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Noninformative priors for the common shape parameter of several inverse Gaussian distributions

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Abstract

In this paper, we develop the noninformative priors for the common shape parameter of several inverse Gaussian distributions. Specially, we want to develop noninformative priors which satisfy certain objective criterion. The probability matching priors and reference priors of the common shape parameter will be developed. It turns out that the second order matching prior does not exist. The reference priors satisfy the first order matching criterion, but Jeffrey's prior is not the first order matching prior. We showed that the proposed reference prior matches the target coverage probabilities in a frequentist sense through simulation study, and an example based on real data is given.

Keywords: Inverse Gaussian distribution, matching prior, reference prior, shape parameter.

1. Introduction

The probability density function (pdf) of inverse gaussian with parameters μ and λ is given by

$$f(x) = \sqrt{\frac{\lambda}{2\pi}} x^{-\frac{3}{2}} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\}, x > 0, \tag{1.1}$$

where $\mu > 0$ is mean parameter and $\lambda > 0$ is the scale parameter. The shape parameter for the inverse Gaussian distribution is defined as η, where $\eta = \lambda/\mu$. Because of the versatility and flexibility in modelling right-skewed data, the inverse gaussian distribution has potentially useful applications in a wide variety of fields such as biology, economics, reliability theory and life testing as discussed in Chhikara and Folks (1989) and Seshadri (1999).

The present paper focuses on noninformative priors for the common shape parameter of several inverse Gaussian distributions. The shape parameter of inverse Gaussian distribution has an important meaning. For example, Chhikara and Folks (1989) derived an exact method of seeking a confidence interval for the ratio of means when the shape parameters of these

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two populations are the same. Consequently, the use of this exact method requires a test to check whether the shape parameters of the two populations are equal or not. And the skewness, the kurtosis and the coefficient of variation (CV) of this distribution involve the shape parameter. As a note, the CV test statistic or its reciprocal plays an important role in medical sciences, physical sciences, social sciences and finance, see Bai *et al.* (2011). Under this premise, a more persuasive inference about some properties for inverse Gaussian distributions can be made. About statistical test of the equality of several inverse Gaussian shape parameters, Niu *et al.* (2014) developed three tests for the equality of inverse Gaussian shape parameters. But the Bayesian inference about the equality of shape parameters based on noninformative priors is not developed yet. To develop the objective Bayesian inference about the equality of shape parameters of inverse Gaussian distributions, it is absolutely necessary to derive the noninformative prior for the common shape parameter.

Usually, the use of subjective priors is ideal when sufficient information from past experience, expert opinion or previously collected data exist. However, often even without adequate prior information, one can perform a Bayesian inference efficiently with a noninformative or default priors. The catalog of such priors has become prohibitively large over years.

We firstly consider Bayesian priors such that the coverage probability of a Bayesian credible interval is asymptotically equivalent to the coverage probability of the frequentist confidence interval up to a certain order. This matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may cite the work of Mukerjee and Dey (1993), DiCiccio and Stern (1994), Datta and Ghosh (1995, 1996) and Mukerjee and Ghosh (1997). An excellent monograph on this topic is due to Datta and Mukerjee (2004) which provides a thorough and comprehensive discussion of various probability matching criteria.

On the other hand, Bernardo (1979) introduced the reference priors which maximizes the Kullback-Leibler divergence between the prior and the posterior. Ghosh and Mukerjee (1992) and Berger and Bernardo (1989,1992) gave a general algorithm to derive a reference prior by splitting the parameters into several groups according to their order of inferential importance. This approach is very successful in various practical problems. Quite often reference priors satisfy the matching criterion described earlier. This approach is very successful in various practical problems (Kang, 2011; Kang *et al.*, 2014).

The outline of the remaining sections is as follows. In Section 2, we develop first order and second order probability matching priors for the common shape parameter. We revealed that the second order matching prior does not exist. It turns out that the one-at-a-time reference prior is a first order matching criterion, but Jeffrey's prior is not a first order matching priors. In Section 3, we provide that the propriety of the posterior distribution for Jeffreys' prior as well as the reference prior. In Section 4, simulated frequentist coverage probabilities under the proposed priors and two real example are given.

