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# CONIC REGULAR FUNCTIONS OF CONIC QUATERNION VARIABLES IN THE SENSE OF CLIFFORD ANALYSIS

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Abstract. The aim of this paper is to research certain properties of conic regular functions of conic quaternion variables in $\mathbb{C}^2$. We generalize the properties of conic regular functions and the Cauchy theorem of conic regular functions in conic quaternion analysis.

## 1. Introduction

We introduce the four dimensional commutative conic quaternions, not quaternions, and its associated function theory and analysis. Conic quaternions have the following advantages: It is a classical four dimensional function theory and has something that is impossible with quaternions and other non-commutative or non-associative systems. Musès [11, 12] discussed specific examples and theorems, specially, the relation of hypernumbers to time, developed in terms of hypernumber computation. Davenport [1] worked with numbers that have four distinct components and constructed a formal algebra formed upon a basis commutative ring and a consistent definition of multiplication and some operators. Kajiwara *etal*. [2, 3] obtained mathematical results of quaternion algebra, properties of several operators in quaternions and regenerations for the inhomogeneous Cauchy Riemann system of quaternion and Clifford analysis. Koriyama *etal*. [8] gave some definitions and properties of regularities of quaternionic functions with regular mappings in a domain in $\mathbb{C}^2$. Nôno [13, 14] and Sudbery [15] gave some properties of quaternionic hyperregular functions and developed theories of quaternionic analysis, by using the exterior differential calculus and the relationship between quaternionic analysis and complex analysis.

We [9, 10] investigated the existence of hyper-conjugate harmonic functions of an octonion number system and some properties of dual quaternion functions. And, we [4, 5, 6] researched the corresponding Cauchy-Riemann systems

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and properties of regularities of functions with values in special quaternions on Clifford analysis. Also, we [7] gave a regular function with values in dual split quaternions and relations between the corresponding Cauchy-Riemann system and a regularity of functions with values in dual split quaternions.

In this paper, we research the properties of conic regular functions of conic quaternion variables in $\mathbb{C}^2$. Also, we generalize certain properties of conic regular functions in conic quaternion analysis for the forms and structures of conic Cauchy-Riemann systems. Also, we investigate the Cauchy theorem of conic regular functions in conic quaternion analysis.

## 2. Preliminaries

The field of quaternions,

$$\mathcal{CQ} = \{Z = x_0 + x_1e_1 + x_2e_2 + x_3e_3 |\ x_l(l = 0, 1, 2, 3) \in \mathbb{R}\}, \tag{1}$$

is a four dimensional commutative $\mathbb{R}$-field generated by four base elements

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

with the following commutative multiplication rules:

$$e_0^2 = e_2^2 = 1,\ e_1^2 = e_3^2 = -1,\ e_1e_2 = e_3,\ e_2e_3 = e_1,\ e_3e_1 = -e_2.$$

The element $e_0$ is the identity of $\mathcal{CQ}$ and $e_1$ identifies the imaginary unit $\sqrt{-1}$ in the $\mathbb{C}$-field of complex numbers. A conic quaternion $Z$ given by (1) is regarded as

$$Z = z_1 + z_2e_2 \in \mathcal{CQ},$$

where $z_1 = x_0 + x_1e_1$ and $z_2 = x_2 + x_3e_1$ are complex numbers in $\mathbb{C}$. Conic quaternions form a commutative, associative, and distributive arithmetic. Also, conic quaternions contain non-trivial idempotents and zero divisors, but no nilpotents. They are isomorphic to tessarines and to bicomplex numbers. Thus, we identify $\mathcal{CQ}$ with $\mathbb{C}^2$.

We use three cases of the conic quaternion conjugate numbers as follows:

(i) $Z^{\dagger_1} = z_1 - z_2e_2$,
(ii) $Z^{\dagger_2} = \overline{z_1} + \overline{z_2}e_2$,
(iii) $Z^{\dagger_3} = \overline{z_1} - \overline{z_2}e_2$.

