THE JONES POLYNOMIAL OF KNOTS WITH SYMMETRIC UNION PRESENTATIONS

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Abstract. A symmetric union is a diagram of a knot, obtained from diagrams of a knot in the 3-space and its mirror image, which are symmetric with respect to an axis in the 2-plane, by connecting them with 2-tangles with twists along the axis and 2-tangles with no twists. This paper presents an invariant of knots with symmetric union presentations, which is called the minimal twisting number, and the minimal twisting number of $10_{42}$ is shown to be two. This paper also presents a sufficient condition for non-amplicehairality of a knot with a certain symmetric union presentation.

1. Introduction

A symmetric union, which is a generalized operation of the connected sum of a knot in the 3-space and its mirror image, was first introduced by Kinoshita and Terasaka [6]. They showed that the Alexander polynomial depends only on the parity of the number of half-twists of a trivial tangle on the symmetry axis and that the determinant is independent of the number of half-twists. In recent years, Lamm [7] generalized their results and also considered the relationship between a symmetric union and a ribbon knot. (See [3] for the definition.) It is easily seen that every knot with a symmetric union presentation is a ribbon knot. On the other hand, the converse question is still open. Lamm showed that every ribbon knot with minimal crossing number $\leq 10$ has a symmetric union presentation, except $10_{87}$ [7]. The knot $10_{87}$ was also shown to be a symmetric union later in [2] and it is known that all two-bridge ribbon knots can be represented as symmetric unions. In fact, Lamm [8] has shown that certain three infinite families of two-bridge ribbon knots can be presented as a symmetric union. Recently, Lisca [11] has shown that the three families contain all two-bridge ribbon knots.

Let $\overline{V}(t) = V_L(t)/V_{O_n}(t)$ for an oriented link $L$ of $n$ components, where $V_L(t)$ and $V_{O_n}(t)$ are the Jones polynomial of $L$ and the $n$-component trivial link $O^n$ respectively. (See Section 2 for the definition.) We set $i = \sqrt{-1}$. In this
paper we show the following formula for the Jones polynomial of knots with symmetric union presentations and its topological properties.

**Theorem 1.1.** Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then

$$V_{\overline{K}}(t) = (-1)^m t^{-m} V_{D_K \cup D_K^*(\infty, 0)}(t) + (1 - (-1)^m t^{-m}) V_{D_K \cup D_K^*(\infty, \infty)}(t).$$

**Corollary 1.2.** Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$ and $s$ be a positive integer. If $m \equiv 0 \mod s$, then

$$V_{\overline{K}}(-\exp(2\pi i s)) = V_{\overline{K}}(-\exp(2\pi i)) V_{K^*}(-\exp(2\pi i)).$$

A knot is called **amphicheiral** if it is isotopic to its mirror image. By Theorem 1.1, we have the following.

**Theorem 1.3.** Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then

$$t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1}) = (t^m + (-1)^m) V_K(t) V_K(t^{-1}).$$

In particular, if $\overline{K}$ is amphicheiral, then $V_{\overline{K}}(t) = V_K(t) V_K(t^{-1})$.

Now we restrict to the special values of the Jones polynomial. First we consider the values of the first derivative of the Jones polynomial at $-1$. We denote a (Laurent) polynomial $f(t)$ evaluated at $r$ by $[f(t)]_{t=r}$.

**Theorem 1.4.** Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then

$$\left[\frac{d}{dt} V_{\overline{K}}(t)\right]_{t=-1} = m \{ V_K(-1)^2 - V_{D_K \cup D_K^*(\infty, \infty)}(-1) \}.$$ 

**Corollary 1.5.** Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then $\left[\frac{d}{dt} V_{\overline{K}}(t)\right]_{t=-1} \equiv 0 \mod 8|m|.$

Next we consider the values of the first derivative of the Jones polynomial at $i$.

**Theorem 1.6.** Let $\overline{K}$ be a knot with a symmetric union presentation of the form $D_K \cup D_K^*(\infty, m)$. Then we have $\left[\frac{d}{dt} V_{\overline{K}}(t)\right]_{t=i} = a + bi$ where $a, b \in 4\mathbb{Z}$. In particular, if $m$ is even $(1 \leq i \leq k)$ or $\overline{K}$ is amphicheiral, then $\left[\frac{d}{dt} V_{\overline{K}}(t)\right]_{t=i} \in 4\mathbb{Z}$.

