beta(t) \leq 1$ for all $t \in [0, T]$. The set of all admissible controls is denoted by $\mathcal{A}$.

We use the HARA utility function $U(v) = U_{hara}(\eta, p, q; v)$ or the extended logarithmic utility function $U(v) = U_{log}(\eta, q; v)$ defined by (1.5) or (1.10), respectively. For the surplus process $(V^\beta(t))_{t\in[0,T]}$ given by (1.4), put

$$J^\beta(t, s, v) = E[U(V^\beta(T)) \mid S(t) = s, V^\beta(t) = v] \tag{2.1}$$

for all $(t, s, v) \in [0, T] \times R^1 \times R^1$, where $E[X|A]$ is the conditional expectation of a random variable X given an event A. In stochastic optimal control theory it is important to find the optimal value function

$$H(t, s, v) = \sup_{\beta\in\mathcal{A}} J^\beta(t, s, v) \tag{2.2}$$

and the optimal strategy $\beta^*(\cdot)$ such that

$$J^{\beta^*}(t, s, v) = H(t, s, v). \tag{2.3}$$

From now we will drive a formula for $\beta^*(t)$. The Hamilton-Jacobi-Bellman (HJB) equation associated with our optimization problem (2.2) and (2.3) is

$$\begin{aligned} 0 =& H_t + \mu s H_s + (rv+\mu_0)H_v + \frac{1}{2}ks^{2\gamma+2}H_{ss} \\ &+ \sup_\beta \left\{ \beta(\mu - r)vH_v + \beta k^2 s^{2\gamma+1} v H_{sv} + \frac{1}{2}\beta^2 k^2 s^{2\gamma} v^2 H_{vv} \right\} \end{aligned} \tag{2.4}$$

with the boundary condition $H(T,s,v) = U(v)$, where $H_t, H_v, H_s, H_{vv}, H_{ss}, H_{sv}$ denote partial derivative of first and second orders with respect to time, stock price and wealth parameters. It is easy to show that the optimal strategy β^* is given by

$$\beta^* = -\frac{(\mu - r)H_v + k^2 s^{2\gamma+1}H_{sv}}{vk^2 s^{2\gamma} H_{vv}}. \tag{2.5}$$

Inserting (2.5) into (2.4), we obtain the following second order partial differential equation for the optimal value function H:

$$\begin{aligned} 0 =& H_t + \mu s H_s + (rv+\mu_0)H_v \\ &+ \frac{1}{2}k^2 s^{2\gamma+2}H_{ss} - \frac{[(\mu-r)H_v + k^2 s^{2\gamma+1}H_{sv}]^2}{2k^2 s^{2\gamma}H_{vv}}. \end{aligned} \tag{2.6}$$

To get an explicit expression for the optimal strategy β^* given by (2.5), we have to solve this nonlinear equation, but it is very difficult. So, by applying a Legendre transform, we transform this equation into a linear partial differential equation of which the solution gives an explicit expression for β^*.

Let $f : R^n \to R^1$ be a convex function. A Legendre transform on R is defined by

$$L(z) = \max_x \{f(x) - zx\}. \tag{2.7}$$

The function $L(z)$ is called the Legendre dual of the function $f(x)$. If $f(x)$ is strictly convex, the maximum in the above equation will be attened at just one point, which we denote by x_0. It is attained at the unique solution to the first order condition

$$\frac{df(x)}{dx} - z = 0. \tag{2.8}$$

So we may write

$$L(z) = f(x_0) - zx_0. \tag{2.9}$$

Following Jonsson and Sircar [5], a Legendre transform can be defined by

$$\widehat{H}(t,s,z) = \sup_{v>0}\{H(t,s,v) - zv \mid 0 < v < \infty\}, \tag{2.10}$$

where $0 < t < T$ and $z > 0$ denotes the dual variable to v. The value of v where this optimum is attained is denoted by $g(t,s,z)$, so that

$$g(t,s,z) = \inf_{v>0}\{v \mid H(t,s,v) \geq zv + \widehat{H}(t,s,z)\}. \tag{2.11}$$