2. The noninformative priors

Let $x_{ij}, i = 1, \cdots, k, j = 1, \cdots, n_i$, be independent random samples from inverse Gaussian distributions with the mean parameter μ_i and the common shape parameter η. Then the likelihood function is

$$L(\eta, \mu_1, \cdots, \mu_k) \propto \eta^{\frac{n}{2}} \prod_{i=1}^{k} \mu_i^{\frac{n_i}{2}} \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{\eta(x_{ij}-\mu_i)^2}{2\mu_i^2 x_{ij}}\right\}, \tag{2.1}$$

where $n = \sum_{i=1}^{k} n_i$, $\mu_i > 0, i = 1, \cdots, k$ and $\eta > 0$. We want to make a Bayesian inference about the common shape parameter η based on noninformative priors.

2.1. The probability matching priors

For a prior π, let $\theta_1^{1-\alpha}(\pi; \mathbf{X})$ denote the $(1-\alpha)$th posterior quantile of θ_1, that is,

$$P^{\pi}[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})|\mathbf{X}] = 1-\alpha, \tag{2.2}$$

where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_{k+1})^T$ and θ_1 is the parameter of interest. We want to find priors π for which

$$P_{\theta}[\theta_1 \le \theta_1^{1-\alpha}(\pi; \mathbf{X})] = 1-\alpha + o(n^{-r}). \tag{2.3}$$

for some $r > 0$, as n goes to infinity. Priors π satisfying (2.3) are called matching priors. If $r = 1/2$, then π is referred to as the first order matching prior, while if $r = 1$, π is referred to as the second order matching prior.

In order to find such matching priors π, let

$$\theta_1 = \eta \text{ and } \theta_{i+1} = \mu_i^{-1}(2+\eta^{-1}), i = 1, \cdots, k.$$

With this parametrization, the likelihood function of parameters $(\theta_1, \cdots, \theta_{k+1})$ for the model (2.1) is given by

$$L(\theta_1, \cdots, \theta_{k+1}) \propto \theta_1^{\frac{n}{2}}(2+\theta_1^{-1})^{\frac{n}{2}} \prod_{i=1}^{k} \theta_{i+1}^{-\frac{n_i}{2}} \exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{\theta_1(x_{ij}-\theta_{i+1}^{-1}(2+\theta_1^{-1}))^2}{2\theta_{i+1}^{-1}(2+\theta_1^{-1})x_{ij}}\right\}. \tag{2.4}$$

Based on (2.4), the Fisher information matrix is given by

$$\mathbf{I}(\theta_1, \theta_2, \cdots, \theta_{k+1}) = \begin{pmatrix} \frac{n}{\theta_1^2(2+\theta_1^{-1})} & 0 & 0 & \cdots & 0 \\ 0 & \frac{n_1(1+2\theta_1)}{2\theta_2^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n_k(1+2\theta_1)}{2\theta_{k+1}^2} \end{pmatrix}. \tag{2.5}$$

From the above Fisher information matrix $\mathbf{I}$, θ_1 is orthogonal to $\theta_2, \cdots, \theta_{k+1}$ in the sense of Cox and Reid (1987). Following Tibshirani (1989), the class of the first order probability matching prior is characterized by

$$\pi_m^{(1)}(\theta_1, \cdots, \theta_{k+1}) \propto \theta_1^{-1}(2+\theta_1^{-1})^{-\frac{1}{2}} d(\theta_2, \cdots, \theta_{k+1}), \tag{2.6}$$

where $d(\theta_2, \cdots, \theta_{k+1}) > 0$ is an arbitrary function differentiable in its argument.

The class of prior given in (2.6) can be narrowed down to the second order probability matching priors as given in Mukerjee and Ghosh (1997). The second order probability matching prior is of the form (2.6), and d must satisfy the following additional differential equation (2.7), namely

$$\frac{1}{6}d(\theta_2)\frac{\partial}{\partial\theta_1}\{I_{11}^{-\frac{3}{2}}L_{1,1,1}\}+\sum_{i=2}^{k+1}\frac{\partial}{\partial\theta_i}\{I_{11}^{-\frac{1}{2}}L_{11i}I^{ii}d(\theta_2,\cdots,\theta_{k+1})\}=0, \tag{2.7}$$