**Remark 1.7.** Theorem 1.1, Corollary 1.2, Theorem 1.4 and Theorem 1.6 can be generalized to the case of $D \cup D^*(\infty, m_1, \ldots, m_k)$. However we do not give the detail since the goal of this paper is to give an obstruction for a knot to have a symmetric union presentation $D_K \cup D_K^*(\infty, m)$. 
In this paper, all knots and links are oriented unless otherwise stated. In Section 2, we give the definitions of the Jones polynomial and a symmetric union. In Section 3, we shall prove Theorem 1.1, Corollary 1.2 and Theorem 1.3. In Section 4, we shall prove Theorem 1.4 and Corollary 1.5 and observe a topological property of the Jones polynomial with respect to a symmetric union in special cases. In Section 5, we shall prove Theorem 1.6 and show a calculation in the case of the example in Section 4. In Section 6, we introduce the minimal twisting number of a knot with a symmetric union presentation. It is the smallest number of trivial tangles (with twists) appearing on the axis of a symmetric union presentation of a knot, the minimum taken over all symmetric union presentations for the knot. We shall show that the minimal twisting number of a knot 10_{12} is equal to two. In Section 7, we consider the amphicheirality of symmetric unions.

Acknowledgements. This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Young Scientists (B), 2011-2014(23740046).

2. Definitions

Definition 2.1. Let $K$ be a knot in the 3-space. We denote a diagram of $K$ by $D_K$. The bracket polynomial of a diagram of a knot $K$, $\langle D_K \rangle$ can be defined as a polynomial which satisfies the following identities.

i) $\langle \bigcirc \bigcirc \rangle = 1$,

ii) $\langle D_K \sqcup \bigcirc \bigcirc \rangle = -(A^2 + A^{-2})\langle D_K \rangle$,

iii) $\langle \bigtriangleup \bigtriangleup \rangle = A\langle \bigtriangleup \rangle + A^{-1}\langle \bigtriangleup \rangle$.

We define $V_{D_K}(t) \in \mathbb{Z}[t^{1/2}, t^{-1/2}]$ by $V_{D_K}(t) = \{(-A^3)^{-\omega(D_K)}\langle D_K \rangle\}_{t^{1/2}, t^{-1/2}}$ for any diagram $D_K$ for $K$, where $\omega$ is the writhe of the diagram. (The writhe is the number of positive crossings of $D_K$ minus the number of negative crossings of $D_K$.) It is shown that $V_{D_K}(t)$ is an invariant of the link [5], [9]. Then we denote $V_{D_K}(t)$ by $V_K(t)$ and call it the Jones polynomial of $K$.

Here we define a symmetric union in [7] as follows. We denote the tangles made of half twists by integers $n \in \mathbb{Z}$ and the horizontal trivial tangle by $\infty$ as in Figure 1.

Definition 2.2. Let $D$ be an unoriented knot diagram and $D^*$ the diagram $D$ reflected at an axis in the plane. If in the symmetric placement of $D$ and $D^*$ we replace the tangles $T_i = 0, (i = 0, \ldots, k)$ on the symmetry axis by $T_0 = \infty$ and $T_i = m_i \in \mathbb{Z}$ for $i = 1, \ldots, k$, we call the result a symmetric union of $D$ and $D^*$ and write $D \sqcup D^*(\infty, m_1, \ldots, m_k)$. The partial knot $\hat{K}$ of the symmetric union is the knot given by the diagram $D$. See Figure 1 for an illustration of the definition.

If a knot $K$ has a diagram $D \sqcup D^*(\infty, m_1, \ldots, m_k)$, then the diagram is called a symmetric union presentation for $K$ and we say that the knot $K$ is a symmetric union.
3. A formula of the Jones polynomial

Proof of Theorem 1.1. By using a skein relation of Kauffman bracket polynomial, we have

\[
\langle D_K \cup D_K^* (\infty, m) \rangle = A^{m/|m|} \langle D_K \cup D_K^* (\infty, m-1) \rangle + F_1 \langle D_K \cup D_K^* (\infty, \infty) \rangle
\]

\[
= (A^{m/|m|})^{|m|} \langle D_K \cup D_K^* (\infty, 0) \rangle + (F_1 + \cdots + F_{|m|}) \langle D_K \cup D_K^* (\infty, \infty) \rangle
\]

\[
= A^m \langle D_K \cup D_K^* (\infty, 0) \rangle + (\sum_{j=1}^{|m|} F_j) \langle D_K \cup D_K^* (\infty, \infty) \rangle,
\]

where each \(F_j\) is the polynomial obtained by applying the skein relation and a finite number of type I Reidemeister moves [9] to the bracket polynomial. In fact, a single type I Reidemeister move changes the bracket polynomial by a factor of \(-A^{\pm 3}\) ([9], Lemma 3.2).