The function $\widehat{H}$ is related to g by

$$g = -\widehat{H}_z, \tag{2.12}$$

so we can take either one of the two function g and $\widehat{H}$ as the dual of H. By the descriptions of (2.9) and (2.10), we have

$$H_v = z \tag{2.13}$$

and hence

$$\widehat{H}(t,s,z) = H(t,s,g) - zg,\ g(t,s,z) = v. \tag{2.14}$$

By differentiating (2.13) and (2.14) with respect to t, s and z, we obtain

$$\begin{aligned} &H_t = \widehat{H}_t,\ H_s = \widehat{H}_s,\ H_v = z,\ \widehat{H}_z = -g \\ &H_{ss} = \widehat{H}_{ss} - \frac{\widehat{H}_{sz}^2}{\widehat{H}_{zz}},\ H_{vv} = -\frac{1}{\widehat{H}_{zz}}, H_{sv} = -\frac{\widehat{H}_{sz}}{\widehat{H}_{zz}}. \end{aligned} \tag{2.15}$$

At the terminal time, we denote

$$\begin{aligned} \widehat{U}(z) &= \sup_{v>0}\{U(v) - zv\}, \\ G(z) &= \inf_{v>0}\{v | U(v) \geq zv + \widehat{U}(z)\}. \end{aligned}$$

As a result, we have

$$G(z) = (U')^{-1}(z). \tag{2.16}$$

Since $H(T,s,v) = U(v)$, we can define

$$g(T,s,z) = \inf_{v>0}\{v | U(v) \geq zv + \widehat{H}(T,s,z)\}$$

and

$$\widehat{H}(T,s,z) = \sup_{v>0}\{U(v) - zv\},$$

so that

$$g(T,s,z) = (U')^{-1}(z). \tag{2.17}$$

Substituting (2.13), (2.14) and (2.15) into (2.6) and differentiating $\widehat{H}$ with respect to z, we get

$$\begin{aligned} &g_t - rg - \mu_0 + rsg_s + \left(\frac{(\mu - r)^2 z}{k^2 s^{2\gamma}} - rz\right) g_z \\ &+ \frac{1}{2}ks^{2\gamma+2}g_{ss} - (\mu - r)szg_{sz} + \left(\frac{(\mu - r)^2 z^2}{2k^2 s^{2\gamma}}\right) g_{zz} = 0 \end{aligned} \tag{2.18}$$

and, from (1.5) and (2.17), we can see that the boundary condition is

$$g(T,s,z) = \frac{1-p}{q}\left(z^{\frac{1}{p-1}} - \eta\right) \tag{2.19}$$

in the case that $U(v) = U_{hara}(\eta, p, q; v)$ or

$$g(T,s,z) = \frac{1}{zq} - \frac{\eta}{q} \tag{2.20}$$

in the case that $U(v) = U_{log}(\eta, q; v)$. This is a linear boundary problem that we have wanted. Moreover, inserting (2.13), (2.14) and (2.15) into (2.5), we have

$$\beta^* = \frac{-(\mu - r)zg_z + k^2 s^{1+2\gamma} g_s}{k^2 s^{2\gamma} g}. \tag{2.21}$$

3. Optimal solution for the extended logarithmic utility

In this section we find the solution $g_{log}(\eta, q; t, s, z)$ of the linear partial differential equation (2.18) with boundary condition (2.20) in the case that $U(v) = U_{log}(\eta, q; v)$ and give an explicit expression for the corresponding optimal strategy $\beta^*_{log}(\eta, q; t)$ from (2.21). That is, we prove the following result.