Then we have three cases of the analogous norm as follows:

(i) $ZZ^{\dagger_1} = z_1^2 + z_2^2 = (x_0 + x_1e_1)^2 + (x_2 + x_3e_1)^2$,
(ii) $ZZ^{\dagger_2} = z_1\overline{z_1} + z_2\overline{z_2} + (z_1\overline{z_2} + z_2\overline{z_1})e_2 = (x_0 + x_2e_2)^2 + (x_1 + x_3e_2)^2$,
(iii) $ZZ^{\dagger_3} = z_1\overline{z_1} - z_2\overline{z_2} - (z_1\overline{z_2} - z_2\overline{z_1})e_2 = (x_0 + x_3e_3)^2 + (x_1 - x_2e_3)^2$.

Consider the following differential operators:

$$\begin{aligned}\frac{\partial}{\partial Z} &:= \frac{\partial}{\partial z_1} + e_2\frac{\partial}{\partial z_2} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}\Big),\\ \frac{\partial}{\partial Z^{\dagger_1}} &= \frac{\partial}{\partial z_1} - e_2\frac{\partial}{\partial z_2} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}\Big),\\ \frac{\partial}{\partial Z^{\dagger_2}} &= \frac{\partial}{\partial \overline{z_1}} + e_2\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}\Big),\\ \frac{\partial}{\partial Z^{\dagger_3}} &= \frac{\partial}{\partial \overline{z_1}} - e_2\frac{\partial}{\partial \overline{z_2}} = \frac{1}{2}\Big(\frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}\Big),\end{aligned}$$

where $\frac{\partial}{\partial z_1}$, $\frac{\partial}{\partial \overline{z_1}}$, $\frac{\partial}{\partial z_2}$, $\frac{\partial}{\partial \overline{z_2}}$ are usual differential operators used in complex analysis.

## 3. Some properties of conic regular functions on $\mathcal{CQ}$

Let $\Omega$ be a bounded open set in $\mathcal{CQ}$. A function $f(Z)$ is defined on $\Omega$ with values in $\mathcal{CQ}$ as follows:

$$\begin{aligned}&f(Z) : \Omega \to \mathcal{CQ}\\ &f(Z) = f(z_1 + z_2e_2) = f_1(z_1, z_2) + f_2(z_1, z_2)e_2,\end{aligned}$$

where

$$f_1(z_1, z_2) = u_0(x_0, x_1, x_2, x_3) + u_1(x_0, x_1, x_2, x_3)e_1$$

and

$$f_2(z_1, z_2) = u_2(x_0, x_1, x_2, x_3) + u_3(x_0, x_1, x_2, x_3)e_1$$

are complex valued functions with real valued functions $u_l$ $(l = 0, 1, 2, 3)$.

**Definition 1.** Let $\Omega$ be an open set in $\mathcal{CQ}$. A function $f(Z)$ is said to be then1st conic regular in $\Omega$, if it admits a conic derivative at each point, i.e. if the limit

$$f'(Z_0) := \lim_{Z\to Z_0} \frac{f(Z) - f(Z_0)}{Z - Z_0}$$

exists and is finite for any $Z_0$ in $\Omega$. The limit will be called the derivative of $f$ and denoted by $f'(Z_0)$.

By the definition of a conic regular function, since the limit has results in any pathes,

$$\begin{aligned}f'(Z_0) &= \lim_{\substack{z_1\to z_1^0\\ z_2=z_2^0}} \Big(\frac{f_1(z_1, z_2) - f_1(z_1^0, z_2^0)}{z_1 - z_1^0} + e_2\frac{f_2(z_1, z_2) - f_2(z_1^0, z_2^0)}{z_1 - z_1^0}\Big)\\ &= \lim_{\substack{z_2\to z_2^0\\ z_1=z_1^0}} e_2\Big(\frac{f_1(z_1, z_2) - f_1(z_1^0, z_2^0)}{z_2 - z_2^0} + \frac{f_2(z_1, z_2) - f_2(z_1^0, z_2^0)}{z_2 - z_2^0}\Big).\end{aligned}$$

That is,

$$f' = \frac{\partial f_1}{\partial z_2} e_2 + \frac{\partial f_2}{\partial z_2} = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1} e_2.$$

Therefore, we have a system such that

$$\frac{\partial f_1}{\partial z_1} = \frac{\partial f_2}{\partial z_2}, \ \frac{\partial f_2}{\partial z_1} = \frac{\partial f_1}{\partial z_2}, \tag{2}$$

which is called the 1st conic Cauchy-Riemann system.