Now we calculate a formula of \(\sum_{j=1}^{|m|} F_j\) by considering the unknot instead of \(K\) as follows. We assume that \(D_K\) is a diagram as in Figure 2 so that we have a symmetric union of the unknot. We denote the diagram by \(D_o\).

Then the resultant symmetric union is a diagram of the unknot with \(r\) crossings where \(r = |m|\) such that it can be transformed into a diagram of the unknot with no crossings by \(r\) type I Reidemeister moves. Thus we have

\[
\langle D_o \cup D_o^* (\infty, m) \rangle = (-A^{-3m/|m|})^{|m|} = (-1)^{|m|} A^{-3m},
\]

\[
\langle D_o \cup D_o^* (\infty, 0) \rangle = 1,
\]
\[ \langle D_o \cup D_o^*(\infty, \infty) \rangle = -A^{-2} - A^2. \]

Then we have
\[ \sum_{j=1}^{|m|} F_j = \frac{\langle D_o \cup D_o^*(\infty, m) \rangle - A^m \langle D_o \cup D_o^*(\infty, 0) \rangle}{\langle D_o \cup D_o^*(\infty, \infty) \rangle} \]
\[ = \frac{(-1)^{|m|}A^{-3m} - A^m}{-A^{-2} - A^2} = \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2}. \]

Since \( \omega(D_K \cup D_K^*(\infty, m)) = -m \), we obtain that
\[ V_{D_K \cup D_K^*(\infty, m)}(A) = (-A^3)^m \langle D_K \cup D_K^*(\infty, m) \rangle \]
\[ = (-A^3)^m \{ A^m \langle D_K \cup D_K^*(\infty, 0) \rangle + \frac{(-A^{-3})^m - A^m}{-A^{-2} - A^2} \langle D_K \cup D_K^*(\infty, \infty) \rangle \} \]
\[ = (-1)^m A^4 \langle D_K \cup D_K^*(\infty, 0) \rangle + \frac{1 - (-1)^m A^4}{-A^{-2} - A^2} \langle D_K \cup D_K^*(\infty, \infty) \rangle. \]

Using \( t^{1/2} = A^{-2} \), we have
\[ V_{D_K \cup D_K^*(\infty, m)}(t) \]
\[ = (-1)^m t^{-m} V_{D_K \cup D_K^*(\infty, 0)}(t) + (1 - (-1)^m t^{-m}) V_{D_K \cup D_K^*(\infty, \infty)}(t). \]

**Remark 3.1.** By a result of Eisermann ([1], Theorem 1), we know that \( V_{D_K \cup D_K^*(\infty, \infty)}(t) \) in the statement of Theorem 1.1 is always a Laurent polynomial.

**Proof of Corollary 1.2.** By Theorem 1.1 and a property that the Jones polynomial is multiplicative under connected sum of knots ([9], p. 29), we have
\[ V_K(- \exp(\frac{2\pi i}{s})) = V_{D_K \cup D_K^*(\infty, 0)}(- \exp(\frac{2\pi i}{s})) \]
\[ = V_K(- \exp(\frac{2\pi i}{s}) \cdot V_K(- \exp(\frac{2\pi i}{s})). \]
Proof of Theorem 1.3. The first part of the theorem is obtained as follows. By Theorem 1.1, we have
\[
\begin{align*}
t^m V_{\mathcal{K}}(t) + (-1)^m V_{\mathcal{K}}(t^{-1}) \\
= t^m ((-1)^m t^{-m} V_{D_K \cup D_\infty(\infty,0)}(t) + (1 - (-1)^m t^{-m}) V_{D_K \cup D_\infty(\infty,\infty)}(t)) \\
+ (-1)^m ((-1)^m t^{-m} V_{D_K \cup D_\infty(\infty,0)}(t^{-1}) + (1 - (-1)^m t^{-m}) V_{D_K \cup D_\infty(\infty,\infty)}(t^{-1})) \\
= (-1)^m V_{D_K \cup D_\infty(\infty,0)}(t) + (t^m - (-1)^m) V_{D_K \cup D_\infty(\infty,\infty)}(t) \\
+ t^m V_{D_K \cup D_\infty(\infty,0)}(t) + ((-1)^m - t^{-m}) V_{D_K \cup D_\infty(\infty,\infty)}(t) \\
= (t^m + (-1)^m) V_{D_K \cup D_\infty(\infty,0)}(t) \\
= (t^m + (-1)^m) V_{\mathcal{K}}(t) \cdot V_{\mathcal{K}}(t^{-1}).
\end{align*}
\]