Theorem 3.1. *Let $U(v) = U_{log}(\eta, q; v)$ in (2.1). Then the solution $g_{log}(\eta, q; t, s, z)$ of the linear partial differential equation (2.18) with boundary condition (2.20) is given by*

$$g_{log}(\eta, q; t, s, z) = \frac{1}{qz} - \frac{\eta}{q} e^{-r(T-t)} - \mu_0 \bar{a}_{\overline{T-t|}}, \tag{3.1}$$

where

$$\bar{a}_{\overline{T-t|}} = \frac{1}{r}\left(1 - e^{-r(T-t)}\right).$$

(Remark that the right hand side of (3.1) does not depend on s.) And the corresponding optimal strategy $\beta^*_{log}(t)$ is given by

$$\begin{aligned} \beta^*_{log}(\eta, q; t) &= \frac{\mu - r}{qzk^2 s^{2\gamma} g_{log}(\eta, q; t, s, z)} \\ &= \frac{\mu - r}{k^2 s^{2\gamma}\left(1 - \eta z e^{-r(T-t)} - qz\mu_0 \bar{a}_{\overline{T-t|}}\right)}. \end{aligned} \tag{3.2}$$

Proof. We try to find a solution of the boundary problem (2.18) and (2.20) in the following form:

$$g_{log}(\eta, q; t, s, z) = \frac{1}{qz} m(t, s) - \frac{\eta}{q} n(t, s) + a(t), \tag{3.3}$$

with the boundary conditions given by $a(T) = 0$, $m(T, s) = 1$ and $n(T, s) = 0$. Let $g(t, s, z) = g_{log}(\eta, q; t, s, z)$ for notational simplification. Then

$$\begin{aligned} &g_t = \frac{1}{qz} m_t - \frac{\eta}{q} n_t + a'(t), \;\; g_s = \frac{1}{qz} m_s - \frac{\eta}{q} n_s, \\ &g_z = -\frac{1}{gz^2} m, \;\; g_{ss} = \frac{1}{qz} m_{ss} - \frac{\eta}{q} n_{ss}, \\ &g_{sz} = -\frac{1}{gz^2} m_s, \;\; g_{zz} = -\frac{2}{gz^3} m. \end{aligned}$$

Substituting these derivatives into (2.18), we have

$$\begin{aligned}\frac{1}{qz}&\left[m_t+\mu s m_s+\frac{1}{2}ks^{2\gamma+2}m_{ss}\right]\\&\qquad+\frac{\eta}{q}\left[-n_t+rn-rsn_s-\frac{1}{2}ks^{2\gamma+2}n_{ss}\right]\\&\qquad\qquad\qquad+a'-ra-\mu_0=0.\end{aligned}$$

We can split this equation into the following three equations:

$$\begin{cases}m_t+\mu s m_s+\dfrac{1}{2}ks^{2\gamma+2}m_{ss}=0,\\ m(T,s)=1,\end{cases}\tag{3.4}$$

$$\begin{cases}n_t-rn+rsn_s+\dfrac{1}{2}ks^{2\gamma+2}n_{ss}=0,\\ n(T,s)=1\end{cases}\tag{3.5}$$

and

$$\begin{cases}a'-ra-c=0,\\ a(T)=0\end{cases}\tag{3.6}$$

First we solve (3.4). Consider a special case that $m(t,s)=m(t)$ is a constant in s. Then (3.4) becomes

$$m'(t)=0,\ m(T)=1$$

which has a unique solution $m(t)=1,\ 0\le t\le T$. Now let us consider in general case. Because (3.4) has no constant term, we can try to find a solution in the following form:

$$m(t,s)=A(t)e^{B(t)y},\ y=s^{-2\gamma}$$

with the boundary condition given by $A(T)=1$ and $B(T)=0$. Then

$$\begin{aligned}m_t&=A'e^{By}+AB'ye^{By},\\ m_s&=-2\gamma ys^{-1}ABe^{By},\\ m_{ss}&=4\gamma^2y^2s^{-2}AB^2e^{By}+(4\gamma^2+2\gamma)ys^{-2}ABe^{By}.\end{aligned}$$

Substituting these derivatives into (3.4) and multiplying e^{-By}, we have

$$\begin{aligned}y&\left[AB'-2\mu\gamma AB+2k^2\gamma^2AB^2\right]\\&\qquad+A'+k^2(2\gamma^2+\gamma)AB=0.\end{aligned}$$

Again we can split this equation into two ordinary differential equations as follows:

$$B'-2\mu\gamma B+2k^2\gamma^2B^2=0,\ B(T)=0,\tag{3.7}$$

$$A'+k^2(2\gamma^2+\gamma)AB=0,\ A(T)=1.\tag{3.8}$$

The equation (3.7) has no solution, and so does (3.8). Thus (3.4) has a unique solution $m(t,s) = 1$ only when $m(t,s) = m(t)$ is a constant in s. Therefore we have $m(t,s) = 1$ in (3.3).