*Remark* 1. In detail, for the system (2), we have

$$\begin{cases} \dfrac{\partial u_0}{\partial x_0} + \dfrac{\partial u_0}{\partial x_0} = \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3}, \\ \dfrac{\partial u_1}{\partial x_0} - \dfrac{\partial u_0}{\partial x_1} = \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3}, \\ \dfrac{\partial u_2}{\partial x_0} + \dfrac{\partial u_3}{\partial x_1} = \dfrac{\partial u_0}{\partial x_2} + \dfrac{\partial u_1}{\partial x_3}, \\ \dfrac{\partial u_3}{\partial x_0} - \dfrac{\partial u_2}{\partial x_1} = \dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_0}{\partial x_3}. \end{cases}$$

*Remark* 2. From the definition of differential operators, we have the following equations:

$$\begin{aligned} \frac{\partial f}{\partial Z} &= \Big(\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2}\Big) + \Big(\frac{\partial f_2}{\partial z_1} + \frac{\partial f_1}{\partial z_2}\Big) e_2, \\ \frac{\partial f}{\partial Z^{\dagger_1}} &= \Big(\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}\Big) + \Big(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\Big) e_2, \\ \frac{\partial f}{\partial Z^{\dagger_2}} &= \Big(\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big) e_2, \\ \frac{\partial f}{\partial Z^{\dagger_3}} &= \Big(\frac{\partial f_1}{\partial \overline{z_1}} - \frac{\partial f_2}{\partial \overline{z_2}}\Big) + \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}}\Big) e_2. \end{aligned}$$

**Definition 2.** Let $\Omega$ be an open set in $\mathcal{CQ}$. A function $f = f_1 + f_2 e_2$ is the 2nd conic regular in $\Omega$ if and only if :

(i) $f_1$ and $f_2$ are continuously differential functions in $\Omega$,

(ii) $f$ satisfies the following equation

$$\frac{\partial f}{\partial Z^{\dagger_2}} = 0.$$

Moreover, from the condition (ii) of Definition 2, we have the following system

$$\frac{\partial f_1}{\partial \overline{z_1}} = -\frac{\partial f_2}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial \overline{z_1}} = -\frac{\partial f_1}{\partial \overline{z_2}}$$

which is said to be the 2nd conic Cauchy-Riemann system on $\Omega$.

**Definition 3.** Let $\Omega$ be an open set in $\mathcal{CQ}$. A function $f = f_1 + f_2 e_2$ is the 3rd conic regular in $\Omega$ if and only if :

(i) $f_1$ and $f_2$ are continuously differential functions in $\Omega$,
(ii) $f$ satisfies the following equation

$$\frac{\partial f}{\partial Z^{\dagger_3}} = 0.$$

Moreover, from the condition (ii) of Definition 3, we have the following system

$$\frac{\partial f_1}{\partial \overline{z_1}} = \frac{\partial f_2}{\partial \overline{z_2}}, \ \frac{\partial f_2}{\partial \overline{z_1}} = \frac{\partial f_1}{\partial \overline{z_2}},$$

which is said to be the 3rd conic Cauchy-Riemann system on $\Omega$.