The latter part of the theorem follow immediately from the first part because $V_{\mathcal{K}}(t) = V_{\mathcal{K}}(t^{-1})$ if $\mathcal{K}$ is amphicheiral ([9], p. 29).

\[\Box\]

4. Evaluation of the derivative at $-1$

Proof of Theorem 1.4. By Theorem 1.1, we know that
\[
\frac{d}{dt} V_{\mathcal{K}}(t) = \frac{d}{dt} ((-1)^m t^{-m} V_{D_K \cup D_\infty(\infty,0)}(t)) \\
+ \frac{d}{dt} ((1 - (-1)^m t^{-m}) V_{D_K \cup D_\infty(\infty,\infty)}(t)).
\]

Since
\[
\frac{d}{dt} ((-1)^m t^{-m} V_{D_K \cup D_\infty(\infty,0)}(t)) = V_{D_K}(t) \cdot V_{D_K}(t^{-1})
\]
and
\[
\frac{d}{dt} ((1 - (-1)^m t^{-m}) V_{D_K \cup D_\infty(\infty,\infty)}(t)) = - \frac{d}{dt} V_{D_K}(t^{-1}) |_{t^{-1}} = 0,
\]
we have $[\frac{d}{dt} (V_{D_K}(t) \cdot V_{D_K}(t^{-1}))]_{t^{-1}} = 0.$

Then
\[
[\frac{d}{dt} ((-1)^m t^{-m} V_{D_K \cup D_\infty(\infty,0)}(t))]_{t^{-1}} = \frac{d}{dt} ((-1)^m t^{-m}) |_{t^{-1}} (V_{D_K}(-1))^2 = m(V_{D_K}(-1))^2.
\]

On the one hand, we have
\[
[\frac{d}{dt} ((1 - (-1)^m t^{-m}) V_{D_K \cup D_\infty(\infty,\infty)}(t))]_{t^{-1}} = -m [V_{D_K \cup D_\infty(\infty,\infty)}(t)]_{t^{-1}} + \frac{d}{dt} V_{D_K \cup D_\infty(\infty,\infty)}(t) |_{t^{-1}} = -m V_{D_K \cup D_\infty(\infty,\infty)}(-1).
\]

Therefore we have
\[
[\frac{d}{dt} V_{\mathcal{K}}(t)]_{t^{-1}} = m V_{D_K}(-1)^2 - m V_{D_K \cup D_\infty(\infty,\infty)}(-1). \quad \Box
\]
Here we need the following theorem due to Eisermann to prove Corollary 1.5.

**Theorem 4.1** ([1]). If $K$ is a ribbon link, then $\nabla_K(-1) \equiv 1 \mod 8$.

**Proof of Corollary 1.5.** By Theorem 1.4, we have

$$\left\lfloor \frac{d}{dt} V_K(t) \right\rfloor_{t=-1} = mV_K(-1)^2 - m\nabla_{D_K \cup D^*_K}(\infty, \infty)(-1)$$

$$= mV_{D_K \cup D^*_K}(\infty, 0)(-1) - m\nabla_{D_K \cup D^*_K}(\infty, \infty)(-1).$$

By Theorem 4.1, we know that $V_{D_K \cup D^*_K}(\infty, 0)(-1)$ and $\nabla_{D_K \cup D^*_K}(\infty, \infty)(-1) \equiv 1 \mod 8$. Thus we have $m(V_{D_K \cup D^*_K}(\infty, 0)(-1) - \nabla_{D_K \cup D^*_K}(\infty, \infty)(-1)) \equiv 0 \mod 8|m|$. □

**Example 4.2.** Let $K_m$ ($m \in \mathbb{Z}$) be a symmetric union as described in Figure 3.

![Figure 3](image)

Then by Theorem 1.4, $\left\lfloor \frac{d}{dt} V_{K_m}(t) \right\rfloor_{t=-1} = m \cdot 9 - m = 8m$. Thus we know that $K_m$ cannot be expressed as $D_K \cup D^*_K(\infty, n)$ if $m$ is not divisible by $|n|$. In particular, if $m$ is odd, then $K_m$ cannot have a symmetric union presentation $D_K \cup D^*_K(\infty, n)$ where $n$ is even.

**5. Evaluation of the derivative at $\sqrt{-1}$**

We say that a Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ is symmetric if it satisfies that $f(t) = f(t^{-1})$. For example, the Jones polynomial of an amphicheiral knot is known to be symmetric ([9], p. 29).

**Proposition 5.1.** Let $K$ be a knot. Then we have

$$\left\lfloor \frac{d}{dt} V_K(t) \right\rfloor_{t=\sqrt{-1}} = a + bi,$$

where $a, b \in 2\mathbb{Z}$.