Next, by the same argument as above, (3.5) has a solution only when $n(t,s) = n(t)$ is a constant in s. In this case, (3.5) becomes

$$n'(t) - rn(t) = 0, \ n(T) = 1$$

and has a unique solution $n(t) = e^{-r(T-t)}$. Therefore we have $n(t,s) = e^{-r(T-t)}$ in (3.3). And we can easily see that the solution of the equation (3.12) is given by

$$a(t) = -\frac{\mu_0}{r}\left(1 - e^{-r(T-t)}\right) = -\mu_0 \bar{a}_{\overline{T-t|}}.$$

This ends the proof of the equality (3.1). Finally, applying (3.1) to (2.21), we can obtain (3.2). The proof of Theorem 3.1 is complete.

□

Remark 1. Consider $U(v) = U_{log}(0,1;v) = \ln v$. Then, by Theorem 3.1, the solution $g_{log}(0,1;t,s,z)$ of (2.18) with boundary condition $g(T,s,z) = (U')^{-1}(z) = \frac{1}{z}$ is given by

$$g_{log}(0,1;t,s,z) = \frac{1}{z} - \mu_0 \bar{a}_{\overline{T-t|}}$$

and the corresponding optimal strategy $\beta^*_{log}(0,1;t)$ is given by

$$\beta^*_{log}(0,1;t) = \frac{\mu - r}{k^2 s^{2\gamma}\left(1 - z\mu_0 \bar{a}_{\overline{T-t|}}\right)}.$$

We can see that these results coincide with ones by Xaio et al. [8].

4. A convergence of optimal strategies

Let $g_{hara}(\eta,p,q;t,s,z)$ be the solution of the linear partial differential equation (2.18) with boundary condition (2.19) in the case that $U(v) = U_{hara}(\eta,p,q;v)$ and $\beta^*_{hara}(\eta,p,q;t)$ the corresponding optimal strategy given by (2.21). Since $U'_{hara}(v) = U'_{mhara}(v)$ for all $v \in \left(\frac{-(1-p)\eta}{q}, \infty\right)$, two optimal strategies corresponding to $U_{hara}(v)$ and $U_{mhara}(v)$ are same, i.e., $\beta^*_{hara}(t) = \beta^*_{mhara}(t)$. By the definition of the extended logarithmic utility function, we know $U_{mhara}(\eta,p,q;v)$ converges to $U_{log}(\eta,q;v)$ as $p \to 0$. In this section, we prove that $\beta^*_{hara}(\eta,p,q;t)$ goes to $\beta^*_{log}(\eta,q;t)$ as $p \to 0$. To do this, we need the following theorem by Jung and Kim [6].

Theorem 4.1. *The solution $g_{hara}(\eta, p, q; t, s, z)$ of the partial differential equation (2.18) with the terminal condition (2.19) is given by*

$$\begin{aligned} g_{hara}(\eta, p, q; t, s, z) =& \frac{1-p}{q} b(p; t, s) \left(z^{\frac{1}{p-1}} - \eta \right) \\ &+ \eta c(p; t, s) + a(t), \end{aligned} \tag{4.1}$$

where

$$a(t) = -\frac{\mu_0}{r} \left(1 - e^{-r(T-t)} \right) = -\mu_0 \overline{a}_{\overline{T-t|}}, \tag{4.2}$$