where

$$\begin{aligned} L_{1,1,1} &= E\left[\left(\frac{\partial\log L}{\partial\theta_1}\right)^3\right]=-\frac{n(3+6\theta_1+8\theta_1^2)}{\theta_1^5(2+\theta_1^{-1})^3}, \\ L_{11i} &= E\left[\frac{\partial^3\log L}{\partial\theta_1^2\partial\theta_i}\right]=-\frac{n_{i-1}\theta_1^{-3}\theta_i^{-1}}{(2+\theta_1^{-1})^2}, i=2,\cdots,k+1, \\ I_{11} &= \frac{n}{\theta_1^2(2+\theta_1^{-1})}, I^{ii}=\frac{2\theta_i^2}{n_{i-1}(1+2\theta_1)}, i=2,\cdots,k+1, \end{aligned}$$

and I^{ij} is an element of the inverse of the matrix $\mathbf{I}$. Then (2.7) simplifies to

$$\begin{aligned} &\frac{(2+\theta_1^{-1})^{\frac{5}{2}}}{6\theta_1^{-3}}\frac{\partial}{\partial\theta_1}\left\{-\frac{3(3+6\theta_1+8\theta_1^2)}{n^{\frac{1}{2}}\theta_1^2(2+\theta_1^{-1})^{\frac{3}{2}}}\right\} \\ +\;&\frac{1}{d(\theta_2,\cdots,\theta_{k+1})}\sum_{i=2}^{k+1}\frac{\partial}{\partial\theta_i}\left\{-2n^{-\frac{1}{2}}\theta_i d(\theta_2,\cdots,\theta_{k+1})\right\}=0. \end{aligned} \tag{2.8}$$

Note that the first term in the left hand side of (2.8) depends only on the model, whereas the second term involves the prior. Also, the first term depends only on θ_1 and the second only on $\theta_2,\cdots,\theta_k$ and θ_{k+1}. There can be no solution to (2.8) unless the first term is a constant. However the first term of (2.8) is not a constant. Therefore, this rules out the existence of the second order matching priors.

2.2. The reference priors

Reference priors introduced by Bernardo (1979), and extended further by Berger and Bernardo (1992) have become very popular over the years for the development of noninformative priors.

Now, we derive the reference priors for different groups of ordering of inferential importance of $(\theta_1,\cdots,\theta_{k+1})$. Then due to the orthogonality of the parameters, following Datta and Ghosh (1995), choosing rectangular compacts for each $\theta_1,\cdots,\theta_{k+1}$ when θ_1 is the parameter of interest, the reference priors are given by as follows.

If θ_1 is the parameter of interest, then the reference prior distributions for different groups of ordering of $(\theta_1, \cdots, \theta_{k+1})$ are:

Group ordering	Reference prior	
$\{(\theta_1, \cdots, \theta_{k+1})\}$,	$\pi_J \propto \theta_1^{\frac{k}{2}-1}(2+\theta_1^{-1})^{\frac{k-1}{2}} \prod_{i=2}^{k+1} \theta_i^{-1}$,	(2.9)
$\{\theta_1, \theta_2, \cdots, \theta_{k+1}\}, \{\theta_1, (\theta_2, \cdots, \theta_{k+1})\}$,	$\pi_r \propto \theta_1^{-1}(2+\theta_1^{-1})^{-\frac{1}{2}} \prod_{i=2}^{k+1} \theta_i^{-1}$.	(2.10)

Remark 2.1 In the above reference priors, the one-at-a-time reference prior, π_r, satisfies the first order matching criterion. And Jeffreys' prior, π_J, is not the first order matching prior.

Note that the matching priors (2.6) include many different matching priors because of the selection of the function d. And for some functions, there does not seem to be any improvement in the coverage probabilities with these posteriors. Thus we consider the first order matching prior where the function d is $\prod_{i=2}^{k+1} \theta_i^{-1}$. This matching prior is given by

$$\pi_m^{(1)}(\theta_1, \cdots, \theta_{k+1}) \propto \theta_1^{-1}(2+\theta_1^{-1})^{-\frac{1}{2}} \prod_{i=2}^{k+1} \theta_i^{-1}.$$

and this is the one-at-a-time reference prior.

