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SOME RECURRENCE RELATIONS FOR THE JACOBI POLYNOMIALS $P_n^{(\alpha,\beta)}(x)$

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Abstract. We use some known contiguous function relations for ${}_2F_1$ to show how simply the following three recurrence relations for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$:
(i) $(\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x) = (\beta+n)\,P_n^{(\alpha,\beta-1)}(x) + (\alpha+n)\,P_n^{(\alpha-1,\beta)}(x)$;
(ii) $2\,P_n^{(\alpha,\beta)}(x) = (1+x)\,P_n^{(\alpha,\beta+1)}(x) + (1-x)\,P_n^{(\alpha+1,\beta)}(x)$;
(iii) $P_{n-1}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta-1)}(x) + P_n^{(\alpha-1,\beta)}(x)$
can be established.

1. Introduction and Preliminaries

Contiguous function relations for the Gauss's hypergeometric series ${}_2F_1$ play an important role in its theoretical and applicable senses. Let $F := {}_2F_1\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\; z\right]$. Then the contiguous relations for F are defined as follows:

$$F(a+) = {}_2F_1\left[\begin{matrix} a+1,\, b; \\ c; \end{matrix}\; z\right] \quad \text{and} \quad F(a-) = {}_2F_1\left[\begin{matrix} a-1,\, b; \\ c; \end{matrix}\; z\right].$$

Likewise $F(b+)$, $F(b-)$, $F(c+)$, and $F(c-)$ are defined.

Gauss presented the following fifteen contiguous function relations between F and any two of its contiguous functions which are linear and whose coefficients are at most linear in z (see, *e.g.*, [1, p. 71]):

$$(a-b)F = a\,F(a+) - bF(b+); \tag{1.1}$$

$$(a-c+1)F = a\,F(a+) - (c-1)F(c-); \tag{1.2}$$

$$[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+); \tag{1.3}$$

$$(1-z)F = F(a-) - c^{-1}(c-b)zF(c+); \tag{1.4}$$

$$(1-z)F = F(b-) - c^{-1}(c-a)zF(c+); \tag{1.5}$$

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$$[2a-c+(b-a)z]F = a(1-z)F(a+) - (c-a)F(a-); \tag{1.6}$$

$$(a+b-c)F = a(1-z)F(a+) - (c-b)F(b-); \tag{1.7}$$

$$(c-a-b)F = (c-a)F(a-) - b(1-z)F(b+); \tag{1.8}$$

$$(b-a)(1-z)F = (c-a)F(a-) - (c-b)F(b-); \tag{1.9}$$

$$[1-a+(c-b-1)z]F = (c-a)F(a-) - (c-1)(1-z)F(c-); \tag{1.10}$$

$$[2b-c+(a-b)z]F = b(1-z)F(b+) - (c-b)F(b-); \tag{1.11}$$

$$[b+(a-c)z]F = b(1-z)F(b+) - c^{-1}(c-a)(c-b)zF(c+); \tag{1.12}$$

$$(b-c+1)F = bF(b+) - (c-1)F(c-); \tag{1.13}$$

$$[1-b+(c-a-1)z]F = (c-b)F(b-) - (c-1)(1-z)F(c-); \tag{1.14}$$

$$[(c-1)+(a+b+1-2c)z]F = (c-1)(1-z)F(c-) - c^{-1}(c-a)(c-b)zF(c+). \tag{1.15}$$

When the contiguous function relations for ${}_2F_1$ regarding two parameters are defined, for example, as follows:

$$F(a-,b+) = {}_2F_1\left[\begin{matrix} a-1,\, b+1; \\ c; \end{matrix}\; z\right] \quad \text{and} \quad F(b+,c+) = {}_2F_1\left[\begin{matrix} a,\, b+1; \\ c+1; \end{matrix}\; z\right],$$

the following two other contiguous function relations are satisfied (see, *e.g.*, [1, p. 72]):

$$F = F(a-,b+) + c^{-1}(b+1-a))zF(b+,c+) \tag{1.16}$$

and

$$(c-1-b)F = (c-a)F(a-,b+) + (a-1-b)(1-z)F(b+). \tag{1.17}$$

Using the above mentioned contiguous function relations for ${}_2F_1$, several recurrence relations for the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are given (see, *e.g.*, [1, pp. 263-265]). Among them, we are interested in the following three recurrence relations:

$$(\alpha+\beta+n)\, P_n^{(\alpha,\beta)}(x) = (\beta+n)\, P_n^{(\alpha,\beta-1)}(x) + (\alpha+n)\, P_n^{(\alpha-1,\beta)}(x); \tag{1.18}$$

$$2\, P_n^{(\alpha,\beta)}(x) = (1+x)\, P_n^{(\alpha,\beta+1)}(x) + (1-x)\, P_n^{(\alpha+1,\beta)}(x); \tag{1.19}$$

$$P_{n-1}^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta-1)}(x) + P_n^{(\alpha-1,\beta)}(x). \tag{1.20}$$

In this note, we are aiming at giving simple proofs for the recurrence relations (1.18), (1.19) and (1.20).

2. Outline of the proofs given in [1]

In this section we outline of the proofs of (1.18), (1.19) and (1.20) given in [1].

We begin by a known expression for $P_n^{(\alpha,\beta)}(x)$:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n,\, 1+\alpha+\beta+n; \\ 1+\alpha; \end{matrix}\; \frac{1-x}{2}\right]. \tag{2.1}$$

Let $D \equiv \frac{d}{dx}$ and $D^k \equiv \frac{d^k}{dx^k}$ $(k \in \mathbb{N} := \{1, 2, 3, \ldots\})$. Then it is easy to see that

$$D\, P_n^{(\alpha,\beta)}(x) = \frac{1}{2}\,(1+\alpha+\beta+n)\, P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{2.2}$$

Iteration of (2.2) yields

$$D^k\, P_n^{(\alpha,\beta)}(x) = 2^{-k}\,(1+\alpha+\beta+n)_k\, P_{n-k}^{(\alpha+k,\beta+k)}(x) \quad (0 < k \leq n;\; n,\, k \in \mathbb{N}). \tag{2.3}$$

On the other hand, we use the Pfaff-Kummer transformation for ${}_2F_1$ (see [1, p. 60, Theorem 20]; see also [2, p. 67, Eq.(19)]):

$${}_2F_1\left[\begin{matrix} a,\, b; \\ c; \end{matrix}\; x\right] = (1-x)^{-a}\, {}_2F_1\left[\begin{matrix} a,\, c-b; \\ c; \end{matrix}\; -\frac{x}{1-x}\right], \tag{2.4}$$

valid for $|x| < 1$ and $|x/(1-x)| < 1$, in (2.1) to obtain

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,\, \beta-n; \\ 1+\alpha; \end{matrix}\; \frac{x-1}{x+1}\right]. \tag{2.5}$$

Differentiating both sides of (2.5) with respect to x gives

$$\begin{aligned} D\, P_n^{(\alpha,\beta)}(x) =& n(1+x)^{-1}\, P_n^{(\alpha,\beta)}(x) + \frac{(2+\alpha)_{n-1}\,(\beta+n)}{(n-1)!\,(x+1)}\left(\frac{x+1}{2}\right)^{n-1} \\ & \times {}_2F_1\left[\begin{matrix} -n+1,\, -\beta-n+1; \\ 1+(1+\alpha); \end{matrix}\; \frac{x-1}{x+1}\right]. \end{aligned} \tag{2.6}$$

We find from (2.6) that

$$(x+1)\, D\, P_n^{(\alpha,\beta)}(x) = n\, P_n^{(\alpha,\beta)}(x) + (\beta+n)\, P_{n-1}^{(\alpha+1,\beta)}(x). \tag{2.7}$$

Similarly as in getting (2.7), starting with another known expression for $P_n^{(\alpha,\beta)}(x)$:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\beta)_n}{n!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix} -n,\, -\alpha-n; \\ 1+\beta; \end{matrix}\; \frac{x+1}{x-1}\right], \tag{2.8}$$

we can obtain

$$(x-1)\, D\, P_n^{(\alpha,\beta)}(x) = n\, P_n^{(\alpha,\beta)}(x) - (\alpha+n)\, P_{n-1}^{(\alpha,\beta+1)}(x). \tag{2.9}$$

Incorporating (2.7) with (2.9) yields

$$2\, D\, P_n^{(\alpha,\beta)}(x) = (\beta+n)\, P_{n-1}^{(\alpha+1,\beta)}(x) + (\alpha+n)\, P_{n-1}^{(\alpha,\beta+1)}(x). \tag{2.10}$$

Finally, using (2.2) to remove the $D\,P_n^{(\alpha,\beta)}(x)$ from (2.10) and replacing n, α, and β by $n+1$, $\alpha-1$, and $\beta-1$, respectively, in the resulting identity gives the first desired recurrence relation (1.18).