$$b(p; t, s) = A(p; t) e^{B(p;t)s^{-2\gamma}}, \tag{4.3}$$

$$c(p; t, s) = C(p; t)s^{-2\gamma} + D(p; t). \tag{4.4}$$

Here

$$A(p; t) = \left(\frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda_- e^{2\gamma^2(\lambda_- - \lambda_+)(T-t)}} \right)^{\frac{2\gamma+1}{2\gamma}} e^{\left(\gamma(2\gamma+1)\lambda_- + \frac{rp}{1-p}\right)(T-t)}, \tag{4.5}$$

$$B(p; t) = k^{-2} \frac{\lambda_- - \lambda_- e^{2\gamma^2(\lambda_- - \lambda_+)(T-t)}}{1 - \frac{\lambda_-}{\lambda_+} e^{2\gamma^2(\lambda_- - \lambda_+)(T-t)}} = k^{-2} I(t) \tag{4.6}$$

with

$$\lambda_\pm = \lambda_\pm(p) = \frac{(\mu - rp) \pm \sqrt{(1-p)(\mu^2 - r^2 p)}}{2\gamma(1-r)}, \tag{4.7}$$

and

$$\begin{aligned} C(p; t) = & \frac{-e^{(2\gamma+1)rt}}{q} \int_t^T e^{-(2\gamma+1)ru} (1-p) A(p; u) e^{B(p;u)s^{-2\gamma}} \\ & \times \left[B'(p; u) - 2\gamma r B(p; u) + 2\gamma^2 k^2 B^2(p; u) \right] du, \end{aligned} \tag{4.8}$$

$$\begin{aligned} D(p; t) = -e^{rt} & \int_t^T \frac{1}{q} e^{-ru} \Big\{ (1-p) e^{B(p;u)s^{2\gamma}} \\ & \times \left[k^2(2\gamma^2 + \gamma) A(p; u) B(p; u) - r A(p; u) + A'(p; u) \right] \\ & - k^2 (2\gamma^2 + \gamma) q C(p; u) \Big\} du. \end{aligned} \tag{4.9}$$

The following theorem is our main result.

Theorem 4.2. *We have*

$$(i) \ \lim_{p \to 0} g_{hara}(\eta, p, q; ts, z) = g_{log}(\eta, q; t, s, z) \ a.s.,$$

$$(ii) \ \lim_{p \to 0} \beta^*_{hara}(\eta, p, q; t) = \beta^*_{log}(\eta, q; t) \ a.s..$$

Proof. We consider the functions and the stochastic processes in (4.2) $\sim$ (4.9). By the definition of $\lambda_\pm(p)$, we get $\lim_{p \to 0} \lambda_+(p) = \frac{\mu}{\gamma(1-r)}$ and $\lim_{p \to 0} \lambda_-(p) = 0$. Thus $\lim_{p \to 0} A(p; t) = 1$ and $\lim_{p \to 0} B(p; t) = 0$, and hence $\lim_{p \to 0} b(p; t, s) = 1$ a.s.. We can check that, not only $B(p; t)$, but $B'(p; t)$ also goes to 0, and so does the integrand in (4.8). This shows that $\lim_{p\to 0} C(p;t) = 0$ a.s.. Now, since $A'(p;t) \to 0$ as $p \to 0$, we have

$$\begin{aligned}\lim_{p\to 0} D(p;t) &= -e^{rt}\int_t^T \frac{1}{q}e^{-ru}\{-r\}du \\ &= -\frac{1}{q}e^{rt}\left(e^{-rT} - e^{-rt}\right) \\ &= -\frac{1}{q}\left(e^{-r(T-t)} - 1\right) \ a.s..\end{aligned}$$

Thus it holds

$$\begin{aligned}\lim_{p\to 0} c(p;t,s) &= \lim_{p\to 0} C(p;t)s^{-2\gamma} + \lim_{p\to 0} D(p;t) \\ &= -\frac{1}{q}\left(e^{-r(T-t)} - 1\right) \ a.s..\end{aligned}$$