Remark 2.2 In the original parametrization $(\eta, \mu_1, \cdots, \mu_k)$, Jeffreys' prior and the reference prior are given by

$$\pi_J(\eta, \mu_1, \cdots, \mu_k) \quad \propto \quad \eta^{-\frac{1}{2}}(1+2\eta)^{\frac{k-1}{2}} \prod_{i=1}^{k} \mu_i^{-1}, \tag{2.11}$$

$$\pi_r(\eta, \mu_1, \cdots, \mu_k) \quad \propto \quad \eta^{-\frac{1}{2}}(1+2\eta)^{-\frac{1}{2}} \prod_{i=1}^{k} \mu_i^{-1}, \tag{2.12}$$

respectively.

3. Implementation of the Bayesian procedure

We investigate the propriety of posteriors for a general class of priors which includes the reference priors and Jeffreys' prior. We consider the class of priors

$$\pi(\theta_1, \cdots, \theta_{k+1}) \propto \theta_1^a (2+\theta_1^{-1})^b \prod_{i=2}^{k+1} \theta_i^{-c}, \tag{3.1}$$

where $-\infty < a, b < \infty$ and $c > 0$. The following general theorem can be proved.

Theorem 3.1 The posterior distribution of $(\theta_1, \cdots, \theta_{k+1})$ under the prior (3.1) is proper if $n + 2a - 2b - 2k + 2kc + 2 > 0$, $n_i + \frac{a-b+1}{k} + 2c - 3 > 0$, $a - b + 1 > 0$, for $b + k - kc < 0$ and $n_i + \frac{k+a+1-kc}{k} + 2c - 3 > 0$, $k + a + 1 - kc > 0$ for $b + k - kc \geq 0$.

Proof: Note that the joint posterior for $\theta_1, \cdots, \theta_k$ and θ_{k+1} given $\mathbf{x}$ is given by

$$\pi(\theta_1, \cdots, \theta_{k+1}|\mathbf{x}) \propto \theta_1^{\frac{n}{2}+a}(2+\theta_1^{-1})^{\frac{n}{2}+b}\prod_{i=1}^{k}\theta_{i+1}^{-\frac{n_i}{2}-c}\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\theta_1(x_{ij}-\theta_{i+1}^{-1}(2+\theta_1^{-1}))^2}{2\theta_{i+1}^{-1}(2+\theta_1^{-1})x_{ij}}\right\}. \quad (3.2)$$

Let $\theta_{i+1} = \mu_i^{-1}(2+\theta_1^{-1}), i = 1, \cdots, k$. Thus we get

$$\begin{aligned}
&\pi(\theta_1, \mu_1, \cdots, \mu_k|\mathbf{x}) \\
\propto\ & \theta_1^{\frac{n}{2}+a}(2+\theta_1^{-1})^{b+k-kc}\prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}+c-2}\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{ij}}\right\} \\
\leq\ & \theta_1^{\frac{n}{2}+a}(2+\theta_1^{-1})^{b+k-kc}\prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}+c-2}\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{max}}\right\} \\
\equiv\ & \pi_1(\theta_1, \mu_1, \cdots, \mu_k|\mathbf{x}). \qquad (3.3)
\end{aligned}$$

where $x_{max} = \max\{x_{11}, \cdots, x_{kn_k}\}$. If $b+k-kc<0$ then

$$\begin{aligned}
&\pi_1(\theta_1, \mu_1, \cdots, \mu_k|\mathbf{x}) \\
=\ & \theta_1^{\frac{n}{2}+a-b-k+kc}(2\theta_1+1)^{b+k-kc}\prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}+c-2}\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{max}}\right\} \\
\leq\ & \theta_1^{\frac{n}{2}+a-b-k+kc}\prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}+c-2}\exp\left\{-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{max}}\right\} \\
\equiv\ & \pi_2(\theta_1, \mu_1, \cdots, \mu_k|\mathbf{x}). \qquad (3.4)
\end{aligned}$$