**Theorem 3.1.** *Let $\Omega$ be an open set in $\mathcal{CQ}$ and let $f(Z) = f_1(z_1, z_2) + f_2(z_1, z_2)e_2 \in \mathcal{C}^1(\Omega)$. Then $f$ is 1st conic regular in $\Omega$ if and only if it satisfies the system*

$$\frac{\partial f}{\partial Z^{\dagger_1}} = 0.$$

*Proof.* By Remarks 1 and 2, the system

$$\frac{\partial f}{\partial Z^{\dagger_1}} = 0$$

is equivalent to Equation (2). That is, since we have the equation

$$0 = \frac{\partial f}{\partial Z^{\dagger_1}} = \Big(\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}\Big) + \Big(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2}\Big)e_2, \tag{3}$$

it satisfies the system

$$\frac{\partial f}{\partial Z^{\dagger_1}} = 0.$$

Conversely, by Equation (3), we obtain the result. □

**Corollary 3.2.** *Let $\Omega$ be an open set in $\mathcal{CQ}$ and let $f(Z) = f_1(z_1, z_2) + f_2(z_1, z_2)e_2 \in \mathcal{C}^1(\Omega)$. Then $f$ is conic regular in $\Omega$ if and only if it satisfies the systems either*

$$\frac{\partial f}{\partial Z^{\dagger_2}} = \frac{\partial f}{\partial x_0} + e_3 \frac{\partial f}{\partial x_3} \quad \text{or} \quad \frac{\partial f}{\partial Z^{\dagger_2}} = e_1 \frac{\partial f}{\partial x_1} + e_2 \frac{\partial f}{\partial x_2}.$$

*Proof.* From Remarks 1 and 2, we have some different terms of the following polynomials

$$\frac{\partial f}{\partial Z^{\dagger_1}}, \frac{\partial f}{\partial Z^{\dagger_2}}, \frac{\partial f}{\partial Z^{\dagger_3}},$$

such that

$$\begin{cases} \dfrac{\partial f}{\partial x_0} = \dfrac{\partial f}{\partial z_1} + \dfrac{\partial f}{\partial \overline{z_1}} \ , \ \dfrac{\partial f}{\partial x_1} = \Big(\dfrac{\partial f}{\partial z_1} - \dfrac{\partial f}{\partial \overline{z_1}}\Big)e_1, \\ \dfrac{\partial f}{\partial x_2} = \dfrac{\partial f}{\partial z_2} + \dfrac{\partial f}{\partial \overline{z_2}} \ , \ \dfrac{\partial f}{\partial x_3} = \Big(\dfrac{\partial f}{\partial z_2} - \dfrac{\partial f}{\partial \overline{z_2}}\Big)e_1. \end{cases} \tag{4}$$

By the definition of differential operators, we obtain the results. □

**Corollary 3.3.** *Let $\Omega$ be an open set in $\mathcal{CQ}$ and let $f(Z) = f_1(z_1, z_2) + f_2(z_1, z_2)e_2 \in \mathcal{C}^1(\Omega)$. Then $f$ is the 1st conic regular in $\Omega$ if and only if it satisfies the systems either*

$$\frac{\partial f}{\partial Z^{\dagger_3}} = \frac{\partial f}{\partial x_0} - e_2\frac{\partial f}{\partial x_2} \quad or \quad \frac{\partial f}{\partial Z^{\dagger_3}} = e_1\frac{\partial f}{\partial x_1} - e_3\frac{\partial f}{\partial x_3}.$$

*Proof.* Arranging and calculating terms of (4), we obtain the results. □

We let a differential form

$$\omega_1 := dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} + e_2 dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}.$$

**Theorem 3.4.** *Let $\Omega$ be a domain in $\mathcal{CQ}$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $\overline{U} \subset \Omega$. If a function $f$ is the 1st conic regular in $\Omega$, then*

$$\int_{bU} \omega_1 f = 0,$$

*where $\omega_1 f$ is the product on $\mathcal{CQ}$ of the form $\omega_1$ on the function $f(Z)$.*