**Proof.** By results of Jones [4], $V_K(t)$ can be divided by $(t-1)(t^3-1)$. Let $V_K(t) = (t-1)(t^3-1)w(t)$. Then $V_K(t) = (t-1)^2(t^2+t+1)w(t)$ and $\frac{d}{dt} V_K(t) =$
2(t - 1)(t^2 + t + 1)w(t) + (t - 1)^2 \frac{d}{dt}((t^2 + t + 1)w(t))$. So \( \frac{d}{dt}V_L(t)|_{t=1} = -2(i + 1)w(i) - 2i(\frac{d}{dt}((t^2 + t + 1)w(t)))|_{t=1} \).

**Proof of Theorem 1.6.** By Theorem 1.1, we have

\[
\frac{d}{dt}V_L(t) = \frac{d}{dt}((-1)^{m-t^m})V(D_{K} \cup D_{K})(\infty, 0)(t) + (-1)^{m-t^m}\frac{d}{dt}V(D_{K} \cup D_{K})(\infty, 0)(t) \\
\quad + \frac{d}{dt}(1 - (-1)^{m-t^m})V(D_{K} \cup D_{K})(\infty, \infty)(t) \\
\quad + (1 - (-1)^{m-t^m})\frac{d}{dt}V(D_{K} \cup D_{K})(\infty, \infty)(t).
\]

Since \( V_L(i) = (\sqrt{2})^{n(L)-1} \) for any ribbon knot \( L \) [13], where \( n(L) \) is the number of components of \( L \), we know that \( V(D_{K} \cup D_{K})(\infty, \ell)(i) = 1 (\ell = 0, \infty) \). Then we have

\[
\left[ \frac{d}{dt}V_L(t) \right]_{t=1} = \left[ \frac{d}{dt}((-1)^{m-t^m}) \right]_{t=1} + \left[ (-1)^{m-t^m}\frac{d}{dt}V(D_{K} \cup D_{K})(\infty, 0)(t) \right]_{t=1} \\
\quad + \left[ \frac{d}{dt}(1 - (-1)^{m-t^m}) \right]_{t=1} \\
\quad + \left[ (1 - (-1)^{m-t^m})\frac{d}{dt}V(D_{K} \cup D_{K})(\infty, \infty)(t) \right]_{t=1}.
\]

Since \( \left[ \frac{d}{dt}((-1)^{m-t^m}) \right]_{t=1} + \left[ \frac{d}{dt}(1 - (-1)^{m-t^m}) \right]_{t=1} = 0 \), we have

\[
\left[ \frac{d}{dt}V_L(t) \right]_{t=1} = \left[ (-1)^{m-t^m}\frac{d}{dt}V(D_{K} \cup D_{K})(\infty, 0)(t) \right]_{t=1} \\
\quad + \left[ (1 - (-1)^{m-t^m})\frac{d}{dt}V(D_{K} \cup D_{K})(\infty, \infty)(t) \right]_{t=1}.
\]

To prove the theorem, we show that

\[
\left[ \frac{d}{dt}V(D_{K} \cup D_{K})(\infty, 0)(t) \right]_{t=1}, \left[ \frac{d}{dt}V(D_{K} \cup D_{K})(\infty, \infty)(t) \right]_{t=1} \in 4\mathbb{Z}
\]
as follows. First we need the following lemma.

**Lemma 5.2.** Let \( a(t) \) be a symmetric Laurent polynomial. If \( a(1) = 0 \), then \( a(t) \) can be divided by \( (t^{1/2} - t^{-1/2})^2 \).

**Proof.** By the assumption, we know that \( a(t) \) can be divided by \( t - 1 \) by the factor theorem. Since \( a(t) \) is symmetric, \( a(t) \) can be also divided by \( t^{1-1} \). \( \square \)

By the fact that \( V_L(1) = (\sqrt{2})^{n(L)-1} \) for any link \( L \) [10], we know that

\[
V(D_{K} \cup D_{K})(\infty, 0)(1) = V(D_{K} \cup D_{K})(\infty, \infty)(1) = 1.
\]