Let $\frac{n}{2}+a-b-k+kc+1>0$. Thus integrating with respect to θ_1, then we get

$$\begin{aligned}
\pi_2(\mu_1, \cdots, \mu_k|\mathbf{x}) &\propto \prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}+c-2}\left[\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(x_{ij}-\mu_i)^2}{2\mu_i x_{max}}\right]^{-\frac{n_i}{2}-\frac{a-b-k+kc+1}{k}} \\
&\leq \prod_{i=1}^{k}\mu_i^{\frac{n_i}{2}+c-2}\left[\sum_{j=1}^{n_i}\frac{(x_{ij}-\mu_i)^2}{2\mu_i x_{max}}\right]^{-\frac{n_i}{2}-\frac{a-b-k+kc+1}{k}} \\
&\propto \prod_{i=1}^{k}\mu_i^{n_i+c+\frac{a-b-k+kc+1}{k}-2}\left[1+\frac{(\bar{x}_i-\mu_i)^2}{s_i^2/n_i}\right]^{-\frac{n_i}{2}-\frac{a-b-k+kc+1}{k}}, \quad (3.5)
\end{aligned}$$

where $s_i^2 = \sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2$. Therefore the integral of (3.5) is finite if $a-b+1>0$ and

$n_i + \frac{a-b+1}{k} + 2c - 3 > 0, i = 1, \cdots, k$. Next if $b + k - kc \geq 0$ then

$$\begin{aligned} & \pi_1(\theta_1, \mu_1, \cdots, \mu_k|\mathbf{x}) \\ \leq \quad & C_1 2^{b+k-kc} \theta_1^{\frac{n}{2}+a} \prod_{i=1}^{k} \mu_i^{\frac{n_i}{2}+c-2} \exp\left\{ -\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{max}} \right\} \\ + \quad & C_1 \theta_1^{\frac{n}{2}+a-b-k+kc} \prod_{i=1}^{k} \mu_i^{\frac{n_i}{2}+c-2} \exp\left\{ -\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{max}} \right\}, \end{aligned} \tag{3.6}$$

where C_1 is a constant. Note that the second term of the posterior (3.6) is proportional to the posterior (3.4). Therefore by the same method for the case of $b + k - kc < 0$, we can show that the integral of (3.6) is finite if $k + a + 1 - kc > 0$, $n_i + \frac{k+a+1-kc}{k} + 2c - 3 > 0$ and $a - b + 1 > 0$, $n_i + \frac{a-b+1}{k} + 2c - 3 > 0$. This completes the proof. □

Theorem 3.2 Under the prior (3.1), the marginal posterior density of θ_1 is given by

$$\begin{aligned} & \pi(\theta_1|\mathbf{x}) \\ \propto \quad & \prod_{i=1}^{k} \int_0^{\infty} \theta_1^{\frac{n}{2}+a} (2+\theta_1^{-1})^{b+k-kc} \prod_{i=1}^{k} \mu_i^{\frac{n_i}{2}+c-2} \exp\left\{ -\sum_{i=1}^{k}\sum_{j=1}^{n_i} \frac{\theta_1(x_{ij}-\mu_i)^2}{2\mu_i x_{ij}} \right\} d\mu_i. \end{aligned} \tag{3.7}$$

Note that actually the marginal density of θ_1 requires two dimensional integration. Therefore, we have the marginal posterior density of θ_1, and so it is easy to compute the marginal moments of θ_1. In Section 4, we investigate the frequentist coverage probabilities for Jeffreys prior π_J and the reference prior π_r, respectively .

4. Numerical studies

4.1. Simulation study

We evaluate the frequentist coverage probability by investigating the credible interval of the marginal posterior density of θ_1 under the reference prior and Jeffreys' prior given in Section 3 for several configurations of $(\eta, \mu_1, \cdots, \mu_k)$ and $(n_1, \cdots, n_k)$. That is to say, the frequentist coverage of a $(1-\alpha)$th posterior quantile should be close to $1-\alpha$. This is done numerically. Tables 4.1 and 4.2 give numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the our prior. The computation of these numerical values is based on the following algorithm for any fixed true $(\eta, \mu_1, \cdots, \mu_k)$ and any prespecified probability value α. Here α is 0.05 (0.95). Let $\theta_1^{\alpha}(\pi; \mathbf{X})$ be the αth posterior quantile of θ_1 given $\mathbf{X}$. Then the frequentist coverage probability of this one sided credible interval of α is

$$P_{(\eta,\mu_1,\cdots,\mu_k)}(\alpha; \theta_1) \quad = \quad P_{(\eta,\mu_1,\cdots,\mu_k)}(0 < \theta_1 \leq \theta_1^{\alpha}(\pi; \mathbf{X})). \tag{4.1}$$