Next, setting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, and $z=\frac{1}{2}(1-x)$ in (1.16) and (1.17), and using (2.1), after some simplification, we, respectively, get the following identities involving $P_n^{(\alpha,\beta)}(x)$:

$$\frac{1}{2}(2+\alpha+\beta+2n)\,(x-1)\,P_n^{(\alpha+1,\beta)}(x)=(n+1)\,P_{n+1}^{(\alpha,\beta)}(x)-(1+\alpha+n)\,P_n^{(\alpha,\beta)}(x) \tag{2.11}$$

and

$$\frac{1}{2}(2+\alpha+\beta+2n)\,(x+1)\,P_n^{(\alpha,\beta+1)}(x)=(n+1)\,P_{n+1}^{(\alpha,\beta)}(x)+(1+\beta+n)\,P_n^{(\alpha,\beta)}(x). \tag{2.12}$$

Further, using (1.1) in the same way as in getting (2.11) and (2.12), and replacing β by $\beta-1$ in the resulting identity gives

$$(\alpha+\beta+2n)\,P_n^{(\alpha,\beta-1)}(x)=(\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x)+(\alpha+n)\,P_{n-1}^{(\alpha,\beta)}(x). \tag{2.13}$$

Joining (1.18) and (2.13) to eliminate the term $P_n^{(\alpha,\beta-1)}(x)$ yields

$$(\alpha+\beta+2n)\,P_n^{(\alpha-1,\beta)}(x)=(\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(x)-(\beta+n)\,P_{n-1}^{(\alpha,\beta)}(x). \tag{2.14}$$

Finally, eliminating the terms $P_{n+1}^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ from two combinations of [(2.11) and (2.12)] and [(2.13) and (2.14)], respectively, gives the desired identities (1.19) and (1.20).

It is remarked in passing that the contiguous relation (1.17) can be obtained by simply replacing b by $b+1$ in (1.9).

3. New and simple derivations of the results (1.18) to (1.20)

In this section, in comparison with rather lengthy proofs in Section 2, we show how the chosen recurrence relations (1.18) to (1.20) can be established in a very simple way.

Setting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, and $z=\frac{1}{2}(1-x)$ in the contiguous relation (1.13) and using (2.1) gives

$$(1+\beta+n)\,P_n^{(\alpha,\beta)}(x)=(1+\alpha+\beta+n)\,P_n^{(\alpha,\beta+1)}(x)-(\alpha+n)\,P_n^{(\alpha,\beta+1)}(x),$$

which, upon replacing β by $\beta-1$, immediately yields (1.18).

Setting $a=-n$, $b=1+\alpha+\beta+n$, $c=1+\alpha$, and $z=\frac{1}{2}(1-x)$ in the contiguous relation (1.5) and using (2.1) is immediately seen to yield (1.19).

Replacing c by $c+1$ in (1.2) gives the following contiguous relation:

$$c\,F=(c-a)\,F(c+)+a\,F(a+,\,c+). \tag{3.1}$$

Setting $a = -n$, $b = 1 + \alpha + \beta + n$, $c = 1 + \alpha$, and $z = \frac{1}{2}(1 - x)$ in (3.1) and using (2.1) is easily seen to prove the last desired result (1.20).

Concluding Remark. In this note, by using contiguous function relations for ${}_2F_1$, we have shown how simply the three known recurrence relations for the Jacobi polynomial can be proved. Recently a good deal of progress has been done in obtaining contiguous function relations with two parameters. Using these contiguous function relations, we will be able to establish a large number of recurrence relations for the Jacobi polynomials whose presentations are to be carried out soon.

References

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