Then by Lemma 5.2, we know that

\[
V(D_{K} \cup D_{K})(\infty, 0)(t) - 1 \quad \text{and} \quad V(D_{K} \cup D_{K})(\infty, \infty)(t) - 1
\]
can be divided by \((t^{1/2} - t^{-1/2})^2\). Thus each of \(V_{D_K \cup D_K^*(\infty,0)}(t)\) and 
\(\overline{V_{D_K \cup D_K^*(\infty,0)}}(t)\) has a form of \((t^{1/2} - t^{-1/2})^2w(t) + 1\) where \(w(t)\) is a certain symmetric Laurent polynomial. Then we have
\[
\frac{d}{dt}V_{D_K \cup D_K^*(\infty,\ell)}(t)|_{t=\pm i} = 2w(i) - \frac{d}{dt}w(t)|_{t=\pm i} \quad (\ell = 0, \infty).
\]
Note that \(w(i) = 0\). Actually by the fact that \(V_L(t) = (-\sqrt{2})^{m(L)-1}\) for any proper link \(L\) with trivial Arf invariant \([12]\) and the fact that a symmetric union is a proper ribbon link \([7]\) with trivial Arf invariant, we know that 
\(\overline{V_{D_K \cup D_K^*(\infty,\ell)}}(t)|_{t=\pm i} = 1\). So we have 
\(-2)w(i) = \overline{V_{D_K \cup D_K^*(\infty,\ell)}}(t)|_{t=\pm i} = 1 = 0\). Thus 
\(w(i) = 0\). Now it is easily seen that \(\frac{d}{dt}w(t)|_{t=\pm i} \in 2\mathbb{Z}\) since \(w(t)\) is a symmetric 
Laurent polynomial. Thus \(\left[\frac{d}{dt}V_{D_K \cup D_K^*(\infty,\ell)}(t)\right]|_{t=\pm i} \in 4\mathbb{Z}\). Then we know that 
\(\frac{d}{dt}\overline{V_{K}(t)}|_{t=\pm i}\) has a form \(a + bi\) where \(a, b \in 4\mathbb{Z}\).

In the case when \(m\) is even, both of \([-1)^m t^{-m}\] and \([1 - (-1)^m t^{-m}]\) are integers. Thus we have
\[
\left[\frac{d}{dt}V_{K}(t)\right]|_{t=\pm i} \in 4\mathbb{Z}.
\]

We denote the complex conjugate of a complex number \(x\) by \(\overline{x}\). Then we can easily see the following lemma.

**Lemma 5.3.** For any Laurent polynomial \(a(t)\), we have
\[
\frac{d}{dt}a(t)|_{t=\pm i} = \frac{1}{i} \frac{d}{dt}a(t^{-1})|_{t=\pm i}.
\]

If \(\overline{K}\) is amphicheiral, then 
\(\left[\frac{d}{dt}\overline{V_{K}(t)}\right]|_{t=\pm i} = \left[\frac{d}{dt}\overline{V_{K}(t^{-1})}\right]|_{t=\pm i}\) since 
\(\overline{V_{K}(t)} = V_{\overline{K}}(t^{-1})\) \([9], p. 29\). So by Lemma 5.3, we know that 
\(\left[\frac{d}{dt}\overline{V_{K}(t)}\right]|_{t=\pm i}\) is an integer.

So we have 
\(\left[\frac{d}{dt}\overline{V_{K}(t)}\right]|_{t=\pm i} \in 4\mathbb{Z}\). This completes the proof. \(\square\)

**Example 5.4.** Let \(K_m\) be a knot in Figure 3 again. Then by Theorem 1.1, we have
\[
\frac{d}{dt}V_{K}(t) = \left[(-1)^m t^{-m} \frac{d}{dt}V_{3_1 \# 3_1^*}(t) + (1 - (-1)^m t^{-m}) \frac{d}{dt}V_{O^2}(t)\right]|_{t=\pm i} = i^m \cdot 4 + (1 - i^m) \cdot 0 = 4i^m,
\]
where \(O^2\) is a 2-component trivial link and \(3_1 \# 3_1^*\) is a connected sum of the knot \(3_1\) \([9], p. 5\) and the mirror image \(3_1^*\). Thus by Theorem 1.6, if \(m\) is an odd integer, we know that \(K_m\) cannot be expressed as \(D_K \cup D_K^*(\infty, n)\) where \(n\) is even.
6. The minimal twisting number

In this section, we introduce the minimal twisting number for a knot with a symmetric union presentation.

**Definition 6.1.** We call the number \( k \) of \( D_K \cup D^*_K(\infty, m_1, \ldots, m_k) \) the twisting number of the symmetric union. The minimal twisting number of a knot \( \overline{K} \) with a symmetric union presentation is the smallest number of the twisting numbers of all symmetric union presentations for \( \overline{K} \). We denote it by \( \text{tw}(\overline{K}) \).

By the definition, we have the following.