The computed $P_{(\eta,\mu_1,\cdots,\mu_k)}(\alpha; \theta_1)$ when $\alpha = 0.05(0.95)$ is shown in Table 4.1 and Table 4.2, respectively. In particular, for fixed $(n_1, \cdots, n_k)$ and $(\eta, \mu_1, \cdots, \mu_k)$, we take 10,000 independent random samples of $\mathbf{X} = (\mathbf{X}_1, \cdots, \mathbf{X}_k)$ from the inverse Gaussian distributions.

For this simulation, FORTRAN equipped with IMSL subroutine is used. To compute the integration in (eq:marginal1), the IMSL subroutine GQRUL/DGQRUL is used.

Table 4.1 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

η	μ_1, μ_2, μ_3	n_1, n_2, n_3	π_J	π_r
0.1	0.5, 0.5, 5.0	5,5,5	0.089 (0.980)	0.036 (0.972)
		5,10,10	0.072 (0.968)	0.038 (0.959)
		10,10,10	0.066 (0.966)	0.036 (0.956)
		10,20,20	0.062 (0.954)	0.042 (0.945)
	1.0, 1.0, 5.0	5,5,5	0.091 (0.981)	0.036 (0.974)
		5,10,10	0.065 (0.970)	0.031 (0.960)
		10,10,10	0.068 (0.965)	0.040 (0.955)
		10,20,20	0.061 (0.957)	0.042 (0.946)
	0.1, 1.0, 10.0	5,5,5	0.088 (0.981)	0.034 (0.974)
		5,10,10	0.068 (0.975)	0.034 (0.968)
		10,10,10	0.067 (0.971)	0.036 (0.962)
		10,20,20	0.060 (0.967)	0.041 (0.959)
	1.0, 5.0, 10.0	5,5,5	0.088 (0.982)	0.035 (0.976)
		5,10,10	0.067 (0.971)	0.034 (0.964)
		10,10,10	0.065 (0.971)	0.036 (0.963)
		10,20,20	0.060 (0.958)	0.042 (0.949)
1.0	0.5, 0.5, 5.0	5,5,5	0.136 (0.980)	0.045 (0.950)
		5,10,10	0.110 (0.973)	0.049 (0.944)
		10,10,10	0.100 (0.978)	0.046 (0.953)
		10,20,20	0.083 (0.970)	0.049 (0.951)
	1.0, 1.0, 5.0	5,5,5	0.138 (0.979)	0.048 (0.948)
		5,10,10	0.104 (0.976)	0.046 (0.952)
		10,10,10	0.098 (0.972)	0.049 (0.949)
		10,20,20	0.087 (0.966)	0.053 (0.946)
	0.1, 1.0, 10.0	5,5,5	0.134 (0.980)	0.045 (0.950)
		5,10,10	0.100 (0.973)	0.047 (0.948)
		10,10,10	0.094 (0.972)	0.047 (0.947)
		10,20,20	0.084 (0.970)	0.048 (0.950)
	1.0, 5.0, 10.0	5,5,5	0.135 (0.979)	0.045 (0.949)
		5,10,10	0.113 (0.976)	0.052 (0.951)
		10,10,10	0.105 (0.974)	0.050 (0.951)
		10,20,20	0.089 (0.969)	0.051 (0.948)
10.0	0.5, 0.5, 5.0	5,5,5	0.153 (0.984)	0.047 (0.952)
		5,10,10	0.117 (0.977)	0.047 (0.944)
		10,10,10	0.108 (0.979)	0.045 (0.950)
		10,20,20	0.097 (0.969)	0.052 (0.946)
	1.0, 1.0, 5.0	5,5,5	0.156 (0.984)	0.049 (0.952)
		5,10,10	0.117 (0.976)	0.047 (0.951)
		10,10,10	0.114 (0.977)	0.052 (0.950)
		10,20,20	0.095 (0.975)	0.053 (0.954)
	0.1, 1.0, 10.0	5,5,5	0.161 (0.985)	0.050 (0.948)
		5,10,10	0.121 (0.979)	0.050 (0.950)
		10,10,10	0.110 (0.977)	0.046 (0.949)
		10,20,20	0.093 (0.970)	0.052 (0.946)
	1.0, 5.0, 10.0	5,5,5	0.160 (0.987)	0.048 (0.952)
		5,10,10	0.120 (0.980)	0.049 (0.951)
		10,10,10	0.111 (0.978)	0.050 (0.950)
		10,20,20	0.090 (0.973)	0.048 (0.950)