*Proof.* Since the function $f = f_1 + f_2e_2$ has the equation

$$\begin{aligned} \omega_1 f &= f_1 dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2} + f_2 dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} \\ &\quad +(f_1 dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2} + f_2 dz_1 \wedge d\overline{z_1} \wedge d\overline{z_2})e_2, \end{aligned}$$

we have

$$d(\omega_1 f) = \Big(\frac{\partial f_2}{\partial z_1} - \frac{\partial f_2}{\partial z_1}\Big)dV + \Big(\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2}\Big)e_2 dV,$$

where $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$. Since $f$ is the 1st conic regular function in $\Omega$, $f$ satisfies Equation (2). Hence, we have $d(\omega_1 f) = 0$. Therefore, by Stokes' theorem, we obtain the result. □

**Corollary 3.5.** *Let $\Omega$ be a domain in $\mathcal{CQ}$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $\overline{U} \subset \Omega$. Let*

$$\omega_2 := dz_1 \wedge d\overline{z_1} \wedge dz_2 + e_2 dz_1 \wedge dz_2 \wedge d\overline{z_2}.$$

*If a function $f$ is the 2nd conic regular in $\Omega$, then*

$$\int_{bU} \omega_2 f = 0,$$

*where $\omega_2 f$ is the product on $\mathcal{CQ}$ of the form $\omega_2$ on the function $f(Z)$.*

*Proof.* Since the function $f = f_1 + f_2 e_2$ has the equation

$$\begin{aligned} \omega_2 f &= f_1 dz_1 \wedge d\overline{z_1} \wedge dz_2 + f_2 dz_1 \wedge dz_2 \wedge d\overline{z_2} \\ &\quad + (f_1 dz_1 \wedge dz_2 \wedge d\overline{z_2} + f_2 dz_1 \wedge d\overline{z_1} \wedge dz_2) e_2, \end{aligned}$$

we have

$$d(\omega_2 f) = -\Big(\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\Big) dV - \Big(\frac{\partial f_2}{\partial \overline{z_1}} + \frac{\partial f_1}{\partial \overline{z_2}}\Big) e_2 dV,$$

where $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$. Since $f$ is a the 2nd conic regular function in $\Omega$, $f$ satisfies the 2nd conic Cauchy-Riemann system. Hence, we have $d(\omega_2 f) = 0$. Therefore, by Stokes' theorem, we obtain the result. □

**Corollary 3.6.** *Let $\Omega$ be a domain in $\mathcal{CQ}$ and $U$ be any domain in $\Omega$ with a smooth boundary $bU$ such that $\overline{U} \subset \Omega$. Let*

$$\omega_3 := dz_1 \wedge d\overline{z_1} \wedge dz_2 - e_2 dz_1 \wedge dz_2 \wedge d\overline{z_2},$$

*and a function $f$ is the 3rd conic regular in $\Omega$. Then*

$$\int_{bU} \omega_3 f = 0,$$

*where $\omega_3 f$ is the product on $\mathcal{CQ}$ of the form $\omega_3$ on the function $f(Z)$.*

*Proof.* Since the function $f = f_1 + f_2 e_2$ has the equation

$$\begin{aligned} d(\omega_3 f) &= d\{f_1 dz_1 \wedge d\overline{z_1} \wedge dz_2 - f_2 dz_1 \wedge dz_2 \wedge d\overline{z_2} \\ &\quad + (f_1 dz_1 \wedge dz_2 \wedge d\overline{z_2} - f_2 dz_1 \wedge d\overline{z_1} \wedge dz_2) e_2\} \\ &= \Big(-\frac{\partial f_1}{\partial \overline{z_1}} + \frac{\partial f_2}{\partial \overline{z_2}}\Big) dV - \Big(\frac{\partial f_2}{\partial \overline{z_1}} - \frac{\partial f_1}{\partial \overline{z_2}}\Big) e_2 dV, \end{aligned}$$

where $dV = dz_1 \wedge dz_2 \wedge d\overline{z_1} \wedge d\overline{z_2}$, from which $f$ satisfies the 3rd conic Cauchy-Riemann system in $\Omega$, we have $d(\omega_3 f) = 0$. Therefore, by Stokes' theorem, the result is obtained. □

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