**Proposition 6.2.** The minimal twisting number is an invariant of a knot with a symmetric union presentation.

**Remark 6.3.** Let \( K \) be a knot with a symmetric union presentation. If \( \text{tw}(K) = 0 \), then \( K \) is a connected sum of a knot and its mirror image. If \( \text{tw}(K) = 1 \), then \( K \) is a symmetric union of Kinoshita-Terasaka type [6]. The minimal twisting number is not additive under connected sum in general since \( \text{tw}(K \# K^*) = 0 \) for any knot \( K \).

Now we consider the following problem.

**Problem.** Is there a knot \( K \) with a symmetric union presentation which satisfies \( \text{tw}(K) \geq 2 \)?

**Example 6.4.** For each knot \( K \) in the set of knots \( \{6_1, 8_8, 8_{20}, 9_{46}, 10_3, 10_{22}, 10_{35}, 10_{137}, 10_{140}, 10_{153} \} \), we have \( \text{tw}(K) = 1 \). (See [7] for the knots and their symmetric union presentations.)

**Remark 6.5.** Lamm found symmetric union presentations for all two-bridge ribbon knots [8] and actually, we know that the minimal twisting number of a two-bridge ribbon knot is equal to either one or two according to the result.

By Theorem 1.3, we know that if \( \overline{K} \) is a knot with a symmetric union presentation \( D_K \cup D^*_K(\infty, m) \), which satisfies \( V_K(t) \neq 1 \), then

\[
\frac{t^m V_{\overline{K}}(t) + (-1)^m V_{\overline{K}}(t^{-1})}{t^m + (-1)^m}
\]

is reducible polynomial over \( \mathbb{Z} \). By using this property, we show the following.

**Theorem 6.6.** \( \text{tw}(10_{42}) = 2 \).

**Proof.** Since \( 10_{42} \) has a symmetric union presentation as in Figure 4, we know that \( \text{tw}(10_{42}) \leq 2 \).

Suppose \( 10_{42} \) has a symmetric union presentation \( D_K \cup D^*_K(\infty, m) \).

The Jones polynomial of \( 10_{42} \) is as follows.

\[
V_{10_{42}}(t) = -t^{-5} + 3t^{-4} - 6t^{-3} + 10t^{-2} - 12t^{-1} + 14 - 13t + 10t^2 - 7t^3 + 4t^4 - t^5.
\]
Then we may assume that $f$ is a knot $H$. In this case, we have the following five equations.

$$m$$

By Theorem 1.3, we know that $H$ has a form $V_K(t)V_K(t^{-1})$ for some knot $K$. By calculation, we have

$$H_{10_{42}}^0(t) = -t^{-5} + 3t^{-4} - 7t^{-3} + 10t^{-2} - 12t^{-1} + 15$$

$$H_{10_{42}}^{-1}(t) = -t^{-5} + 4t^{-4} - 6t^{-3} + 10t^{-2} - 13t^{-1} + 13$$

Now we show that $H_{10_{42}}^{\pm 1}(t)$ cannot be $V_K(t)V_K(t^{-1})$ for any knot $K$. Suppose $H_{10_{42}}^1(t)$ has a form $f(t)f(t^{-1})$ for some Laurent polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$. Then we may assume that $f(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$ ($a_0a_5 \neq 0$).

In this case, we have that following five equations.

(1) $a_5a_0 = -1$,
(2) $a_5a_1 + a_4a_0 = 3$,
(3) $a_5a_2 + a_4a_1 + a_3a_0 = -7$,
(4) $a_5a_3 + a_4a_2 + a_3a_1 + a_2a_0 = 10$,
(5) $a_5a_4 + a_4a_3 + a_3a_2 + a_2a_1 + a_1a_0 = -12$,
(6) $a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 = 15$.

By the equation (6), we know that $|a_i| \leq 3$ ($i = 1, 2, 3, 4, 5$) and by the equation (1), we know that $(a_0, a_5) = (1, -1)$ or $(-1, 1)$.

Suppose that $(a_0, a_5) = (-1, 1)$. Then by the equation (3), we have $a_2 + a_4a_1 - a_3 = -7$ and by the equation (2), we have $a_1 = a_4 + 3$. So $(a_1, a_4) = (0, -3)$ or $(1, -2)$ or $(2, -1)$ or $(3, 0)$. If $(a_1, a_4) = (1, -2)$ or $(2, -1)$, then we have

$$\begin{cases} a_2 - a_3 = -5 \\ a_2^2 + a_3^2 = 8 \end{cases}$$

by the equations (3) and (6). Then $a_3 \notin \mathbb{Z}$.
If \((a_1, a_4) = (0, -3)\) or \((3, 0)\), then we have
\[
\begin{aligned}
  a_2 - a_3 &= -7 \\
  a_2^2 + a_3^2 &= 4
\end{aligned}
\]
by the equations (3) and (6). Then \(a_3 \notin \mathbb{Z}\).