Table 4.2 Frequentist coverage probability of 0.05 (0.95) posterior quantiles of θ_1

η	$\mu_1, \cdots, \mu_6$	$n_1, \cdots, n_6$	π_J	π_r
0.1	0.5, 0.5, 1.0, 1.0, 5.0, 5.0	5,5,5,5,5,5	0.110 (0.976)	0.035 (0.958)
		5,5,5,10,10,10	0.090 (0.968)	0.040 (0.950)
		10,10,10,10,10,10	0.083 (0.967)	0.043 (0.949)
		10,10,10,20,20,20	0.073 (0.960)	0.042 (0.944)
	1.0, 1.0, 3.0, 3.0, 5.0, 5.0	5,5,5,5,5,5	0.114 (0.977)	0.037 (0.959)
		5,5,5,10,10,10	0.092 (0.968)	0.042 (0.949)
		10,10,10,10,10,10	0.083 (0.965)	0.044 (0.948)
		10,10,10,20,20,20	0.075 (0.960)	0.045 (0.945)
	0.1, 0.5, 1.0, 4.0, 7.0, 10.0	5,5,5,5,5,5	0.109 (0.977)	0.038 (0.966)
		5,5,5,10,10,10	0.090 (0.972)	0.038 (0.956)
		10,10,10,10,10,10	0.080 (0.968)	0.044 (0.953)
		10,10,10,20,20,20	0.079 (0.963)	0.050 (0.948)
	1.0, 4.0, 7.0, 10.0, 15.0, 20.0	5,5,5,5,5,5	0.113 (0.980)	0.036 (0.966)
		5,5,5,10,10,10	0.090 (0.976)	0.040 (0.963)
		10,10,10,10,10,10	0.080 (0.972)	0.042 (0.959)
		10,10,10,20,20,20	0.076 (0.970)	0.046 (0.958)
1.0	0.5, 0.5, 1.0, 1.0, 5.0, 5.0	5,5,5,5,5,5	0.191 (0.987)	0.053 (0.947)
		5,5,5,10,10,10	0.147 (0.983)	0.047 (0.948)
		10,10,10,10,10,10	0.117 (0.980)	0.046 (0.949)
		10,10,10,20,20,20	0.099 (0.979)	0.045 (0.954)
	1.0, 1.0, 3.0, 3.0, 5.0, 5.0	5,5,5,5,5,5	0.192 (0.987)	0.053 (0.951)
		5,5,5,10,10,10	0.154 (0.984)	0.053 (0.951)
		10,10,10,10,10,10	0.129 (0.980)	0.052 (0.950)
		10,10,10,20,20,20	0.108 (0.977)	0.050 (0.951)
	0.1, 0.5, 1.0, 4.0, 7.0, 10.0	5,5,5,5,5,5	0.189 (0.990)	0.045 (0.949)
		5,5,5,10,10,10	0.149 (0.984)	0.051 (0.951)
		10,10,10,10,10,10	0.127 (0.980)	0.051 (0.950)
		10,10,10,20,20,20	0.109 (0.977)	0.049 (0.951)
	1.0, 4.0, 7.0, 10.0, 15.0, 20.0	5,5,5,5,5,5	0.183 (0.988)	0.049 (0.952)
		5,5,5,10,10,10	0.146 (0.983)	0.049 (0.947)
		10,10,10,10,10,10	0.125 (0.978)	0.046 (0.948)
		10,10,10,20,20,20	0.110 (0.977)	0.050 (0.953)
10.0	0.5, 0.5, 1.0, 1.0, 5.0, 5.0	5,5,5,5,5,5	0.215 (0.991)	0.049 (0.948)
		5,5,5,10,10,10	0.167 (0.986)	0.051 (0.946)
		10,10,10,10,10,10	0.141 (0.985)	0.048 (0.949)
		10,10,10,20,20,20	0.121 (0.984)	0.047 (0.953)
	1.0, 1.0, 3.0, 3.0, 5.0, 5.0	5,5,5,5,5,5	0.221 (0.991)	0.050 (0.948)
		5,5,5,10,10,10	0.167 (0.988)	0.050 (0.953)
		10,10,10,10,10,10	0.142 (0.985)	0.049 (0.952)
		10,10,10,20,20,20	0.121 (0.983)	0.050 (0.952)
	0.1, 0.5, 1.0, 4.0, 7.0, 10.0	5,5,5,5,5,5	0.227 (0.992)	0.055 (0.950)
		5,5,5,10,10,10	0.164 (0.987)	0.050 (0.947)
		10,10,10,10,10,10	0.144 (0.985)	0.051 (0.952)
		10,10,10,20,20,20	0.119 (0.980)	0.050 (0.948)
	1.0, 4.0, 7.0, 10.0, 15.0, 20.0	5,5,5,5,5,5	0.211 (0.990)	0.049 (0.946)
		5,5,5,10,10,10	0.169 (0.989)	0.049 (0.952)
		10,10,10,10,10,10	0.145 (0.983)	0.049 (0.947)
		10,10,10,20,20,20	0.117 (0.977)	0.051 (0.944)