Therefore we get

$$\begin{aligned}&\lim_{p\to 0} g_{hara}(\eta,p,q;t,s,z) \\ &\quad = \frac{1}{q}\left(z^{-1} - \eta\right) - \frac{\eta}{q}\left(e^{-r(T-t)} - 1\right) - \mu_0 \overline{a}_{\overline{T-t|}} \\ &\quad = \frac{1}{qz} - \frac{\eta}{q}e^{-r(T-t)} - \mu_0 \overline{a}_{\overline{T-t|}} \\ &\quad = g_{log}(\eta,q;t,s,z) \ a.s..\end{aligned}$$

Now we prove (ii). Since

$$\frac{\partial}{\partial z} g_{hara}(\eta,p,q;t,s,z) = -\frac{1}{q}b(p;t,s)z^{\frac{1}{p-1}-1}$$

and $\lim_{p\to 0} b(p;t,s) = 1$ a.s., it holds

$$\lim_{p\to 0}\frac{\partial}{\partial z} g_{hara}(\eta,p,q;t,s,z) = -\frac{1}{q}z^{-2} \ a.s..$$

And since $\lim_{p\to 0} A(p;t) = 1$, $\lim_{p\to 0} B(p;t) = 0$ and $\lim_{p\to 0} C(p;t) = 0$ a.s.,

$$\begin{aligned}&\frac{\partial}{\partial s} g_{hara}(\eta,p,q;t,s,z) \\ &\quad = \frac{1-p}{q}b_s(p;t,s)\left(z^{\frac{1}{p-1}} - \eta\right) + \eta c_s(p;t,s) \\ &\quad = \frac{1-p}{q}(-2\gamma)A(p;t)B(p;t)s^{-2\gamma-1}e^{B(p;t)s^{-2\gamma}}\left(z^{\frac{1}{p-1}} - \eta\right) \\ &\quad\quad - 2\eta\gamma C(p;t)s^{-2\gamma} \\ &\quad \to 0 \ a.s.\end{aligned}$$

holds as $p \to 0$. From (2.21) and (3.2) we have

$$\begin{aligned}\lim_{p\to 0} \beta^*_{hara}(\eta, p, q; t) =& \frac{\mu - r}{qzk^2 s^{2\gamma} g_{log}(\eta, q; t, s, z)} \\ =& \beta^*_{log}(\eta, q; t) \ a.s..\end{aligned}$$

This completes the proof. □

5. Conclusions

The modified HARA utility functions with parameters with η, p and q converge to the extended logarithmic utility function with η and q as p goes to 0. Jung and Kim [6] found an explicit expression of the optimal investment strategy corresponding to the HARA utility function with parameters η, p and q. In this paper, we have found an explicit expression of the optimal investment strategy corresponding to the extended logarithmic utility function with η and q. Xiao et al. [8] proved this problem in the case that $\eta = 0$ and $q = 1$. And we have proved an a.s. convergence of the HARA solutions to the extended logarithmic one as p goes to 0.

References

[1] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford Univ. Press (1998).

[2] P. Devolder, M.B. Princep and I.D. Fabian, *Stochastic optimal control of annuity contracts*, Insurance Math. Econom. **33** (2003), 227-238.

[3] J. Gao, *Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model*, Insuance Math. Econom. **45** (2009), 9-18.

[4] M. Gu, Y. Yang, S. Li and J. Zhang, *Constant elasticity of variance model for proportional reinsurance and investment strategies*, Insuance Math. Econom. **46** (2010), 580-587.

[5] M. Jonsson and R. Sircar, *Optimal investment problems and volatility homogenization approximations*, NATO Science Series II, Vol. 75, Kluwer (2002), 255-281.

[6] E.J. Jung and J.H. Kim, *Optimal investment strategies for the HARA utility under the constant elasticity of variance model*, Insuance Math. Econom. **51** (2012), 667-673.

[7] B. Øksendal, *Stochastic Differential Equations*, Springer-Verlag Berlin Heidelberg (1998).

[8] J. Xiao, Z. Hong and C¿ Qin, *The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts*, Insuance Math. Econom. **40** (2007), 302-310.

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