Suppose that \((a_0, a_3) = (1, -1)\). Then by the equation (2), we have \(a_4 = a_1 + 3\). So \((a_1, a_4) = (0, 3)\) or \((-1, 2)\) or \((-2, 1)\) or \((-3, 0)\). By the equation (3), we have \(-a_2 + a_4 a_1 + a_3 = -7\). If \((a_1, a_4) = (-1, 2)\) or \((-2, 1)\), then we have
\[
\begin{aligned}
  a_2 - a_3 &= 5 \\
  a_2^2 + a_3^2 &= 8
\end{aligned}
\]
by the equations (3) and (6). Then \(a_3 \notin \mathbb{Z}\).

If \((a_1, a_4) = (0, 3)\) or \((-3, 0)\), then we have
\[
\begin{aligned}
  a_3 - a_2 &= -7 \\
  a_2^2 + a_3^2 &= 4
\end{aligned}
\]
by the equations (3) and (6). Then \(a_3 \notin \mathbb{Z}\).

From the above, we have contradiction and we know that \(H_{10_{42}}^1(t)\) does not have a form \(f(t)f(t^{-1})\) for some Laurent polynomial \(f(t)\). Next suppose that \(H_{10_{42}}^{-1}(t)\) has a form \(f(t)f(t^{-1})\) for some polynomial \(f(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0\) \((a_0 \cdot a_5 \neq 0)\). Then we have that following five equations.

1. \(a_5 a_0 = -1\),
2. \(a_5 a_1 + a_4 a_0 = 4\),
3. \(a_5 a_2 + a_4 a_1 + a_3 a_0 = -6\),
4. \(a_5 a_3 + a_4 a_2 + a_3 a_1 + a_2 a_0 = 10\),
5. \(a_5 a_4 + a_4 a_3 + a_3 a_2 + a_2 a_1 + a_1 a_0 = -13\),
6. \(a_5^2 + a_4^2 + a_3^2 + a_2^2 + a_1^2 = 13\).

By using the same argument above, we have \(\{a_i\} \leq 3\) \((i = 1, 2, 3, 4, 5)\) and \((a_0, a_3) = (1, -1)\) or \((-1, 1)\). Then in each case, we can show that \(a_3 \notin \mathbb{Z}\). Thus we know that \(H_{10_{42}}^{-1}(t)\) cannot be \(V_K(t)V_K(t^{-1})\) for any knot \(K\). This completes the proof. □

Remark 6.7. We can also see that \(\text{tw}(9_{27}) = \text{tw}(10_{99}) = \text{tw}(10_{123}) = 2\). (See [7] for the knots and their symmetric union presentations.)

7. On the amphicheirality of symmetric unions

In this section, we consider symmetric unions related to the following problem.

**Problem (Kirby Problem 1.88(C) (Jones)).** Is there a non-trivial knot with the same Jones polynomial as the unknot?

Symmetric unions often appear when we construct a non-trivial knot with trivial Alexander polynomial (See [9] for the definition). For example, a knot with symmetric union presentation \(D_O \cup D'O(\infty, m_1, \ldots, m_k)\) with even integers \(m_1, \ldots, m_k\) has trivial Alexander polynomial [7]. (Here \(O\) is the unknot.) Then
it is natural to expect to have a non-trivial knot with trivial Jones polynomial from among such symmetric unions.

By using Theorem 1.3, we have the following.

**Proposition 7.1.** If there exists a non-trivial, amphicheiral knot with a symmetric union presentation $D_O \cup D'_O(\infty, m)$, then the Jones polynomial is trivial and, in particular, the answer to the above problem is affirmative.

**Remark 7.2.** For example, $10_{153}$ has a presentation $D_O \cup D'_O(\infty, m)$ (See [7]), but it is not amphicheiral since it has non-trivial Jones polynomial.

Finally, we give more examples of symmetric unions concerning Theorem 1.3.

**Example 7.3.** Let $K_{r,s}$ ($r, s \geq 1$) be a knot with a symmetric union presentation as described in Figure 5. We denote the partial knot of the symmetric union by $K_c$. Then we can find that $K_{r,s}$ is amphicheiral and the Jones polynomial of $K_{r,s}$ is the same as that of $K_c \# K^*_c$ by making use of a result of Watson [14]. Actually, we know that the Khovanov homology of $K_{r,s}$ is equivalent to that of $K_c \# K^*_c$ by the result.

**References**


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