Tables 4.1 and 4.2 indicate that the reference prior meets the target coverage probabilities better than Jeffreys' prior. We also note that the reference prior provides good cov-

erage in small sample size, and the results are less sensitive to the change of the values of $(\eta, \mu_1, \cdots, \mu_k)$. Thus, we recommend to use the reference prior.

4.2. Example

This example is taken from Niu *et al.* (2014). This data set can be obtained from `http://lib.stat.cmu.edu/DASL/Datafiles/Crash.html` originally provided by National Transportation Safety Administration. The data came from the trials in which stock automobiles were crashed into a wall at 35MPH, with dummies in the driver and front passenger seat. The information of injury variables on how the crash affected the dummies was collected, namely the extent of head injuries, chest deceleration, and left and right femur load. To illustrate our methods, here we only consider the problem of comparing the left femur load among three car makes, namely Dodge, Honda and Hyundai. Tian (2006) showed that the variable left femur load can be fitted by an inverse Gaussian distribution. Further Tian (2006) concluded that left femur loads were not different among these three car makers based on the proposed generalized test method. Ye *et al.* (2010) considered the hypothesis about whether the common mean of these populations are equal to 8.5.

The summary statistics of this data set are: $n_1 = 8, n_2 = 7, n_3 = 5$, $x_1 = 8.578, x_2 = 8.053, x_3 = 15.968$, $v_1 = 0.0254, v_2 = 0.0214, v_3 = 0.0164$. Niu *et al.* (2014) investigated whether the shape parameters of these three populations are equal or not. They showed that the null hypothesis (the common shape parameter) can not be rejected with the $p-$value 0.854.

For this data set, maximum likelihood estimate and asymptotic 90% confidence interval of θ_1 is given in Table 4.3. Also Bayes estimates and the 90% credible intervals based on the reference prior and Jeffreys' prior given in Table 4.3. The Bayes estimates based on Jeffreys' prior and the MLE give the similar results, but the Bayes estimate based on the reference prior is slightly different. Note that Jeffreys' prior does not meet the target coverage probabilities. Also the credible interval based on the reference prior is slightly shorter than the credible intervals of Jeffreys' prior and asymptotic confidence interval of MLE.

Table 4.3 MLE, Bayes estimate and credible interval for θ_1

MLE	π_J	π_r
4.702 (2.163, 7.240)	4.629 (2.367, 7.449)	3.922 (1.859, 6.540)

5. Concluding remarks

In the inverse Gaussian models, we have found the matching prior and the reference prior for the common shape parameter. For the common shape parameter, we revealed that the reference prior satisfies a first order matching criterion, but Jeffreys' prior does not. As illustrated in our numerical study, the reference prior seems to be the best appropriate results than Jeffreys' prior in the sense of asymptotic frequentist coverage property. Thus we recommend the use of the reference prior for Bayesian inference of the common shape parameter in the inverse Gaussian distributions.

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